Increasing Supply Chain Robustness through Process Flexibility and Inventory

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Abstract

We study a hybrid strategy that uses both process flexibility and inventory to mitigate risks of plant disruption. In this setting, a firm allocates inventory before disruption while facing demand uncertainties; demand is realized after disruption, and the firm schedules its production using process flexibility to minimize lost sales. This interplay between process flexibility and inventory is modeled as a two-stage robust optimization problem. We show that the robust optimization model can be formulated as a linear program and solved efficiently. Using analytical and numerical analysis, we study the impact of different process flexibility designs on the firm’s inventory decisions. Our analysis reveals three important insights regarding the impact of process flexibility on (i) total inventory level; (ii) freedom in inventory placement; and (iii) inventory allocation strategy. In particular, we find that under designs with high degrees of process flexibility, the firm allocates more inventory to low demand variability products and less inventory to high demand variability products.

Keywords: supply chain risk mitigation strategies, process flexibility, inventory, robust optimization

1 Introduction

Over the past decade, supply chain disruption has emerged as an important business challenge. Indeed, as reported by Munich Re, the number of financially significant natural catastrophes have been steadily going up over the last decades (Hedde 2014). For instance, the 2011 flood in Thailand reduced Intel’s quarterly revenue target by $1 billion.1 Toyota’s production in Japan, to give another example, declined 31.4% in the first six months after the 2011 Japanese earthquake, as compared with its forecast.2

To safeguard against future disruptions, firms have been adopting various risk mitigation strategies to improve resiliency of their supply chains. In this study, we focus on two risk
mitigation strategies: holding additional inventory and adding (process) flexibility. While it is well documented that both strategies can improve supply chain resiliency, the performance of a hybrid risk mitigation strategy that combines inventory and process flexibility is not well understood. Therefore, in this study, we seek to answer the following two questions. Given different levels of flexibility, (i) how much inventory is needed to achieve a required level of resiliency, and (ii) how does one optimally allocate inventory to the various products?

Next, we briefly explain how process flexibility and inventory are used as risk mitigation strategies.

1.1 Risk Mitigation Strategies

Risk mitigation inventory, also known as protective inventory, is the additional inventory that is dedicated to supply chain disruption and hence is independent of lead time, the review policy or the details of the inventory management policy used on a day-to-day basis. Holding inventory beyond cycle stock and safety stock has been identified in a number of papers as an important tool for dealing with supply chain disruption (see literature review in §1.3). Unfortunately, if disruptions last a long period of time, the firm needs to hold a large amount of inventory and hence pay high inventory costs (Tomlin and Wang 2011).

Process flexibility is defined as the “ability to build different types of products in the same manufacturing plant or on the same production line at the same time” (Jordan and Graves 1995). For example, under full flexibility design, each plant is capable of producing all products; while under dedicated design, each plant is capable of producing just a single product, see Figure 1. With process flexibility, the firm is in a better position to match available capacity with variable demand. Unfortunately, implementing full flexibility can be very expensive since each plant needs to be capable of producing all products (Simchi-Levi 2010), and as a result, partial flexibility designs are considered. Under such designs, each plant is capable of producing just a few products.

![Figure 1: Process Flexibility Designs](image)

Intuitively, in the wake of a plant disruption, process flexibility allows the firm to “route” its excess capacity to the disrupted plants. However, while process flexibility can help the firm to safeguard against disruption, it is only viable when the firm has plenty of excess capacity in its plants. When plants are highly utilized, process flexibility is no longer an effective strategy to mitigate the impact of disruptions.

In sum, both inventory and process flexibility have their limitations as disruption mitigation strategies. This motivates us to consider a hybrid strategy which combines inventory and process flexibility. In such hybrid strategy, inventory provides excess capacity during disruption, thus
improving the firm’s ability to reroute production using process flexibility; Vice versa, the existence of process flexibility greatly reduces the amount of inventory required. The synergy between inventory and process flexibility makes the hybrid strategy more compelling than either of the two strategies individually.

1.2 Overview and Summary of Results

Robust Optimization Model for Inventory Allocation. In §2, we propose a two-stage robust optimization problem to model the firm’s risk mitigation strategies. In the first stage, given a process flexibility design, the firm optimizes its inventory levels for all products to ensure that demand shortage does not exceed a certain level subject to plant disruptions. Uncertainties in plant disruption and product demand are modeled using uncertainty sets, which are chosen to approximate probabilistic settings. In the second stage, after disruption happens and demands are realized, the firm uses both inventory and process flexibility to minimize demand shortage. In particular, the firm may leverage its manufacturing flexibility and choose a production schedule to minimize lost sales.

We show that the robust optimization problem can be reformulated as a linear programming problem (§3). This linear program can be solved quickly for systems with a moderate number of products. In special cases where demand and disruption uncertainty sets satisfy certain symmetric properties, we give a closed-form characterization of the optimal inventory decision. Using this characterization, we analyze a family of different flexibility designs and show the impact of these designs on the firm’s inventory decisions (§4). In particular, we consider three aspects: (i) total inventory level, (ii) freedom in inventory placement, and (iii) inventory allocation strategy.

Total Inventory Levels. In §4.1, we characterize the exact total inventory levels for a family of flexibility designs known as the K-chains. Using this characterization, we observe that while changing from a dedicated network to a 2-chain design provides a large portion of benefit, there is also significant benefit achieved by increasing the degree of flexibility beyond the 2-chain design, even when demand variability is relatively low. We also observe that the minimum inventory level associated with full flexibility can be often achieved by having limited degrees of flexibility.

Freedom of Inventory Placements. In §4.2, we show that for many process flexibility designs, the optimal solution that minimizes total inventory is not unique. In other words, flexibility provides the firm with freedom of placing inventory to mitigate disruption. For example, our analysis shows that for many different flexibility designs, the firm may choose to hold inventory for just a few products, instead of holding inventory for every product.

Inventory Allocation Strategy. In §4.3, we find that when the firm has a high degree of flexibility, it may be optimal to hold more inventory for products with lower demand variability, and to hold less inventory for products with higher demand variability. A similar result is observed in stochastic settings where we specify probability distributions of plant disruption and product demands (§5.3). This observation is in contrast with classical inventory management theory, which states that higher demand variance implies that more inventory is needed in order to achieve the same service level. Indeed, we show that classical theory remains true when the
firm has a low degree of flexibility. But when plants are highly flexible, it is more effective to satisfy high variability demand using plant capacity, and hence more inventory is allocated to low variability demand.

The three insights described above are mostly obtained for systems where the number of plants is equal to the number of products. Although such settings are not typical to real world systems, as Jordan and Graves (1995) argues, understanding these ideal settings provides insight into realistic scenarios. Indeed, in §5.4 we consider an example where the number of plants is different than the number of products. This example is based on a real dataset with 16 vehicle models and 8 assembly plants at General Motors presented in Jordan and Graves (1995). Using numerical experiments, we find that all three main insights are carries over to this example.

1.3 Related Literature

There is rich literature on supply chain risk mitigation using inventory management strategies. Most of these papers (e.g., Meyer et al. 1979, Song and Zipkin 1996, Arreola-Risa and DeCroix 1998) assume a single product setting. More recently, researchers have also studied inventory mitigation strategies under multi-period, multi-echelon settings (e.g., Bollapragada et al. 2004, DeCroix 2013).

Academics have also considered hybrid strategies where firms use both dual sourcing and inventory to mitigate risks (e.g., Güler and Parlar 1997, Tomlin 2006). Our paper is related to the multiple sourcing/inventory mitigation literature in the following sense: A supply chain design where multiple plants are able to produce the same product can be viewed as a multiple sourcing strategy. However, dual sourcing is mostly studied in a single product system, while our model includes multiple sources of supply (plants) and multiple products.

Process flexibility is also referred to as “mix flexibility” or “product flexibility”. The study of partial process flexibility started with the seminal work of Jordan and Graves (1995). In their paper, Jordan and Graves propose the long chain (a.k.a. 2-chain) design, and empirically observe that while in the long chain each plant is capable of producing just a few products, this strategy has almost the same expected sales as that of full flexibility. More recently, there has been a series of theoretical developments in explaining the effectiveness of long chain and sparse flexibilities designs (e.g., Chou et al. 2010, 2011, Simchi-Levi and Wei 2012, Wang and Zhang 2013, Simchi-Levi and Wei 2014). In parallel, the academic community also investigated the (optimal) mix between dedicated and fully flexible resources (e.g., Fine and Freund 1990, Bish and Wang 2004, Tomlin and Wang 2005, Goyal and Netessine 2011), and the properties of more general flexible resource selection problems (e.g., Bassamboo et al. 2010).

Of particular interest to us is the research that observed flexibility as an effective tool to safeguard against supply disruption. Tomlin and Wang (2005) considers a risk mitigation strategy that uses a combination of mix-flexibility and dual sourcing. Tang and Tomlin (2008) suggests process flexibility as one of the five types of flexibility strategies that can be used to mitigate supply chain disruptions. And finally, Sodhi and Tang (2012) lists flexible manufacturing processes as one of the eleven robust supply chain strategies.

To the best of our knowledge, this paper is the first to consider a hybrid strategy combining partial process flexibility and inventory as a way to mitigate against supply disruption in multi-products systems. The hybrid strategy requires the firm to allocate inventory before disruption (ex-ante), and to schedule its flexible production after disruption (ex-post). Finally, in our
model, the inventory allocation decisions are made to ensure that the demand shortage is always lower than some threshold under a set of scenarios. This modeling feature is common in the robust optimization literature (e.g., Ben-Tal et al. 2009, Bandi and Bertsimas 2012). Also, it is similar in spirit to the model studied in Graves and Willems (2000), where the firm commits to a certain service level under a range of uncertainties.

2 Model

We consider a firm that has \( M \) plants, denoted by \( S_i \) for \( i = 1, \ldots, M \), and produces \( N \) products, denoted by \( T_j \) for \( j = 1, \ldots, N \). A plant can produce one or multiple products, and the flexibility design specifies which products each plant can make. More formally, the flexibility design can be represented as a bipartite network, where a link \((S_i, T_j)\) exists if and only if plant \( i \) is able to produce product \( j \). We refer to the set of such links as \( \mathcal{F} \). For a given flexibility design \( \mathcal{F} \) and a subset of product \( A \subset \{T_1, \ldots, T_N\} \), we define \( P_F(A) = \{S_i : (S_i, T_j) \in \mathcal{F}, T_j \in A\} \) as the plants that can produce at least one of these products.

The plant capacities and product demands are uncertain quantities. The uncertainties in plant capacities capture plant disruptions, while the uncertainties in product demands capture the fluctuation of market demand. In addition to flexibility, the firm also uses inventory to protect the supply chain. Such inventory is referred to as risk mitigation inventory, and \( s_j \) is used to denote the inventory of product \( j \).

The firm faces a two-stage decision problem. In the first stage, given the existing flexibility structure, the firm determines the level of risk mitigation inventory for each product before disruption happens and before product demands are realized. In the second stage, the firm observes realized product demands and available plant capacities after disruption, and adjusts its production schedule to minimize demand shortage.

The interplay between risk mitigation inventory and flexibility is a key factor that we capture in our model. Inventory and flexibility are closely related, because the firm’s demand shortage in the second stage is determined by the inventory decision in first stage as well as its flexibility design. Although we didn’t include flexibility as a decision variable, we are going to consider a family of different flexibility designs and analyze how different designs change the firm’s inventory decisions.

We first define the firm’s second stage problem of minimizing shortage in §2.1, and then outline the complete two-stage problem in §2.2.

2.1 Shortage Function

After disruption happens, suppose that the firm observes available capacity \( c = (c_1, c_2, \ldots, c_M) \) for each plant and realized demand \( d = (d_1, d_2, \ldots, d_N) \) for each product. We define \( \Pi(\mathcal{F}, s, c, d) \) to be the minimum demand shortage, given flexibility design \( \mathcal{F} \), inventory vector (decision from the first stage) \( s = (s_1, s_2, \ldots, s_N) \), capacities \( c \) and demands \( d \).

Let \( x_{ij} \) be the units of product \( j \) produced by plant \( i \), and \( l_j \) be the shortage of product \( j \).
Π(ℱ, s, c, d) is expressed as the following optimization problem.

\[
Π(ℱ, s, c, d) = \min_i \sum_{j=1}^{N} l_j \\
\sum_{i: (S_i, T_j) \in ℱ} x_{ij} + l_j \geq d_j - s_j, \quad \forall 1 \leq j \leq N \\
\sum_{j: (S_i, T_j) \in ℱ} x_{ij} \leq c_i, \quad \forall 1 \leq i \leq M \\
l_j, x_{ij} \geq 0.
\]

The first constraint is the definition of shortage or lost sales for each product. The second constraint specifies that the sum of units produced at each plant cannot exceed its realized capacity. Without loss of generality, we assume that any product consumes one unit of capacity, otherwise we can re-scale the units for each product. Throughout the paper, Π(ℱ, s, c, d) will be referred to as the shortage function.

We develop two properties of Π(ℱ, s, c, d) for later use.

**Lemma 1.** Π(ℱ, s, c, d) is convex in s, c and d.

**Lemma 2.** The optimal value of Π(ℱ, s, c, d) can be rewritten as

\[
Π(ℱ, s, c, d) = \max_{A \subset \{T_1, \ldots, T_N\}} \sum_{T_j \in A} (d_j - s_j) - \sum_{S_i \in P_{A}(A)} c_i 
\]

Lemma 2 provides us with a combinatorial expression for the shortage function. The expression is similar to a min-cut type formulation in the network flow context. This is because of the formulation Π(ℱ, s, c, d) can be viewed as a network flow problem (shown in Appendix A).

### 2.2 Robust Optimization Model for Inventory Decision

In the first stage, the firm faces the problem of minimizing total inventory while ensuring that the shortage is at most δ. In what follows, δ is referred to as the shortage allowance. We use ℰc ⊂ ℜ+M to denote the capacity uncertainty set, and ℰd ⊂ ℜ+N to denote the demand uncertainty set. Let s = (s1, s2, ..., sN) be the vector of inventory for all products. The optimization problem can be formulated as the following worst-case (WM) model:

**Problem-WM:**

\[
\min_N \sum_{j=1}^{N} s_j \\
s.t. Π(ℱ, s, c, d) \leq δ, \forall c \in ℰc, d \in ℰd \\
s_j \geq 0, \forall 1 \leq j \leq N.
\]

The robust optimization approach that we take differs from many risk mitigation methods proposed in literature, which require firms to identify probabilities of disruption events. We show in the next subsection that the uncertainty sets can be chosen using the Central Limit Theorem to mimic the behavior of stochastic uncertainties, and therefore, Problem-WM can be viewed as an approximation of the stochastic model. To ensure the insights obtained from Problem-WM
is not restrictive to the worst-case modelling assumption, we use numerical experiments to show that our insights are also valid for its stochastic counterpart (§5).

The main reason we study the worst-case model is because its computational and analytical tractability. In §3, we show that worst-case model can be solved efficiently using a linear program even when the elements in the uncertainty sets are infinite. In §4, we show that if the uncertainty set satisfies a certain symmetric property, we can characterize the optimal solution and thus provide analytical expression for the optimal inventory levels.

The worst-case model also provides several other advantages compared to its stochastic counterpart. First, it is difficult and often impossible to accurately assess the probability of a plant disruption, because disruption can occur from so many different sources, whether from natural disaster, epidemics or factory fire (Simchi-Levi 2010). Because the inventory level can be very sensitive to the probability of disruption, considering the worst-case scenario may offer a more “robust” approach. Second, for small probability events such as plant disruptions, managers might find it useful to understand the maximum possible shortage of demand under wide range of scenarios. This understanding may lead them to better identify scenarios where the supply chain is most vulnerable.

2.3 Choice of Uncertainty Sets

We propose a class of demand uncertainty sets that mimic the properties of stochastic demand distributions. Suppose that the demand for each product \( j \) is denoted by some random variable \( D_j \) with mean \( \mu_j \). For the worst-case model, we consider a class of demand uncertainty sets of the form

\[
U_d = \{ d \mid \sum_{j=1}^{N} d_j \leq \sum_{j=1}^{N} \mu_j + \gamma, \sum_{j=1}^{N} |d_j - \mu_j| \leq \beta, |d_j - \mu_j| \leq \alpha_j, \forall 1 \leq j \leq N \},
\]

with parameters \( \gamma, \beta \) and \( \alpha_j \) for \( 1 \leq j \leq N \). The parameter \( \gamma \) limits the deviation of total realized demand from the total expected demand; the parameter \( \beta \) limits the total absolute (\( L_1 \)) deviation between the realized demand and the mean demand; and finally, the parameters \( \alpha_j \) for \( 1 \leq j \leq N \) limit the deviation of each product’s realized demand from its mean. The values of \( \alpha_j, \beta \) and \( \gamma \) can be determined using probabilistic guidelines such as the Central Limit Theorem (CLT), see Bandi and Bertsimas (2012) for a more detailed discussion on this topic. Appendix B gives an example showing how to select these parameters.

The capacity uncertainty sets are chosen to reflect possible plant disruptions. When the likelihood of plant disruption is low and disruptions at plants are weakly correlated, the chance of disruptions happening at more than one plant is relatively small. In this case, we may let the capacity uncertainty set \( U_c \) to include every scenario where a single plant is disrupted.

Because we are more interested in the capacity uncertainties that arise from disruption, we choose not to model the recurrent production variabilities which may cause a slight increase or decrease to the plant capacities. However, these variabilities can be modelled by again applying CLT-type of constraints, and the theoretical results we develop in the paper apply to capacity uncertainty sets with CLT-type of constraints as well.
3 Computational Algorithm

In the formulation of Problem-WM, for each specific set of capacity and demand, \((c, d)\), \(\Pi(F, s, c, d) \leq \delta\) can be expanded into a set of linear constraints by introducing a set of auxiliary variables \(x_{c, d}\) for each \(c\) and \(d\) (§2.1). Unfortunately, because the number of elements in \(U_c\) and \(U_d\) can be infinite, this would give us an LP with infinitely many constraints and variables. If \(U_c\) and \(U_d\) are polyhedra sets, by convexity of \(\Pi(F, s, c, d)\) (Lemma 1), we just need to consider the extreme points of \(U_c\) and \(U_d\). However, although there are finitely many extreme points, if we simply expand \(\Pi(\cdot)\) using the formulation in §2.1, we would have an astronomical number of variables and constraints even for a system with just a handful of plants and products.

To avoid having a huge number of constraints and variables, we apply Lemma (2) and rewrite Problem-WM as follows.

\[
\text{Problem-WM: } \min \sum_{j=1}^{N} s_j \quad (6)
\]

\[
\text{s.t. } \max_{(c, d) \in U_c \times U_d} \left\{ \frac{1}{\max} \left( \sum_{T_j \in A} (d_j - s_j) - \sum_{S_t \in P_F(A)} c_t \right) \right\} \leq \delta, \forall A \subset \{T_1, \ldots, T_N\} \quad (7)
\]

\[
s_j \geq 0, \forall 1 \leq j \leq N. \quad (8)
\]

Given \(A \subset \{T_1, \ldots, T_N\}\), observe that the values of \(c\) and \(d\) that maximize the left-hand side of Equation (7) are independent of decision variables \(s\). Therefore, we can find \((c^A, d^A)\) for every possible \(A \subset \{T_1, \ldots, T_N\}\) by solving \(2^N\) optimization problems, where \(N\) is the number of products. If \(U_c\) and \(U_d\) are polyhedra, there are \(2^N\) LPs with moderate size. Then, we can solve Problem-WM as an LP with \(2^N\) constraints, where each constraint is associated with a specific \((c^A, d^A)\).

\[
\text{Problem-WM: } \min \sum_{j=1}^{N} s_j \quad (9)
\]

\[
\sum_{T_j \in A} (d_j^A - s_j) - \sum_{S_t \in P_F(A)} c_t^A \leq \delta, \forall A \subset \{T_1, \ldots, T_N\}, \quad (10)
\]

\[
s_j \geq 0, \forall 1 \leq j \leq N. \quad (11)
\]

In our computational experience, Problem-WM can be solved within seconds for \(N \leq 20\). The computational advantage of the worst case model over the stochastic model is quite significant. Indeed, solving a stochastic model with Type 2 service-level constraint (defined in §5) with 10 products and 10 plants takes more than 500 seconds with 10000 random samples from the disruption and demand distributions.\(^3\) By contrast, solving Problem-WM for demand and capacity polyhedron uncertainty sets constructed using the methods described in Section 2.3 takes less than 0.4 second, which is more than 1000 times faster. Finally, note that a stochastic model with Type 1 service-level constraint (defined in Appendix C) is non convex and therefore not even computationally tractable.

\(^3\)The timing is based on Gurobi 6.0.0 LP solver with an Intel Xeon E5440 processor (8 core, 2.83 GHz).
4 Analysis for K-Chain Designs

In order to gain insight and intuition on the inventory allocation strategy as well as the level of inventory required under different degrees of flexibility, in this section, we will study the setting where \( M = N \). To study different degrees of process flexibility, we consider a canonical family of process flexibility designs known as the K-chain, where \( K \) is a strictly positive integer. A \( K \)-chain design is where plant 1 produces product 1 to product \( K \), plant 2 produces product 2 to product \( K + 1 \), and in general plant \( i \) produces products \( i, i + 1, \ldots, i + K - 1 \). Note that the K-chain design is defined in Hopp et al. (2004) as \( K \)-skill chaining, in the context of labor cross-training.

In §4.1 and §4.2, we will assume that the system is symmetric. That is, for any demand instance \( \mathbf{d} \in \mathcal{U}_d \), every permutation of \( \mathbf{d} \) is also in \( \mathcal{U}_d \). Respectively, for any capacity instance \( \mathbf{c} \in \mathcal{U}_c \), every permutation of \( \mathbf{c} \) is in \( \mathcal{U}_c \). For a fixed uncertainty set \( \mathcal{U}_c \) (and \( \mathcal{U}_d \)), we define the quantitative measures \( C^\text{min}(t) \) (and \( D^\text{max}(t) \)) as follows.

**Definition 1.** Define \( C^\text{min}(t) := \min_{\mathbf{c} \in \mathcal{U}_c} \sum_{j=1}^t c_j, \) and \( D^\text{max}(t) := \max_{\mathbf{d} \in \mathcal{U}_d} \sum_{j=1}^t d_j. \)

Because \( \mathcal{U}_c \) and \( \mathcal{U}_d \) are symmetric, \( C^\text{min}(t) \) is the minimum possible sum of \( t \) entries of any \( \mathbf{c} \) in \( \mathcal{U}_c \) and \( D^\text{max}(t) \) is the maximum possible sum of \( t \) entries of any \( \mathbf{d} \) in \( \mathcal{U}_d \). Note that because both \( \mathcal{U}_c \) and \( \mathcal{U}_d \) only contain non-negative vectors, both \( C^\text{min}(t) \) and \( D^\text{max}(t) \) are non-decreasing with \( t \). If \( \mathcal{U}_c \) is convex and \( \mathcal{U}_d \) is concave, then both \( C^\text{min}(t) \) and \( D^\text{max}(t) \) can be computed efficiently by solving convex optimization problems. Moreover, when \( \mathcal{U}_d \) is a symmetric linear polytope with the form

\[
\mathcal{U}_d = \{ \mathbf{d} | \sum_{j=1}^N d_j \leq N + \gamma, \sum_{j=1}^N |d_j - 1| \leq \beta, |d_j - 1| \leq \alpha, \forall 1 \leq j \leq N \},
\]

then we can characterize \( D^\text{max}(t) \). The characterization is discussed in Appendix B.

### 4.1 Total Inventory Required by K-chain

**Lemma 3.** Suppose for any inventory allocation \( \mathbf{s} = (s_1, s_2, \ldots, s_N) \), the rearranged allocation \( \sigma(\mathbf{s}) = (s_2, s_3, \ldots, s_N, s_1) \) (stock \( s_2 \) units of product 1, etc.) is feasible for the Problem-WM, then the optimal inventory allocation can be achieved by allocating inventory equally across all products.

The symmetry of K-chain designs certainly satisfies the condition in Lemma 3, so we can assume the optimal inventory allocation has the property where \( s_j = s \) for all product \( j \). This property turns out to be very important for our analysis for two reasons. First, by restricting ourselves to study inventory allocations satisfying \( s_j = s \) for all product \( j \), we greatly simplifies the original optimization problem to an optimization problem with just a single variable. Second, if \( s_j = s \) for all product \( j \), then both the inventories and the uncertainties are symmetric and this allows us to derive a simple condition for checking the feasibility of \( \mathbf{s} \) in Problem-WM.

Next, we show that for integers \( 1 \leq K \leq N \), if \( \mathcal{F} \) is a K-chain, then we can obtain a closed-form analytical expression for the optimal inventory level.

**Proposition 1.** Let \( \mathcal{F} \) be a K-chain, for some integer \( K \) between 1 and \( N \). Then \( \mathbf{s} \) is an
optimal inventory allocation if \( s_j = s^* \) for all \( 1 \leq j \leq N \) where

\[
s^* = \max \{ \max_{1 \leq t \leq N-K} \left( \frac{D_{\max}(t) - C_{\min}(t + K - 1) - \delta}{t} \right), \max_{N-k < t \leq N} \left( \frac{D_{\max}(t) - C_{\min}(N) - \delta}{t} \right) \}.
\]

Next, we will apply this expression to specific uncertainty sets \( \mathcal{U}_c \) and \( \mathcal{U}_d \) to better understand the impact of \( K \)-chains on the optimal inventory level. In this analysis, we seek understand the sensitivity of the inventory level to different values of \( K \) and different degrees of demand variability.

**Example.** Consider capacity uncertainty set

\[
\mathcal{U}_c = \{ c | (c_1, \ldots, c_N) | c_{i'} = 0, \text{ for some } 1 \leq i' \leq N, \forall i \neq i' \leq N \},
\]

and demand uncertainty set

\[
\mathcal{U}_d = \{ d | \sum_{j=1}^{N} d_j \leq N + 2\sqrt{N} \sigma, \sum_{j=1}^{N} |d_j - 1| \leq \frac{\sqrt{3} \sigma N}{2} + \sigma \sqrt{N}, |d_j - 1| \leq \sqrt{3} \sigma, \forall j = 1 \ldots N \}.
\]

where \( \sigma \) is some parameter between 0 and \( \frac{1}{\sqrt{3}} \). Recall from §2.3, the choice of \( \mathcal{U}_c \) fits the setting where plants are subject to low probability, independent disruptions. The choice of \( \mathcal{U}_d \) fits empirically with the setting where products have i.i.d. uniform distributions over the range \([1 - \sqrt{3} \sigma, 1 + \sqrt{3} \sigma]\), see Appendix B. We test varying values of \( \sigma \) to analyze how different degrees of demand variability affect the inventory level in the \( K \)-chain flexibility design.

We perform a numerical study for \( N = 12 \). Using Appendix B and Proposition 1, we get that for \( K = 1 \), the optimal inventory level for 1-chain is equal to

\[
12 \cdot \max\{1 + \sqrt{3} \sigma - \delta, \frac{6 \sqrt{3} \sigma - (\delta - 1)}{6}, \frac{6 \sqrt{3} \sigma - (1 - \delta)}{10}, \frac{4 \sqrt{3} \sigma + 1 - \delta}{12}, 0\};
\]

for \( 2 \leq K \leq 6 \), the optimal inventory level for \( K \)-chain is equal to

\[
12 \cdot \max\{\frac{6 \sqrt{3} \sigma - (K - 2 + \delta)}{6}, \frac{6 \sqrt{3} \sigma - (1 - \delta)}{10}, \frac{4 \sqrt{3} \sigma + 1 - \delta}{12}, 0\};
\]

and finally, for \( 7 \leq K \leq 12 \), the optimal inventory level for \( K \)-chain is the same as that of full flexibility, which can be expressed as

\[
12 \cdot \max\{\frac{6 \sqrt{3} \sigma - (1 - \delta)}{10}, \frac{4 \sqrt{3} \sigma + 1 - \delta}{12}, 0\}.
\]

Therefore, we can compute the optimal inventory level for any \( K \)-chain with different \( \delta \) and \( \sigma \). In Tables 1, 2 and 3, we output the optimal total inventory levels for 1-chain (dedicated design), 2-chain, 3-chain, 4-chain, 5-chain and 12-chain (full flexibility design) under different variations and different shortage allowances. The following is a summary of the insights obtained from these tables.

First, full flexibility is never required to achieve the optimal inventory level. Indeed, a 7-chain (in a 12-products system) would always require the same amount of inventory as full flexibility, see Equation (16). Moreover, in our numerical examples, the 5-chain design achieves the same
inventory level as full flexibility for all of the parameters we studied.

Second, while changing a dedicated design to a 2-chain design provides a large benefit, the benefit achieved by changing from 2-chain to 3-chain is also significant. This holds even when the demand variability is not high. For example, under medium level of variability ($\sigma = 0.3$) and zero shortage allowance ($\delta = 0$), changing from dedicated to the 2-chain design reduces total inventory by 66%, while changing from the 2-chain to the 3-chain further reduces the total inventory (compared to the inventory of dedicated design) by another 11%. This observation is somewhat different from the insight described in the process literature when no plant disruption is present. In that case, it has been consistently observed that almost all of the benefits are obtained by changing from dedicated to the 2-chain design. One intuitive way to explain the improvement achieved by a 3-chain design over a 2-chain is that while the 2-chain design is very effective in satisfying uncertain demand, the disruption would break the chain. Therefore, a disruption reduces 2-chain’s ability to satisfy uncertain demand. On the other hand, a 3-chain design will still form a chain even if one plant is down.

### 4.2 Inventory Freedom in $K$-chain

In this section, we focus on the following question, is there freedom in selecting different inventory allocations that satisfy the same shortage allowance? We answer this question in affirmative, and characterize the range of all optimal inventory allocations under the $K$-chain flexibility.
Consider a case with very low demand variation. Intuitively, under full flexibility design, the optimal total inventory level that solves Problem-WM is equal to \( D_{\text{max}}(N) - C_{\text{min}}(N) - \delta \), the maximum possible total demand minus the minimum total capacity and the shortage allowance. By Lemma 3, this implies that the optimal inventory placement at each product \( j \) is equal to \( D_{\text{max}}(N) - C_{\text{min}}(N) - \frac{\delta}{N} \).

Next, we make an assumption on demand variability to ensure that this is indeed the optimal inventory allocation.

**Assumption 1.** For any \( 1 \leq j \leq N \), for any \( d \in U_d \), we have \( d_j \geq \frac{D_{\text{max}}(N) - C_{\text{min}}(N) - \delta}{N} \).

With Assumption 1, for any \( d \in U_d \), if the inventory placement at each product \( j \) is equal to \( \frac{D_{\text{max}}(N) - C_{\text{min}}(N) - \delta}{N} \), then all of the inventories can be used to satisfy product demands. Therefore, for any demand \( d \), a fully flexible firm can always first spend all of the inventories to satisfy some of the demand, and then use all of its fully flexible capacities to satisfy the remaining demand. Because the total amount of inventory we have is \( D_{\text{max}}(N) - C_{\text{min}}(N) - \delta \) and it is always used, we will always have enough fully flexible capacities to ensure that the lost sales is lower than the allowance.

To better understand how restrictive Assumption 1 is, let us consider the example where the demand uncertainty set \( U_d \) is defined by Equation (13) of the previous subsection. In that example, Assumption 1 is satisfied if \( \sigma \leq 0.397 \).

Next, we propose a metric for measuring how much freedom exists in inventory allocation of a flexibility design.

**Definition 2.** Assume Assumption 1 holds and fix a flexibility design \( F \). We say \( F \) has degree of freedom of at least \( u \) if any \( s \) that satisfies the equations

\[
0 \leq s_j \leq u, \forall 1 \leq j \leq N, \\
\text{and} \quad \sum_{j=1}^{N} s_j = D_{\text{max}}(N) - C_{\text{min}}(N) - \delta,
\]

is an optimal inventory allocation. We also define \( \text{FREE}(F) \) to be the maximum possible \( u \).

For a given \( F \), if there does not exists an \( s \) satisfying Equations (17) and (18) for some \( u \), then we define \( \text{FREE}(F)= 0 \). Note that with Assumption 1, full flexibility design always has a non-zero degree of freedom by selecting \( u = \frac{D_{\text{max}}(N) - C_{\text{min}}(N) - \delta}{N} \).

Clearly, the larger \( \text{FREE}(F) \), the more freedom \( F \) has in choosing its inventory allocation. Consider the example where \( \delta = 0 \), \( U_c \) satisfies Equation (12) and \( U_d \) satisfies Equation (13) with \( \sigma = 0 \). If \( F \) is a K-chain for any \( K \geq 2 \), then \( \text{FREE}(F) = 1 \). Because the optimal total inventory level is equal to \( D_{\text{max}}(N) - C_{\text{min}}(N) - \delta = 1 \), it means that we can place this 1 unit of inventory at any product and obtain an optimal inventory allocation.

In the next proposition, we provide a set of conditions to check if the \( K \)-chain designs has a degree of freedom of at least \( u \).

**Proposition 2.** Assume Assumption 1 holds, and fix some \( u \leq \min_{d \in U_d} d_1 \). Then for \( K \geq 2 \),
the K-chain design has a degree of freedom of at least $u$ if and only if

$$D^{\text{max}}(t) - C^{\text{min}}(t + K - 1) - \delta \leq 0, \forall 1 \leq t \leq N - n^*$$

(19)

$$D^{\text{max}}(t) - C^{\text{min}}(t + K - 1) - D^{\text{max}}(N) + C^{\text{min}}(N) + (N - t)u \leq 0, \forall N - n^* < t \leq N - K,$$

(20)

where $n^* = \max\{K, \lceil \frac{D^{\text{max}}(N) - C^{\text{min}}(N) - \delta}{u} \rceil \}$.

Using Proposition 2, we can study the range of optimal inventory allocations under K-chain designs, if the optimal inventory level of full flexibility is $D^{\text{max}}(N) - C^{\text{min}}(N) + \delta$.

**Example.** Again, we consider the example from Section 4.1 with $N = 12$, $U_c$ defined by Equation (12) and $U_d$ defined by Equation (13). It is easy to check that with $U_d$ being defined by Equation (13), $\text{FREE}(F)$ cannot be greater than $\min_{d \in U_d} d_1$. Therefore, we can use Proposition 2 to numerically compute the maximum degree of freedom, $\text{FREE}(F)$, for any design $F$ that is a K-chain. In Figure 2, we plot the maximum degree of freedom for K-chains under different demand variabilities for $K = 2, 3, 4, 5$ and 12. The shortage allowance parameter, $\delta$, is picked to equal to 0.6, which is equal to 5% of the total capacity.

We observe that when $\sigma = 0.3$, the total inventory level is roughly 2.5 and the maximum degree of freedom of full flexibility is approximately 0.5. Therefore, there exists an optimal inventory allocation that places inventories at only 5 out of the 12 products. This implies that there exist a sparse optimal inventory allocation, i.e., placing inventories at just a few products.

More surprisingly, the same freedom is also achieved by the 5-chain design, and even the 4-chain design has a maximum degree of freedom of approximately 0.4. This implies that an optimal inventory allocation for 4-chain need to hold inventory at any 7 of the 12 products. Finally, observe that the inventory freedom of 2-chain decreases quickly with increasing demand variabilities.

It is important to note that the freedom of inventory allocation depends crucially on the fact that all products have the same unit holding costs. If some products have different unit holding costs, then there may not be a large range of inventory freedom and in fact, the optimal inventory allocation may be unique. Nevertheless, if a manufacturer has a high degree of flexibility, then the manufacturer can move around its inventories among products that have comparable unit costs without significantly increasing the total inventory cost or decreasing the supply chain robustness.

### 4.3 Inventory Allocation Strategy

The symmetric setting in the previous subsections assume demands for all products are in the same range. However, it is often the case that the firm has some products facing various degrees of variability.

To understand how process flexibility affects inventory decisions when products have different demand variabilities, we consider a stylized setting with $2N$ products with equal expected demand. Products 1, 3, ..., $2N - 1$ have the same high level of variability; products 2, 4, ..., $2N$ have the same low level of variability. Therefore, we refer to products with odd labels as *high variability products*, while products with even labels as *low variability products*. 
In the robust optimization setting, we can model higher variability using larger demand uncertainty set, and lower demand variability using smaller demand uncertainty set. Based on the discussion in §2.3 and Appendix B, we let $\sigma_1, \sigma_2$ be two parameters with $\sigma_1 > \sigma_2$, and consider the following demand uncertainty set

$$\mathcal{U}_d = \left\{ d \mid \sum_{j=1}^{2N} d_j \leq 2N + 2\sqrt{(\sigma_1^2 + \sigma_2^2)N}, \sum_{j=1}^{2N} |d_j - 1| \leq \sqrt{3}(\sigma_1 + \sigma_2)N + \frac{\sqrt{(\sigma_1^2 + \sigma_2^2)N}}{2}, \right.$$  

$$|d_{2j-1} - 1| \leq \sqrt{3}\sigma_1, |d_{2j} - 1| \leq \sqrt{3}\sigma_2, \forall 1 \leq j \leq N \right\}.$$  

The first two constraints are motivated by central limit theorem (CLT) to bound total variability among all products. The last two constraints bound the variability for each individual product. Note that odd and even products have different variability.

Since the demand set $\mathcal{U}_d$ is symmetric for odd products and even products, respectively, we can easily verify that there exists an optimal inventory decision such that every odd product has the same inventory level $s_1$, and every even product has the same inventory level $s_2$. Using the computational algorithm proposed in §3, we compute the optimal inventory level for a network of 12 plants, 6 high variability products and 6 low demand variability products. We test two pairs of parameters: $(\sigma_1 = 0.4, \sigma_2 = 0.2), (\sigma_1 = 0.5, \sigma_2 = 0.3)$.

The result in Figure 3 shows an interesting effect: As one would expect, the firm holds more inventory for high variability demand when the degree of flexibility is small. This is similar to the classical newsvendor model, where more inventory is needed to achieve the same fill rate as the demand variability increases. But surprisingly, when the firm has high degree of flexibility, it is optimal to hold more inventory for products facing lower variability, and to hold less inventory for products with higher variability. We refer to such phenomenon as the \textit{flipping effect}, since the inventory levels of two kinds of products flipped as degree of flexibility, $K$, increases.
The flipping effect can be explained by relating our analysis to Push and Pull strategies (see, e.g., Simchi-Levi 2010). In a Pull strategy, the firm produces to order by applying capacity. In a Push strategy, the firm produces to stock, and then satisfies demand from inventory. When there is high degree of flexibility, high variability products are typically served using Pull (capacity) while low variability products are served using Push (inventory), so most inventory is allocated to the low variability products while less inventory is allocated to the high variability products. When there is low degree of flexibility, capacity is not enough to match supply with demand for high variability products, so more inventory is allocated to the high variability products.

Another observation is that the flipping effect increases when demand variabilities, $\sigma_1$ and $\sigma_2$, both increase. Indeed, in Figure 5, the flipping effect is more significant when $\sigma_1 = 0.5, \sigma_2 = 0.3$ (right panel). To explain this, note that the total inventory increases when both $\sigma_1$ and $\sigma_2$ increase. However, as $\sigma_1$ increases, it is more efficient to satisfy high variability demands with capacity (pull) and less inventory. Therefore, higher percentage of inventory is allocated to low variability products and the flipping effect becomes more significant.

The flipping effect also occurs in probabilistic settings where probability distributions of plant disruption and product demands are specified. We show similar results using numerical experiments in §5.3.

5 Numerical Examples for Stochastic Model

So far, we have mainly focused on a worst-case (robust optimization) model, because the robust optimization model can be computed efficiently and yields analytical solutions for symmetric uncertainty sets. However, as we showed in §2.3, there is a close connection between the stochastic model and the worst-case model, because we choose a certain form of uncertainty set inspired by the probabilistic model. In this subsection, we use numerical examples to verify that many of our insights of the worst-case model also hold for the stochastic model.

Under the stochastic model, we consider the problem that minimizes the total inventory while ensuring that the expected shortage is at most $\delta$. The constraint on the expected shortage is also known as the Type 2 service-level constraint in the supply chain literature. Let $C$ and $D$ be vectors representing random capacities and demands, respectively, and let $s = (s_1, \ldots, s_N)$ be the inventory decision variables for all products. Using the shortage function $\Pi(F, s, C, D)$
The optimization problem is defined as follows:

\[
\text{Problem-SM: } \min \sum_{j=1}^{N} s_j \quad (21)
\]
\[
\text{s.t. } \mathbb{E}_{C,D}[\Pi(F, s, C, D)] \leq \delta, \quad (22)
\]
\[
s_j \geq 0, \forall 1 \leq j \leq N. \quad (23)
\]

5.1 Total Inventory Level

Consider a network of 10 plants and 10 products. We assume all plants have one unit of capacity, and each plant may be disrupted independently with probability \( p = 0.1 \). Each product has demand uniformly distributed between \( 1 - \alpha \) and \( 1 + \alpha \), where \( \alpha \) represents the variability of the demand. Demand distributions are independent for different products. The objective is to determine inventory level of each product such that the expected shortage is no more than 0.2 (fill rate = 98%).

We restrict our attention to the \( K \)-chain designs introduced in §4. We sample 5000 times from demand and capacity distributions, and solve a stochastic program with these samples to determine the optimal inventory level.

In Figure 4, we plot the total inventory level for 2-chain, 3-chain and 10-chain (full flexibility) under different demand variability \( \alpha \). The figure shows that adding flexibility to 2-chain design significantly reduces the inventory. In particular, this is true even for low demand variability. This observation is consistent with the insights from the robust model, see §4.1, Tables 1–3.

5.2 Inventory Placement Freedom

Using the robust model, we found in §4.2 that process flexibility not only reduces the total inventory required, but also offers freedom in placing inventory among different products. We show the same observation in the stochastic setting.

To quantify the firm’s freedom in placing inventory, we consider *sparsity constraints*, that is, constraints specifying that the firm can only hold inventory for a few products. We then compare...
Table 4: Total Inventory Increase under Sparsity Constraint

<table>
<thead>
<tr>
<th>Products</th>
<th>{1,9}</th>
<th>{1,2,4,5,7,8,9}</th>
<th>{1,3,5,7,9}</th>
<th>{1,4,7}</th>
<th>{1,6}</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-chain</td>
<td>3.16%</td>
<td>11.36%</td>
<td>25.43%</td>
<td>Inf</td>
<td>Inf</td>
</tr>
<tr>
<td>3-chain</td>
<td>0.04%</td>
<td>0.10%</td>
<td>0.21%</td>
<td>0.62%</td>
<td>3.26%</td>
</tr>
<tr>
<td>10-chain</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>1.61%</td>
</tr>
</tbody>
</table>

By symmetry, we know that placing inventory equally among all products is always an optimal solution, so we expect the total inventory to increase under the sparsity constraints. If the increase is small, then a sparse inventory solution is near optimal, which implies that the firm has the freedom of placing inventory for only a few products.

Figure 5: Inventory under Asymmetric Demand in K-chains

Figure 5: Inventory under Asymmetric Demand in K-chains

We use the same setting as in the previous subsection, and fix demand variability parameter \( \alpha = 0.3 \) and shortage allowance \( \delta = 0.2 \) (98% fill rate). The results are listed in Table 4. The header row shows the subsets of products that the firm can hold inventory for. The percentage numbers are the increase in total inventory compared with the unconstrained case where any product can hold inventory. “Inf” means the problem is infeasible for the given placement constraint. We find that the 10-chain design (full flexibility) has high degree of freedom: When restricting to holding inventory for only two products, the total inventory increases by merely 1.61%. In addition, the 3-chain design has significantly higher degree of freedom than the 2-chain design.

5.3 Inventory Allocation Strategy

Next, we consider a modified setting where demand distributions are not i.i.d. In particular, we label products from 1 to 10, and assume that products with odd labels have demand uniformly distributed between \( 1 - \alpha \) and \( 1 + \alpha \), while products with even labels have demand uniformly distributed between \( 1 - \beta \) and \( 1 + \beta \), for some \( \beta < \alpha \). The objective is to determine inventory levels of each product so as to achieve an expected fill rate of at least 98%.

We tested two sets of parameters: \( (\alpha = 0.6, \beta = 0.2) \), \( (\alpha = 0.3, \beta = 0.1) \). It is easily verified that all products with lower variability (odd labels) have the same inventory level, and all products with high variability (even labels) have the same inventory level. Figure 5 presents the result for the stochastic model, and it is consistent with the result from the robust model: the figure verifies the flipping effect we observed in §4.3.
5.4 Unbalanced System Example

Finally, we consider an unbalanced system example where the number of plants and products are unequal ($M \neq N$). This example is based on a real dataset of 16 vehicle models and 8 assembly plants at General Motors presented in Jordan and Graves (1995). We use the same plant capacities and expected product demands as in their paper, see the left chart of Figure 6. We assume that product demands are normally distributed random variables truncated at two standard deviations. Product A to F (compact cars) have high variability demands, with a coefficient of variation equal to 0.5 prior to truncation; Product G to P (full-sized and luxury cars) have low variability demand, with a coefficient of variation equal to 0.3 prior to truncation. Each plant is assumed to be disrupted independently with probability 0.1. The expected fill rate is assumed to be 95%. For simplicity, we assume demands are independent.

There is no clear definition of a K-chain when the number of plants and products are unequal. Based on the “chaining” principle, Jordan and Graves (1995) proposed an ad hoc method to add process flexibility to the base design. They first cluster the plants and products into six groups, then add six links to connect six groups and create a “2-chain” design, see the right chart of Figure 6. Following their method, we also create “3-chain” and “4-chain” designs by adding six links at a time, see Figure 7. There are many ways to add six links to the initial design. Our objective here is not to find the “optimal” designs, but to create a family of designs with increasing degree of flexibility, and analyze how inventory decisions change within this family.

Table 5 shows the total inventory required for different designs, as well as the percentage of inventory allocated to high variability products (Product A to F). The observation is consistent with the findings from the balanced systems. While changing from a base design to a 2-chain
Table 5: Inventory Allocation for Asymmetric System.

<table>
<thead>
<tr>
<th>Flexibility Designs</th>
<th>Base Design</th>
<th>2-chain</th>
<th>3-chain</th>
<th>4-chain</th>
<th>Full Flexibility</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Inventory</td>
<td>955.4</td>
<td>430.8</td>
<td>312.2</td>
<td>296.6</td>
<td>290.3</td>
</tr>
<tr>
<td>% High Variability Products</td>
<td>72.54%</td>
<td>47.92%</td>
<td>33.44%</td>
<td>17.55%</td>
<td>4.62%</td>
</tr>
</tbody>
</table>

Table 6: Total Inventory Increase under Sparsity Constraint for the Asymmetric Example

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Base Design</td>
<td>5.30%</td>
<td>36.61%</td>
<td>Inf</td>
<td>Inf</td>
</tr>
<tr>
<td>2-chain</td>
<td>1.09%</td>
<td>3.73%</td>
<td>34.38%</td>
<td>Inf</td>
</tr>
<tr>
<td>3-chain</td>
<td>3.89%</td>
<td>4.12%</td>
<td>8.70%</td>
<td>15.05%</td>
</tr>
<tr>
<td>4-chain</td>
<td>0.51%</td>
<td>1.11%</td>
<td>3.53%</td>
<td>6.31%</td>
</tr>
<tr>
<td>Full Flexibility</td>
<td>0.02%</td>
<td>0.19%</td>
<td>1.14%</td>
<td>3.37%</td>
</tr>
</tbody>
</table>
design reduces total inventory, there is also significant inventory reduction from a 2-chain design to higher degrees of flexibility. Moreover, as the degree of flexibility increases, less inventory is allocated to high variability products.

We also consider sparsity constraints of inventory placement. Table 6 shows the increase of total inventory compared with the unconstrained case where any product can hold inventory. “Inf” means the problem is infeasible for the given placement constraint. Again, we find that when there is high degree of flexibility, the firm can hold inventory for only a few products without significantly increasing total inventory.

6 Time-to-Survive Model

In this section, we propose a supply chain risk metric called Time-to-Survive (TTS), and demonstrate that it can be determined by solving a special case of our robust optimization model introduced in §2.

Given a facility in the supply chain, e.g. a plant or a distribution center, the TTS is defined as the longest time that customer service level is guaranteed if this facility is disrupted. The TTS metric has been already implemented at Ford Motor Company to assess risk exposure in its complex supply chain and evaluate Ford’s risk mitigation strategies (Simchi-Levi et al. 2014a,b).4

The definition of TTS is motivated by the concept of Time-to-Recover (TTR), which is the time for a facility to return to full capacity after a disruption. TTR is widely used to evaluate supply chain risk (see e.g. Hopp et al. 2012). If TTS is greater than TTR, then a disruption in that facility is not going to affect the firm’s service level. On the other hand, when TTS for a specific facility is shorter than TTR, than a disruption at that facility will have an impact on service. Thus, an important challenge in supply chain risk management is to allocate inventory to different products to increase the supply chain’s time to survive.

For this purpose, we define the supply chain TTS as the minimum (worst) TTS among all its facilities. The longer the supply chain TTS, the more robust the supply chain is. Below we show that the problem of allocating inventory to maximize supply chain TTS can be reduced to a special case of the model in §2.

Suppose that plant $i$ has capacity $c_i$ ($i = 1, \ldots, M$), product $j$ has a constant demand rate $d_j$ per unit time ($j = 1, \ldots, N$). Let $r_j$ be the amount of inventory allocated to product $j$, and assume that the sum of inventory among all products cannot exceed a given budget $R$. We define a finite set $\Omega$ as the set of disruption scenarios, and let $\delta^{(\omega)}_i$ be 0 if plant $i$ is disrupted in scenario $\omega \in \Omega$ and 1 otherwise. We then formulate the problem of maximizing supply chain

\footnote{This work was awarded the 2014 INFORMS Daniel H. Wagner Prize for excellence in operations research practice.}
TTS as the following nonlinear program:

\[
TTS = \max_{r, x^{(\omega)}} \quad t
\]

subject to

\[
t \leq \frac{r_j}{d_j - \sum_{i: (i,j) \in F} x_{ij}^{(\omega)}}, \quad \forall 1 \leq j \leq N, \omega \in \Omega
\]

\[
\sum_{j: (i,j) \in F} x_{ij}^{(\omega)} \leq c_i \delta_i^{(\omega)}, \quad \forall 1 \leq i \leq M, \omega \in \Omega
\]

\[
\sum_{j=1}^{N} r_j \leq R, \quad r_j, x_{ij}^{(\omega)} \geq 0.
\]

In this formulation, \(x_{ij}^{(\omega)}\) denotes the production of product \(j\) by plant \(i\) in disruption scenario \(\omega\). The first constraint is the definition of TTS: In any scenario, the supply chain must survive the disruption by continuously supplying all products for at least \(t\) time units. The second constraint requires that production does not exceed plant capacity. The last constraint specifies that total inventory cannot exceed a given budget \(R\).

Let \(s_j = r_j/t\), we can rewrite the TTS model as

\[
\min_{s_j, x_{ij}^{(\omega)} \geq 0} \quad \sum_{j=1}^{M} s_j
\]

subject to

\[
d_j - \sum_{i: (i,j) \in F} x_{ij}^{(\omega)} \leq s_j, \quad \forall 1 \leq j \leq N, \omega \in \Omega
\]

\[
\sum_{j: (i,j) \in F} x_{ij}^{(\omega)} \leq c_i \delta_i^{(\omega)}, \quad \forall 1 \leq i \leq M, \omega \in \Omega.
\]

This is a special case of the model in §2 where there is no demand uncertainty and the shortage allowance equals to \(\delta = 0\).

### 7 Conclusion

We consider a firm using process flexibility and inventory to mitigate risk from plant disruptions, with particular focus on the interplay between the two strategies. This interplay is modeled as a two-stage robust optimization problem. In the first stage, given the firm’s process flexibility design, the firm optimizes its inventory levels for all products in order to guarantee that demand shortage never exceeds some constant, subject to plant disruptions. Uncertainties from plant disruptions and product demands are modeled using uncertainty sets. In the second stage, after disruption occurs, demands are realized, and the firm uses both inventory and process flexibility to minimize demand shortage. In particular, the firm may leverage its process flexibility, and reallocate production capacities to better match supply with demand.

We show that the robust optimization model can be solved efficiently by reformulating it as a linear program. Moreover, we derive analytical results for a canonical family of flexibility designs known as K-chain designs, and find the following managerial insights. First, while a 2-chain design is not as effective as K-chain designs for \(K > 2\), full flexibility is often not needed. Second, with high the degree of flexibility, the firm has freedom in placing inventory to
only a few products without affecting system performance. Finally, when the firm has a high degree of flexibility, more inventory is allocated to products with low demand variability, and less inventory is allocated to products with high demand variability. These insights are verified by numerical experiments in a probabilistic setting for systems where the number of plants is not necessarily equal to the number of products.

References


Appendix

A Theoretical Proofs

A.1 Proof of Lemma 1

Proof of Lemma 1. For any $s^1, c^1, d^1$ and $s^2, c^2, d^2$. Let $x^1, l^1$ and $x^2, l^2$ be the optimal solutions for the optimization problems defining $\Pi(\mathcal{F}, s^1, c^1, d^1)$ and $\Pi(\mathcal{F}, s^2, c^2, d^2)$. For any $0 \leq \lambda \leq 1$, clearly $\lambda x^1 + (1 - \lambda)x^2, \lambda l^1 + (1 - \lambda)l^2$ is feasible for the optimization problem defining $\Pi(\mathcal{F}, s^1 + (1 - \lambda)s^2, c^1 + (1 - \lambda)c^2, d^1 + (1 - \lambda)d^2)$ and therefore, we have

$$\Pi(\mathcal{F}, s^1 + (1 - \lambda)s^2, c^1 + (1 - \lambda)c^2, d^1 + (1 - \lambda)d^2) \leq \lambda \Pi(\mathcal{F}, s^1, d^1, d^1) + (1 - \lambda) \Pi(\mathcal{F}, s^2, d^2, d^2).$$

$\square$

A.2 Proof of Lemma 2

Proof of Lemma 2. First, note that we can view the LP formulation of $\Pi(\mathcal{F}, s, c, d)$ as a network flow problem: Consider a network consisting of all the plant and product nodes, a source node, and a sink node. Create arcs

- from the source node to each plant node $i$ with upper bound $c_i$
- from the source node to each product node $j$ with lower bound 0
- from each plant node $i$ to each product node $j$ with lower bound 0, if $(i, j)$ is in the flexibility design $\mathcal{F}$
- from each product node $j$ to the sink node, with lower bound $d_j - s_j$.

Suppose the arc from the source node to product node $j$ has one unit of cost, and other arcs have zero cost. Let $x_{ij}$ be the flow on arc from plant node $i$ to product node $j$, and $l_j$ be the flow on arc from the source node to product node $j$. Then this minimum cost flow problem is exactly the primal formulation of $\Pi(\mathcal{F}, s, c, d)$.

Using the strong duality theorem, we can express $\Pi(\mathcal{F}, s, c, d)$ by the following dual formulation.

$$\Pi(\mathcal{F}, s, c, d) = \max \sum_{j=1}^{N} (d_j - s_j)q_j - \sum_{i=1}^{M} c_i p_i$$

$$q_j - p_i \leq 0, \quad \forall (S_i, T_j) \in \mathcal{F}$$

$$q_j \leq 1, \quad \forall 1 \leq j \leq N$$

$$p_i, q_j \geq 0, \forall 1 \leq i \leq M, \forall 1 \leq j \leq N.$$

The constraints in the dual formulation (24) are totally unimodular. This is an immediate result as the primal formulation of $\Pi(\mathcal{F}, s, c, d)$ is a network flow problem.

Note that for any feasible solution $p, q, q_j = 1$ immediately implies that $p_i = 1$ for all $(S_i, T_j) \in \mathcal{F}$. Moreover, if $p, q$ is an optimal solution, because $c_i \geq 0$, we can without loss of
Lemma 4. The symmetries in \( U \) sales of \( F \)

to prove Proposition 1, we first develop a technical lemma that characterizes the worst-case lost

\[ A.4 \text{ Proof of Proposition 1} \]

\[ \text{Proof of Lemma 3.} \]

\[ \text{Problem-WM. By assumption of symmetry, if the inventory allocation} \]

\[ \text{feasible for Problem-WM, then by Lemma 1, the convex combination of} \]

\[ \text{where} \]

\[ \text{Let} \]

\[ \text{Also, define} \]

\[ \text{Also, define} \]

\[ \text{A.3 Proof of Lemma 3} \]

\[ \text{Proof of Lemma 3.} \]

\[ \text{If two inventory allocations} \]

\[ \text{are also optimal. Therefore, their convex combination} \]

\[ \text{A.4 Proof of Proposition 1} \]

\[ \text{To prove Proposition 1, we first develop a technical lemma that characterizes the worst-case lost} \]

\[ \text{sales of} \]

\[ \text{generality assume that} \]

\[ \text{Therefore,} \]

\[ \Pi(F, s, c, d) = \max_{A \subset \{T_1, \ldots, T_N\}} \sum_{t_j \in A} (d_j - s_j) - \sum_{S_i \in P_F(A)} c_i. \]

\[ A^* = \arg \max_{A \subset \{T_1, \ldots, T_N\}} D_{\max}(|A|) - |A| \cdot s - C_{\min}(|P_F(A)|), \text{ and define} \]

\[ t^* = |A^*|. \]
Because $C_{\min}(t)$ is nondecreasing with $t$, we have $|P_F(A^*)| = \varphi^t(F)$. Therefore, we have

$$
\max_{(c,d)\in U_c\times U_d} \Pi(F, s, c, d) = D_{\max}(t^*) - t^* \cdot s - C_{\min}(\delta^t(F)) \\
\leq \max_{1 \leq t \leq N} \left( D_{\max}(t) - t \cdot s - C_{\min}(\delta^t(F)) \right).
$$

Combining both inequalities we obtain the desired result. \hfill \square

Now, we are ready to prove Proposition 1.

**Proof of Proposition 1.** Because $F$ is a $K$-chain, for any integer $1 \leq t \leq N$, if we take $A = \{T_1, \ldots, T_t\}$, then $|P_F(A)| = \delta^t(F)$. Moreover, if $1 \leq t \leq N - K + 1$, then $|P_F(A)| = t + K - 1$ and if $N - K + 1 < t \leq N$, then $|P_F(A)| = N$. Thus, we have

$$
\delta^t(F) = \begin{cases} 
  t + K - 1 & \text{if } 1 \leq t \leq N - k + 1, \\
  N & \text{if } N - k + 1 < t \leq N.
\end{cases} \quad (26)
$$

Combining this with Lemma 4, we get that $\max_{(c,d)\in U_c\times U_d} \Pi(F, s, c, d) \leq \delta$ if and only if

$$
D_{\max}(t) - ts - C_{\min}(t + K - 1) \leq \delta, \forall 1 \leq t \leq N - K \quad \text{and} \quad D_{\max}(t) - ts - C_{\min}(N) \leq \delta, \forall N - k \leq t \leq N.
$$

Therefore if $s_j = s^*$ for $1 \leq j \leq N$ is an optimal inventory, then $s^*$ must be the smallest nonnegative quantity so that $s$ satisfies the two equations above. This implies that

$$
s^* = \max\left\{ \max_{1 \leq t \leq N - K} \frac{D_{\max}(t) - C_{\min}(t + K - 1) - \delta}{t}, \max_{N - k \leq t \leq N} \frac{D_{\max}(t) - C_{\min}(N) - \delta}{t}, 0 \right\}.
$$

\hfill \square

### A.5 Proof of Proposition 2

The key to prove Proposition 2 is to view all of the vectors $s$ that satisfies Definition 2 as an uncertainty set itself. That is, let

$$
U_s = \{s| \sum_{j=1}^{N} s_j = D_{\max}(N) - C_{\min}(N) - \delta, 0 \leq s_j \leq u, \forall 1 \leq j \leq N\}.
$$

Also, let $S_{\min}(t) := \min_{s \in U_s} \sum_{j=1}^{t} s_j$.

Note that $U_s$ is symmetric itself. Thus, we have the following lemma which can be seen as an extension to Lemma 4.

**Lemma 5.** Fix an arbitrary flexibility structure $F$. Then

$$
\max_{(c,d,s)\in U_c\times U_d\times U_s} \Pi(F, s, c, d) = \max_{1 \leq t \leq N} \left( D_{\max}(t) - S_{\min}(t) - C_{\min}(\delta^t(F)) \right), \quad (27)
$$

where $\delta^t(F)$ is the minimal value of $|P_F(A)|$ for any $A \subseteq \{T_1, \ldots, T_N\}$ such that $|A| = t$. 

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Proof of Lemma 5. By Equation (1) of Lemma 2,

\[
\max_{(c,d,s) \in U_c \times U_d \times U_s} \Pi(\mathcal{F}, s, c, d) = \max_{A \subset \{T_1, \ldots, T_N\}} \max_{(c,d,s) \in U_c \times U_d \times U_s} \left( \sum_{T_j \in A} (d_j - s_j) - \sum_{S_i \in P_A} c_i \right)
\]

\[
= \max_{A \subset \{T_1, \ldots, T_N\}} \left( D^{\max}(|A|) - S^{\min}(|A|) - C^{\min}(|P_A|) \right)
\]

\[
\geq \max_{1 \leq t \leq N} \left( D^{\max}(t) - S^{\min}(t) - C^{\min}(\delta^t(\mathcal{F})) \right).
\]

Also, define

\[
A^* = \arg \max_{A \subset \{T_1, \ldots, T_N\}} D^{\max}(|A|) - S^{\min}(|A|) - C^{\min}(|P_A|), \text{ and } t^* = |A^*|.
\]

Because \(C^{\min}(t)\) is nondecreasing with \(t\), we have \(|P_A(A^*)| = \delta^{t^*}(\mathcal{F})\). Therefore, we have

\[
\max_{(c,d,s) \in U_c \times U_d} \Pi(\mathcal{F}, s, c, d) = D^{\max}(t^*) - S^{\min}(t^*) - C^{\min}(\delta^{t^*}(\mathcal{F}))
\]

\[
\leq \max_{1 \leq t \leq N} \left( D^{\max}(t) - S^{\min}(t) - C^{\min}(\delta^t(\mathcal{F})) \right).
\]

Combining both inequalities and we obtain the desired result. \(\square\)

Proof of Proposition 2. Combining Equation (26) of Proposition 1 and Lemma 4, we get that

\[
\max_{(c,d) \in U_c \times U_d} \Pi(\mathcal{F}, s, c, d) \leq \delta \text{ if and only if}
\]

\[
D^{\max}(t) - S^{\min}(t) - C^{\min}(t + K - 1) \leq \delta, \forall 1 \leq t \leq N - K \quad (28)
\]

and

\[
D^{\max}(t) - S^{\min}(t) - C^{\min}(N) \leq \delta, \forall N - k \leq t \leq N. \quad (29)
\]

By definition of \(U_s\), \(D^{\max}(t) - S^{\min}(t) - C^{\min}(N) \leq D^{\max}(N) - S^{\min}(N) - C^{\min}(N) = \delta\). Thus, \(\Pi(\mathcal{F}, s, c, d) \leq \delta\) if and only if Equation (28) is satisfied. Note that \(S^{\min}(t)\) can be expressed as follows.

\[
S^{\min}(t) = \begin{cases} 
0 & \text{if } 0 \leq t \leq N - \left\lfloor \frac{D^{\max}(N) - C^{\min}(N) - \delta}{u} \right\rfloor, \\
D^{\max}(N) - C^{\min}(N) - \delta - (N - t)u & \text{if } N - \left\lfloor \frac{D^{\max}(N) - C^{\min}(N) - \delta}{u} \right\rfloor < t \leq N.
\end{cases}
\]

Substitute Equation (30) into Equation (28), and we get our desired result. \(\square\)

B Computing \(D^{\max}(t)\) and Selecting Uncertainty Set Parameters

B.1 Selecting Uncertainty Set Parameters

Below we provide a concrete example of selecting uncertainty set parameters \(\alpha, \beta\) and \(\gamma\.

Suppose product demands are independent and identically distributed (i.i.d.) uniform random variables with mean \(\mu\) and standard deviation \(\sigma\). Let \(D_j\) be the stochastic demand for
product $j$. By the Central Limit Theorem (CLT), we have

$$\lim_{N \to \infty} P\left[\frac{\sum_{j=1}^{N} D_j - N\mu}{\sigma\sqrt{N}} \leq \Omega\right] = P[Z \leq \Omega],$$

where $Z$ is a random variable with a normal distribution of mean 0 and standard deviation 1. For reasonably large $N$ and some $\Omega$ greater than or equal to 2, we get $P[\sum_{j=1}^{N} D_j \leq N\mu + \Omega\sigma\sqrt{N}] \approx 1$. As a result, if we set $\gamma = 2\sigma\sqrt{N}$, the inequality $\sum_{j=1}^{N} D_j \leq \sum_{j=1}^{N} \mu_j + \gamma$ will be satisfied with high probability.

Similarly, we can apply CLT on distributions $\{|D_1 - \mu|, \ldots, |D_N - \mu|\}$. This guides us to set $\beta = \mu' N + 2\sigma' \sqrt{N}$ for the second inequality in $U_d$, where $\mu'$ is the mean of $|D_1 - \mu|$ and $\sigma'$ is the standard deviation of $|D_1 - \mu|$.

For each $1 \leq j \leq N$, we select $\alpha_j$ such that the probability $|D_j - \mu_j| \leq \alpha_j$ is close to 1. The parameter $\alpha_j$ typically depends on the demand distribution. Since we assume demand are uniformly distributed, the support of the distribution is $[\mu - \sqrt{3}\sigma, \mu + \sqrt{3}\sigma]$. Therefore, we can set $\alpha_j = \sqrt{3}\sigma$ for $1 \leq j \leq N$. In sum, we have

$$U_d = \{d | \sum_{j=1}^{N} d_j \leq \mu N + 2\sqrt{N}\sigma, \sum_{j=1}^{N} |d_j - \mu| \leq \frac{\sqrt{3}\sigma N}{2} + \sigma\sqrt{N}, |d_j - \mu| \leq \sqrt{3}\sigma, \forall 1 \leq j \leq N\}.$$ 

The method for choosing parameters $\gamma, \beta$ and $\alpha_j$ described above makes $U_d$ a good empirical fit especially to stochastic uniform demand distribution. For example, for $N = 10$, $D_j$ being uniformly distributed over interval $[0, 2]$ for $j = 1, \ldots, N$, we get $\alpha_j = 1$ for $j = 1, \ldots, N$, and set $\beta = 6.82$, $\gamma = 3.65$ using the CLT guideline. Numerical experiment shows that a sample of $|D_1, \ldots, D_N|$ lies in the above demand uncertainty set about 96% of the time.

### B.2 Computing $D_{\text{max}}(t)$

Here, we describe a general formula for computing $D_{\text{max}}(t)$, when $U_d$ is defined by

$$U_d = \{d | \sum_{j=1}^{N} d_j \leq N + \gamma, \sum_{j=1}^{N} |d_j - 1| \leq \beta, |d_j - 1| \leq \alpha, \forall 1 \leq j \leq N\},$$

where $\alpha, \beta$ and $\gamma$ are real parameters.

**Lemma 6.** Suppose $U_d$ is defined by Equation (32), then

$$D_{\text{max}}(t) = \begin{cases} t(1 + \alpha) & \text{if } 0 \leq t \leq \lfloor \frac{\beta + \gamma}{2\alpha} \rfloor, \\ t + \frac{\beta + \gamma}{2\alpha} & \text{if } \lfloor \frac{\beta + \gamma}{2\alpha} \rfloor < t < N - \lfloor \frac{\beta + \gamma}{2\alpha} \rfloor, \\ t + \gamma + (N - t)\alpha & \text{if } N - \lfloor \frac{\beta + \gamma}{2\alpha} \rfloor \leq t \leq N. \end{cases}$$

**Proof of Lemma 6.** Let $d^*$ be the vector such that

$$d_j^* = \begin{cases} (1 + \alpha) & \text{if } 0 \leq j \leq \lfloor \frac{\beta + \gamma}{2\alpha} \rfloor, \\ 1 + \lfloor \frac{\beta + \gamma}{2\alpha} \rfloor \alpha & \text{if } \lfloor \frac{\beta + \gamma}{2\alpha} \rfloor < t \leq \lfloor \frac{\beta + \gamma}{2\alpha} \rfloor, \\ 1 & \text{if } N - \lfloor \frac{\beta + \gamma}{2\alpha} \rfloor \leq t \leq N - \lfloor \frac{\beta + \gamma}{2\alpha} \rfloor, \\ 1 - \lfloor \frac{\beta + \gamma}{2\alpha} \rfloor \alpha & \text{if } N - \lfloor \frac{\beta + \gamma}{2\alpha} \rfloor < t \leq N - \lfloor \frac{\beta + \gamma}{2\alpha} \rfloor, \\ 1 - \alpha & \text{if } N - \lfloor \frac{\beta + \gamma}{2\alpha} \rfloor < t \leq N. \end{cases}$$

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It is easy to check that \(d^* \in \mathcal{U}_d\).

For any integer \(0 \leq t \leq \lfloor \frac{\beta - \gamma}{\alpha} \rfloor\), for any \(d \in \mathcal{U}_d\) because \(|d_j - 1| \leq \alpha, \forall 1 \leq j \leq N\), clearly \(\sum_{j=1}^t d_j \leq t(1 + \alpha)\). But \(\sum_{j=1}^t d_j^* = t(1 + \alpha)\), and this implies \(D^{\max}(t) = t(1 + \alpha)\).

For any integer \(\lfloor \frac{\beta - \gamma}{\alpha} \rfloor < t < N - \lfloor \frac{\beta - \gamma}{\alpha} \rfloor\), for any \(d \in \mathcal{U}_d\) because \(\sum_{j=1}^N d_j \leq N + \gamma, \sum_{j=1}^N |d_j - 1| \leq \beta\), we must have \(\sum_{j=1}^t d_j \leq t + \frac{\beta - \gamma}{2}\). But \(\sum_{j=1}^t d_j^* = t + \frac{\beta - \gamma}{2}\), we get \(D^{\max}(t) = t + \frac{\beta - \gamma}{2}\).

Finally, for any integer \(N - \lfloor \frac{\beta - \gamma}{\alpha} \rfloor \leq t \leq N\), for any \(d \in \mathcal{U}_d\) because \(\sum_{j=1}^N d_j \leq N + \gamma, |d_j - 1| \leq \alpha, \forall 1 \leq j \leq N\), we must have \(\sum_{j=1}^t d_j \leq t + \gamma + (N - t)\alpha\). But \(\sum_{j=1}^t d_j^* = t + \gamma + (N - t)\alpha\), we get \(D^{\max}(t) = t + \gamma + (N - t)\alpha\).

In §4, we choose \(N = 12\) and \(\mu = 1\), and set \(\gamma = 4\sqrt{3}\sigma, \beta = 8\sqrt{3}\sigma\) and \(\alpha = \sqrt{3}\sigma\). Substitute this into Equation (32), we get

\[
D^{\max}(t) = \begin{cases} 
6(1 + \sqrt{3}\sigma) & \text{if } 1 \leq t \leq 6, \\
 t + 6\sqrt{3}\sigma & \text{if } 7 \leq t \leq 10, \\
 t + (16 - t)\sqrt{3}\sigma & \text{if } 11 \leq t \leq 12.
\end{cases}
\]  

Applying Equation (32), we get that for \(2 \leq K \leq 12\), the optimal inventory level for \(K\)-chain is equal to

\[
12 \cdot \max\{6\sqrt{3}\sigma - (K - 2 + \delta), \frac{6\sqrt{3}\sigma}{10} - \frac{5\sqrt{3}\sigma}{11} - \frac{\delta}{12}, 0\}.
\]  

## C Type 1 Service Level

The Type 1 service level is an event-oriented performance guarantee. In the context of demand shortage, the Type 1 service level ensures that the probability of total shortage being greater than \(\delta\) is less than or equal to \(\epsilon\), for some constants \(\delta\) and \(\epsilon\). Therefore, given that \(\mathbf{C}, \mathbf{D}\) are probabilistic distributions, the optimization problem with Type 1 service level constraint is defined as follows.

\[
\min \sum_{j=1}^N s_j \quad \text{s.t.} \quad \mathbb{P}_{\mathbf{C}, \mathbf{D}}[\Pi(\mathcal{F}, \mathbf{s}, \mathbf{C}, \mathbf{D}) \leq \delta] \geq 1 - \epsilon \tag{36}
\]

\[
s_j \geq 0, \forall 1 \leq j \leq N. \tag{37}
\]

Unfortunately, the service level constraint is non-convex. For example, consider a setting with two products and two plants, where demand for each product is always equal to 1 and the plant capacities are i.i.d. random with each plant having capacity equal to 1 with probability 0.9 and 0 with probability 0.1. In this case, if the firm wants to guarantee that the probability of total shortage being less than 0 is less than or equal to 0.1 (\(\delta = 0\) and \(\epsilon = 0.1\)), then either the inventory allocation \(\mathbf{s}^1 = [1, 0]\) or \(\mathbf{s}^2 = [0, 1]\) is a feasible solution for Equation (36). However, note that \(0.5\mathbf{s}^1 + 0.5\mathbf{s}^2 = [0.5, 0.5]\), a convex combination of \(\mathbf{s}^1\) and \(\mathbf{s}^2\), is not feasible for Equation (36). As a result, because of the non-convex nature of Type 1 service level, solving the optimization problem defined by Equation (35)-(37) is generally very difficult.