

TOPICAL REVIEW

The use of Generalised Functions and Distributions in General Relativity

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Abstract. In this paper we review the extent to which one can use classical distribution theory in describing solutions of Einstein's equations. We show that there are a number of physically interesting cases which cannot be treated using distribution theory but require a more general concept. We describe a mathematical theory of nonlinear generalised functions based on Colombeau algebras and show how this may be applied in general relativity. We end by discussing the concept of singularity in general relativity and show that certain solutions with weak singularities may be regarded as distributional solutions of Einstein's equations.

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1. Introduction

Idealisations play an important role in modelling a wide range of physical situations. In many field theories a particularly useful idealisation is to replace an extended source which is concentrated in a small region of space by a point charge. Such an approximation is physically reasonable provided the internal structure of the source can be neglected. In a similar way sources concentrated near a line or a surface can be described in terms of strings or shells of matter. On trying to describe this idealisation mathematically it is natural to use a delta function to describe the source, and hence in this context distributions arise in a natural way. Distributions are also used to describe a number of other important physical scenarios such as the description of shock waves and the junction conditions between matter and vacuum regions.

In the case of linear field theories such as electrodynamics, distribution theory in fact furnishes a consistent framework which has the following two important features. Firstly, since the Maxwell equations are linear with respect to both sources and fields, one can allow distributional solutions of the field equations as well as classical smooth solutions. Secondly it is guaranteed that smooth charge densities that are close to those of a point charge (in the sense of distributional convergence) produce fields that are close to the Coulomb field. While the first property permits one to have a mathematically sound formulation of concentrated sources it is precisely the latter notion of “limit consistency” which renders the idealisation physically reasonable.

One would like to have a similar mathematical description of concentrated sources in the theory of general relativity. However, general relativity is different from other field theories in two important respects. Firstly the Einstein field equations are nonlinear, so that one cannot simply pass from smooth solutions of the field equations to weak solutions as one can in a linear theory. More precisely the curvature tensor is a nonlinear function of the metric and its first two derivatives. Thus the metric needs to be at least C^2 to guarantee that the curvature is continuous. Mathematically one can weaken this condition to allow a C^{2-} (i.e. first derivative Lipschitz continuous) metric and this is sufficient for most results in differential geometry to remain valid. In certain special situations one can lower the differentiability requirements still further and formulate the field equations in a way which avoids ill-defined products of distributions. For example it is possible to describe shells of matter [35], [14] and gravitational radiation [38] within the context of classical distribution theory. Using the standard definition of the curvature

$$R^a{}_{bcd} = \Gamma^a_{db,c} - \Gamma^a_{cb,d} + \Gamma^a_{cf}\Gamma^f_{db} - \Gamma^a_{df}\Gamma^f_{cb} \quad (1)$$

one sees that one wants the connection to have a (weak) derivative and be locally square integrable in order for the left hand side to make sense as a distribution. This led Geroch and Traschen [25] to introduce a class of “regular metrics” for which the components of the curvature are well defined as a distribution. Geroch and Traschen went on to show that such regular metrics can only have curvature with singular support on a submanifold of co-dimension at most one. Thus for a 4-dimensional spacetime a metric

representing a shell of matter could belong to this class but a string or particle could not. We will consider such distributional solutions of Einstein's equations in more detail in §2. The second important respect in which general relativity is different from other field theories is that one does not have a fixed background metric, but instead the geometry is determined by the field equations. Spacetime is described by a manifold together with a Lorentz metric which is assumed to be sufficiently differentiable for Einstein's equations to be defined. One then detects the presence of singularities by showing that the spacetime is incomplete in some sense. The problem with this approach is that many physically reasonable spacetimes contain singularities according to this definition. For example solutions to Einstein's equations representing cosmic strings are singular. What one would like to do is to lower the differentiability required of the metric to permit a wide class of "distributional geometries" which represent physically reasonable solutions and are also mathematically tractable. Owing to the nonlinear nature of Einstein's equations such distributional geometries will in general require some nonlinear theory of generalised functions. Moreover, as pointed out by Isham [34] distributional solutions are not only important in allowing one to deal with concentrated sources or describing weak singularities but (based on our experience with linear field theories) should also be included in any path integral formulation of quantum gravity.

Although there has long been a desire to allow distributional geometries in general relativity it is only comparatively recently that any real progress has been made in realising this aim. Many of the early attempts (e.g. [67], [74], [4], [5]) used methods which were specifically adapted to the particular problem being considered and whose general applicability was uncertain. More recently a number of different authors (for an overview see [91], [27, Ch. 5.2]) have investigated distributional geometries using an approach based on Colombeau algebras. These were developed by J.F. Colombeau in the 1980's and contain the smooth functions as a subalgebra and the distributions as a linear subspace. The key idea of this approach is regularisation through smoothing and the use of asymptotic estimates with respect to a regularisation parameter. In particular, it provides a mathematically consistent way of multiplying distributions and a unified view on calculations involving various regularisation procedures. An important feature of these algebras is the notion of association which gives a correspondence between elements of the algebra and distributions. This allows one to use the power of the algebras to do mathematical calculations but then use the concept of association to interpret the final result in terms of classical distributions and give the solutions a physical interpretation.

In §2 we review the extent to which it is possible to incorporate nonlinear operations into classical distribution theory. We will show that some very limited operations are permitted and that some apparently reasonable "multiplication rules" lead to inconsistencies. We then consider the implications of this for distributional solutions of Einstein's equations and end the section by looking in detail at the properties of the Geroch-Traschen regular metrics. In §3 we give a brief review of Colombeau theory and explain how it is able to circumvent the result of Laurent Schwartz on the impossibility

of multiplying distributions. Historically one of the first singular solutions of Einstein's equations to be studied from a distributional viewpoint was the Schwarzschild solution which was considered by Parker in 1979 [67]. The key observation is that when written in Kerr-Schild coordinates the coefficients of the metric are locally integrable. The Schwarzschild and Kerr solutions have been studied in these coordinates by Balasin and Nachbagauer in a number of papers and in §4 we describe their work using the language of the (special) Colombeau algebra. If one boosts the Schwarzschild solution with velocity v then one obtains, in the limit that the velocity is that of light, the “ultrarelativistic Schwarzschild solution”. This metric was investigated by Aichelburg and Sexl [1] who showed that it could be thought of as a gravitational shock wave. When one does a similar calculation with the ultrarelativistic limit of the Reissner-Nordström solution one obtains a solution with vanishing electromagnetic field but δ -function energy density. In §5 we show how Steinbauer was able to explain this physically surprising result using Colombeau algebras. In §6 we consider geodesic equations for impulsive gravitational wave spacetimes and show how Colombeau algebras provide an appropriate way of obtaining the results expected on physical ground without the need to make use of ad hoc “rules” for the multiplication of distributions. The study of conical singularities is another area where various authors had used a variety of regularisation procedures to obtain the physically plausible result that a cone has δ -function curvature at the vertex. However this result appeared to be at odds with the results of Geroch and Traschen. In §7 we review the analysis and resolution of the problem by Clarke, Vickers and Wilson using the (full) Colombeau algebra. In §8 we discuss questions of coordinate invariance of Colombeau algebras and review work on global and diffeomorphism invariant versions of the construction leading to a “nonlinear distributional geometry”. In §9 we look at “generalised hyperbolicity” and how weak singularities may be regarded as distributional solutions of Einstein's equations. Finally we give some conclusions and an outlook to future work in §10.

2. Classical Distribution Theory and General Relativity

In this section we briefly review the fundamental problems encountered when one tries to incorporate nonlinear operations into classical distribution theory. We will then examine the inherent limitations this imposes on distributional products and review the consequences for distributional solutions of Einstein's equations.

There has been a long history of using generalised function ideas in physics to model point sources and discontinuous phenomena. Such generalised functions were put on a sound mathematical footing by the development of the theory of distributions through the work of S. L. Sobolev [78] and L. Schwartz [76]. The basic idea is to make distributions dual to a space of smooth “test functions”. To introduce some notation we let $\mathcal{D}(\mathbb{R}^n)$ denote the space of smooth functions of compact support on \mathbb{R}^n . If S is a linear form $S : \mathcal{D}(\mathbb{R}^n) \rightarrow \mathbb{C}$ then we will denote the action of S on $\phi \in \mathcal{D}$ by $\langle S, \phi \rangle$. The vector space of distributions $\mathcal{D}'(\mathbb{R}^n)$ is then defined to be the space of continuous

linear forms on $\mathcal{D}(\mathbb{R}^n)$. In a similar way distributions on an orientable manifold M are defined as continuous linear functionals on the space of compactly supported n -forms i.e. $\mathcal{D}'(M) := [\Omega_c^n(M)]'$. A rich theory of distributional tensor fields (and sections of more general vector bundles) has been developed by Marsden in [61]; for a pedagogical introduction see [27, Ch. 3.1].

The theory of distributions quickly proved to be extremely successful both in applications to the study of linear partial differential equations and in justifying the use of generalised functions in physics. For example the Malgrange-Ehrenpreis theorem shows that any linear PDE with constant coefficients has a fundamental solution in the space of distributions. However the theory soon displayed its natural limitations when in 1957 H. Lewy [51] gave his famous example of a linear partial differential equation with smooth (non-constant) coefficients which does not have a distributional solution. A second difficulty with the theory is that the definition of a distribution as a linear functional does not make it easy to define the product of distributions. This prevents one from using distributions to investigate nonlinear PDEs with singular data or coefficients.

Although the Lewy example of a linear PDE without a distributional solution came as a great surprise, the difficulties that one encounters with the multiplication of distributions are much easier to understand and can be seen by looking at some simple cases. In the following we shall briefly discuss such examples concentrating on the powers of the Heaviside function H and its product with Dirac's delta function δ . If we regard H as a discontinuous function then $H^m = H$ ($m \in \mathbb{N}$). However if we differentiate this formula and use the Leibniz rule for the derivative the following one-line calculation implies the vanishing of the delta function:

$$2H\delta = (H^2)' = (H^3)' = 3H^2\delta = 3H\delta, \text{ so } H\delta = 0 \text{ hence } \delta = 0. \quad (2)$$

Another popular ‘‘multiplication rule’’ is

$$H\delta = \frac{1}{2} \delta. \quad (3)$$

We demonstrate the problematic nature of this rule (and in fact any rule of the form $H\delta = c\delta$, $c \in \mathbb{R}$) by considering the simple ODE

$$y'(t) = \delta(t)y(t), \quad y(-\infty) = 1, \quad (4)$$

(for an amusing discussion of this equation see [30].)

Using the ansatz $y(t) = 1 + \alpha H(t)$ and (3) we find $\alpha\delta = (1 + \alpha/2)\delta$, hence $\alpha = 2$ and the solution takes the form

$$y(t) = 1 + 2H(t). \quad (5)$$

On the other hand a different approach motivated by the requirement of stability under perturbation is to regularise the singular coefficient by a sequence δ_n weakly converging to δ . Then the solution to the regularised equation is given by $y_n(t) = \exp(\int_{-\infty}^t \delta_n(s)ds) + 1$ which converges to

$$\tilde{y}(t) = 1 + (e - 1)H(t), \quad (6)$$

which obviously does not coincide with the previous solution.

The deeper reason behind these and all other inconsistencies in this realm is the incompatibility of the laws of a (commutative) differential algebra with the formulae $H^m = H$ and $H' \neq 0$. This insight was put into stringent form by L. Schwartz himself in his incompatibility result [75], which says that if the vector space \mathcal{D}' of distributions is embedded into a differential algebra $(\mathcal{A}, +, \circ)$ then the following properties are mutually contradictory:

- (i) \mathcal{D}' is linearly embedded into \mathcal{A} and $f(x) \equiv 1$ is the unity in \mathcal{A} .
- (ii) There exist linear derivation operators $\partial_i : \mathcal{A} \rightarrow \mathcal{A}$ satisfying the Leibniz rule.
- (iii) $\partial_i|_{\mathcal{D}'}$ is the usual partial derivative.
- (iv) $\circ|_{C^k \times C^k}$ (for k finite) is the usual pointwise product of functions.

It was this result that led to the idea that it was impossible to multiply distributions. However given that repeatedly differentiating a C^k function eventually produces a distribution it is perhaps unreasonable to insist on (iv) but instead we should only require that the product of smooth functions is the usual pointwise product. We will see in §3 that this is precisely the condition satisfied by Colombeau algebras.

In the rest of this section we examine the extent to which one can apply linear distribution theory in the context of general relativity. After summarising work done in a number of special cases we review a classical paper by Geroch and Traschen [25] in which they set up a “maximal” distributional framework by finding the “largest possible” class of spacetime metrics which allow for a distributional formulation of the field equations, and we discuss its limitations.

Spacetimes involving an energy-momentum tensor supported on a hypersurface of spacetime have long been used in general relativity (see [49, 50, 19] and [35, 65], as well as the references therein). Consider a submanifold S of codimension one dividing spacetime into a “lower” and “upper” part and let the metric be smooth up to and including S from each of its sides but allow for a jump of its first derivatives across S . Writing out the Einstein equations in terms of the extrinsic curvature of S one finds junction conditions very similar to the ones in electrodynamics (see e.g. [73], Ch. 3.7). More precisely, the jump of the extrinsic curvature is interpreted as the surface stress-energy of a surface layer located at S . In the case of S being timelike this arrangement represents a thin shell of matter, while if S is null it may be interpreted as a thin shell of radiation (see e.g. [38]). In [35] W. Israel has given a general formulation of this widely applied approach with the practical advantage that no reference to any special coordinate system is required; the four-dimensional coordinates may be chosen freely and hence may be adapted to possibly different symmetries in the upper and lower part of spacetime.

On the other hand Lichnerowicz [52], has given an alternative description using tensor distributions assuming the existence of an admissible continuous coordinate system across S . This formalism was used by Lichnerowicz [53, 54] and Choquet-Bruhat [12] to study gravitational shock waves. They derived algebraic conditions on

the metric across the shock (the “gravitational Rankine-Hugoniot conditions”) as well as equations governing the propagation of the discontinuities. The respective formalisms of Israel and Lichnerowicz were shown to be equivalent in [59].

The description of gravitational sources supported on two-dimensional submanifolds of spacetime, however, is more delicate. Israel [36] has given conditions under which a sensible treatment of the field of a “thin massive wire” is possible. He isolated a class of “simple line sources” which possess a linear energy-momentum tensor and hence allow a well-defined limit as the wire’s radius shrinks to zero. We will look at line sources in more detail below.

On the other hand Taub [83] has claimed to have generalised Lichnerowicz’s formalism to include gravitational sources supported on submanifolds of arbitrary codimension in spacetime. However, he had to fix the ill-defined products by “multiplication rules”, in particular, by using equation (3).

We now begin to review the systematic approach by Geroch and Traschen in analysing the structure of the nonlinearities of the field equations to see how far one can get avoiding ill-defined distributional products. More precisely, the quest is for a class of metrics allowing for a distributional formulation of the Einstein tensor in order to assign to the spacetime—via the field equations—a distributional energy-momentum tensor representing the “concentrated” source. Note that there are two contradictory demands on this class of metrics: on the one hand these metrics should be “nice enough” to permit the distributional calculation of the curvature entities, while on the other hand they should be “bad enough” to have the Einstein tensor and hence the energy-momentum tensor concentrated on a submanifold of a high codimension in spacetime.

We write out the coordinate formula of the Riemann curvature tensor in terms of the Levi-Civita connection and the connection in terms of the metric

$$R_{abc}{}^d = 2\Gamma_{e[b}\Gamma_{a]c}{}^e + 2\partial_{[b}\Gamma_{a]c}{}^d, \quad (7)$$

$$\Gamma_{bc}{}^a = g^{ae}(\partial_{(b}g_{c)e} - \frac{1}{2}\partial_c g_{bc}). \quad (8)$$

and try to “save” these equations by putting just as much restrictions on the metric tensor as needed to allow for a distributional interpretation of the respective right hand sides. For the first term in (7) it is obviously sufficient to assume $\Gamma_{bc}{}^a$ to be locally square integrable. Since $L_{\text{loc}}^2 \subseteq L_{\text{loc}}^1$ this requirement actually also suffices to interpret the second term in (7) as the weak derivative of the regular distribution $\Gamma_{bc}{}^a$. Furthermore from equation (8) we see that it is sufficient to demand g^{ab} to be bounded locally almost everywhere in order to produce locally square integrable $\Gamma_{bc}{}^a$ from locally square integrable first weak derivatives of g_{ab} . This motivates the following definition.

2.1 Definition. *A symmetric tensor field g_{ab} on a four-dimensional manifold M is called a g -regular metric if g_{ab} and $g^{ab} \in L_{\text{loc}}^\infty \cap H_{\text{loc}}^1$.*

In the above definition L_{loc}^∞ denotes the space of locally bounded functions and H_{loc}^1 denotes the Sobolev space of functions which are locally square integrable and also have

locally square integrable first (weak) derivative. Note that although the above definition appears to be stronger than that in [25] it is actually equivalent to the original one where it was merely demanded that g_{ab} as well as g^{ab} are locally bounded almost everywhere, and the first weak derivatives of g_{ab} are locally square integrable. In fact, these assertions imply $\partial_c g^{ab} \in L^2_{\text{loc}}$ since $g^{ab} = \text{cof}(g_{ab}) / \det g_{ab}$, where $\text{cof}(g_{ab})$ denotes the cofactor of g_{ab} . We also note that the above conditions will always hold for a C^{1-} metric, for such a metric will always admit a locally bounded weak derivative. For a detailed discussion of the relationship between gt-regularity and some other regularity conditions in the context of axial and cylindrical symmetry we refer to [93], chaps. 2.3–2.5.

A further important remark on the notion of gt-regular metrics is in order. While the definition is coordinate invariant with the manifold fixed beforehand, in the case we are most interested in i.e. when we are dealing with a singular spacetime in general relativity, the situation is different. We are not given in advance a coordinate system that includes the singularity. So the question of whether a singular metric is gt-regular or not depends crucially on the choice of the differentiable structure which is imposed on the manifold to include the singular region.

To see that gt-regular metrics actually allow for the distributional formulation of Einstein's equations we show that one can build a distributional Einstein tensor. To do this we need to show that the tensor product of the contravariant metric with the Riemann tensor makes sense as a distribution. By writing the terms involving the second derivative as a total derivative we may write this in the form

$$g^{ef} R_{abc}{}^d = 2g^{ef} \Gamma_{m[b}^d \Gamma_{a]c}^m + 2\partial_{[b}(g^{ef} \Gamma_{a]c}^d) - 2(\partial_{[b} g^{ef}) \Gamma_{a]c}^d. \quad (9)$$

Now the first term involves a product $L^\infty_{\text{loc}} \times L^1_{\text{loc}}$ hence stays locally integrable. The second term involves a weak derivative of an L^1_{loc} -tensor field hence may be interpreted as a distribution and the third term is a product of two locally square integrable fields so is also locally integrable.

Before discussing convergence for gt-regular metrics we briefly introduce tensor distributions. Distributional sections of vector bundles and, in particular, distributional tensor fields can be defined as continuous linear forms on suitable test section spaces but are most easily viewed just as sections with distributional coefficients, that is

$$\mathcal{D}'_s(M) = \mathcal{D}'(M) \otimes \mathcal{T}_s^r(M), \quad (10)$$

where \mathcal{D}'_s and \mathcal{T}_s^r denote the spaces of distributional and smooth (r, s) -tensor fields respectively.

We can now discuss an appropriate notion of convergence for gt-regular metrics. As already indicated above we would like the Einstein tensor of a sequence of metrics approximating a gt-regular one to approximate the Einstein tensor of the gt-regular metric. The natural notion of weak convergence for a sequence of locally square integrable tensor fields $((\mu_{j_1 \dots j_s}^{i_1 \dots i_r})_n)_n$ is convergence locally in square integral, i.e.

$$(\mu_{j_1 \dots j_s}^{i_1 \dots i_r})_n \rightarrow 0 \quad (n \rightarrow \infty) \text{ iff } \int (\mu_{j_1 \dots j_s}^{i_1 \dots i_r})_n (\mu_{l_1 \dots l_s}^{k_1 \dots k_r})_n t_{i_1 \dots i_r k_1 \dots k_r}^{j_1 \dots j_s l_1 \dots l_s} \rightarrow 0 \in \mathbb{C} \quad (11)$$

for all smooth compactly supported $(2s, 2r)$ -tensor densities $t_{i_1 \dots i_r k_1 \dots k_r}^{j_1 \dots j_s l_1 \dots l_s}$. Using this notion of convergence one may prove the following theorem.

2.2 Theorem. (Convergence of gt-regular metrics)

Let g_{ab} and $((g_{ab})_n)_n$ be a gt-regular metric and a sequence of gt-regular metrics respectively and let

- (i) $((g_{ab})_n)_n$ and $((g^{ab})_n)_n$ be locally uniformly bounded, and
- (ii) $(g_{ab})_n \rightarrow g_{ab}$, $(g^{ab})_n \rightarrow g^{ab}$, and $(\partial_a g_{bc})_n \rightarrow \partial_a g_{bc}$ locally in square integral.

Then $(R_{abc}{}^d)_n \rightarrow R_{abc}{}^d$ in $\mathcal{D}'_3(M)$ and hence $(G_{ab})_n \rightarrow G_{ab}$ in $\mathcal{D}'_2(M)$.

We note that the space of gt-regular metrics is complete with respect to the notion of convergence defined by hypotheses (i) and (ii). Moreover, let g_{ab} be a continuous gt-regular metric then there exists a sequence of smooth metrics $((g_{ab})_n)_n$ converging to g_{ab} in the sense of (i) and (ii).

Before actually checking which class of gravitational sources may be described by gt-regular metrics we start with the following heuristic consideration of the behaviour of gt-metrics. Suppose S is a d dimensional submanifold of a 4-dimensional spacetime M and the metric g_{ab} is smooth on $M \setminus S$ but some of its components diverge as one approaches S . What order of divergence is allowed if g_{ab} is to be gt-regular? Let r be a typical distance from S measured by some background Riemannian metric h_{ab} and suppose the components of g_{ab} diverge at the rate of r^{-s} for some positive number s . Then the weak derivatives of g_{ab} diverge like r^{-1-s} while the volume element is proportional to r^{3-d} . In order for the derivatives of the metric to be locally square integrable we therefore require that $2(-s - 1) + 3 - d > -1$, and hence that

$$s < 1 - \frac{d}{2}. \quad (12)$$

Hence we see that the components of gt-regular metrics must grow more slowly than a rate of $r^{-1+d/2}$ as one approaches a d -dimensional submanifold in 4-dimensional spacetime. In particular, the larger the codimension of the submanifold the more strongly the components of the metric may diverge. However, as shown by the following theorem, there are severe constraints on the dimension of S .

2.3 Theorem. (Concentrated sources from gt-regular metrics)

Let S be a submanifold of dimension $d = (0, 1, 2, 3)$ of a four-dimensional manifold M and let $T_{j_1 \dots j_s}^{i_1 \dots i_r} \neq 0$ a tensor distribution satisfying

- (i) $\text{supp}(T_{j_1 \dots j_s}^{i_1 \dots i_r}) \subseteq S$, and
- (ii) $T_{j_1 \dots j_s}^{i_1 \dots i_r}$ is the sum of a locally integrable tensor field and the weak derivative of a locally square integrable tensor field

(hence is of the form of the Riemann tensor of a gt-regular metric). Then $d = 3$.

This theorem fits the picture described earlier in this section, i.e. that gravitating sources with their support concentrated on a 3-dimensional submanifold have been

treated successfully in the literature while sources concentrated on submanifolds of higher codimension have turned out to be more subtle to deal with. However, it should also be emphasised that although the gt-regularity conditions are sufficient for the curvature to make sense as a distribution they are certainly not necessary as shown by the examples below, so that the above result should not be interpreted as implying that a spacetime with distributional curvature can only have the source confined to a 3-dimensional submanifold. Indeed in certain algebraically special situations—and using a preferred coordinate system—some of the curvature quantities may be defined for non gt-regular metrics. For example impulsive *pp*-waves [68] have been treated extensively using distributions.

Summing up, gt-regularity provides us with a large class of “badly behaved” metrics which nevertheless allows one to formulate the field equations in a “stable” way. We are, however, sailing close to the wind as may be seen from the fact that energy conservation may not be formulated in general for gt-regular metrics. Indeed, the left hand side of the Bianchi identities $\nabla_{[a}R_{bc]de}$ involves a product of the distributional coefficients of the Riemann tensor with the non-smooth Christoffel symbols and this is only well defined if one imposes additional conditions on the metric. Similarly, gt-regular metrics may not be used to raise or lower the indices of a general tensor distribution since the tensor product would again involve a multiplication of distributions.

By looking in more detail at the combination of terms that one has in the expression for the curvature, Garfinkle in [24] has generalised the formalism of Geroch and Traschen to include a slightly more general class of metrics. However, in extending the class of metrics in this way one can no longer establish the convergence theorem which one has for gt-metrics. One is therefore forced to give up the requirement of “limit consistency”. This is another indication that with the Geroch Traschen definition of regularity one has gone about as far as possible using conventional distribution theory. Staying strictly within the mathematically and physically consistent setting given by this theory, one has to restrict oneself to a class of metrics that excludes physically interesting cases such as strings and point particles. If one wants to describe more general gravitational sources the nonlinearity of the field equations forces one to go beyond the limits of classical distribution theory and face true conceptual problems. A consistent framework allowing for nonlinear operations on singular (e.g. distributional) objects is provided by Colombeau’s algebras of generalised functions, which we introduce in the next section.

3. A brief review of Colombeau theory

In this section we give a brief introduction to Colombeau algebras. For more details see [16], [17], [11], [27]. As we said in the previous section the definition of distributions as linear functionals is not well suited to formulate a definition of multiplication. However it is common to visualise the Dirac delta function as the limit of a sequence of smooth functions, all with integral one, whose support gets concentrated at the origin. In fact

it is possible to give these ideas a precise mathematical formulation and an alternative sequential approach to distribution theory was developed by Mikusiński [63] as early as 1948 (see also Temple [84]). In this approach a distribution is an equivalence class of weakly converging sequences of smooth functions modulo weak zero-sequences. Working with a more subtle quotient construction Colombeau was able to construct a differential algebra \mathcal{A} satisfying conditions (i)–(iii) of §2 but with (iv) replaced by

(iv') $\circ|_{C^\infty \times C^\infty}$ is the usual pointwise product of smooth functions.

In order to introduce the basic concepts we will start by describing the special (or simplified) Colombeau algebra on \mathbb{R}^n . The basic idea is to consider generalised functions as 1-parameter families of smooth functions $\{f_\epsilon\}$. Our basic space will thus be

$$\mathcal{E}(\mathbb{R}^n) = \{\{f_\epsilon\} : 0 < \epsilon \leq 1, f_\epsilon \in C^\infty(\mathbb{R}^n)\}. \quad (13)$$

We now want to consider how we can represent a function f of finite differentiability as an element of this space. If we start with some $\Phi \in \mathcal{D}(\mathbb{R}^n)$ with integral one then we can rescale this to obtain a family of functions

$$\Phi_\epsilon(x) = \frac{1}{\epsilon^n} \Phi\left(\frac{x}{\epsilon}\right) \quad (14)$$

with the property that $\Phi_\epsilon \rightarrow \delta$ in \mathcal{D}' as $\epsilon \rightarrow 0$. Hence if we take the convolution of f with Φ_ϵ we obtain a family

$$f_\epsilon(x) = \frac{1}{\epsilon^n} \int f(y) \Phi\left(\frac{y-x}{\epsilon}\right) d^n y. \quad (15)$$

of smooth functions that converge to f in \mathcal{D}' as ϵ tends to zero. We will refer to Φ in the above expression as a molifier.

Of course we can also apply the above formula to a smooth function f . However for a smooth function we can also represent f as an element of \mathcal{E} by considering the constant family $f_\epsilon(x) = f(x)$. For the case of a smooth function we would like both these possible representations to be equivalent. Using a Taylor series expansion to compare the difference between these expressions we are lead to define two representations to be equivalent if they differ by a “negligible function” which is defined as a 1-parameter family of functions which on any compact set vanishes faster than any given positive power of ϵ . Since we are trying to construct a *differential algebra* we also require the derivatives of f to satisfy this property and the resulting set \mathcal{N} to be an ideal. Clearly \mathcal{N} is not an ideal in $\mathcal{E}(\mathbb{R}^n)$, but by restricting this space to “moderate functions” $\mathcal{E}_M(\mathbb{R}^n)$ which grow no faster than some inverse power of ϵ , one does have an ideal and we may define the differential algebra \mathcal{G} as the quotient.

3.1 Definition.

(i) (Moderate functions)

$$\mathcal{E}_M(\mathbb{R}^n) := \{\{f_\epsilon\} : \forall K \subset\subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}_0^n, \exists p \in \mathbb{N} \text{ such that} \\ \sup_{x \in K} |D^\alpha f_\epsilon(x)| \leq O(\epsilon^{-p}) \text{ as } \epsilon \rightarrow 0\}.$$

(ii) (Negligible functions)

$$\mathcal{N}(\mathbb{R}^n) := \{ \{f_\epsilon\} : \forall K \subset\subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}_0^n, \forall q \in \mathbb{N} \text{ such that} \\ \sup_{x \in K} |D^\alpha f_\epsilon(x)| \leq O(\epsilon^q) \text{ as } \epsilon \rightarrow 0 \}.$$

(iii) (Special algebra)

$$\mathcal{G}(\mathbb{R}^n) := \mathcal{E}_M(\mathbb{R}^n) / \mathcal{N}(\mathbb{R}^n)$$

Note $K \subset\subset \mathbb{R}^n$ indicates that K is compact and we have also employed the standard multi-index notation for $D^\alpha f$.

Thus a nonlinear generalised function f denoted by $f = [\{f_\epsilon\}]$ is an equivalence class of moderate sequences of smooth functions modulo negligible ones; it is represented by a moderate sequence of smooth functions $\{f_\epsilon\}$. The space $\mathcal{E}_M(\mathbb{R}^n)$ is a differential algebra with pointwise operations and since the space of negligible functions is a differential ideal, \mathcal{G} is also a commutative differential algebra.

The vector space of distributions is now embedded into the algebra \mathcal{G} by convolution with a molifier: More precisely we choose a molifier Φ which for technical reasons (not to be discussed here) is a Schwartz function and has all moments vanishing i.e. $\int \Phi(x)x^\alpha dx = 0 \forall |\alpha| \geq 1$. Then we embed $T \in \mathcal{D}'$ with compact support as

$$\iota(T) = [\{T * \Phi_\epsilon\}]. \tag{16}$$

Distributions which are not compactly supported are embedded via a localised version of (16) using a standard sheaf theoretic construction.

As remarked earlier one of the advantages of the Colombeau approach is that one may frequently interpret the results in terms of distributions using the concept of association or weak equivalence. A generalised function f is said to be associated to a distribution $T \in \mathcal{D}'$ if for one (hence any) representative $\{f_\epsilon\}$ we have

$$\forall \phi \in \mathcal{D}, \quad \lim_{\epsilon \rightarrow 0} \int f_\epsilon(x)\phi(x)d^n x = \langle T, \phi \rangle \tag{17}$$

and we then write $f \approx T$. Note that not all elements of \mathcal{G} are associated to distributions.

More generally we say for two generalised functions $f \approx g$ if

$$\forall \phi \in \mathcal{D}, \quad \lim_{\epsilon \rightarrow 0} \int (f_\epsilon(x) - g_\epsilon(x))\phi(x)d^n x = 0 \tag{18}$$

for one (hence any) pair of representatives $\{f_\epsilon\}, \{g_\epsilon\}$. Association is an equivalence relation which respects addition and differentiation. It also respects multiplication by smooth functions but by the Schwartz impossibility results cannot respect multiplication in general.

The algebra presented above is the simplest of the Colombeau algebras and can be readily generalised to arbitrary manifolds (see §8). However it does suffer from the disadvantage that the embedding ι of distributions and of functions of finite differentiability is not canonical but depends on the choice of molifier Φ (see above). Thus one has to appeal to mathematical or physical arguments outside the theory to

justify a particular representation of a non-smooth function. We discuss these matters in §8 where we also present the construction of so-called full Colombeau algebras which do possess a canonical embedding of distributions. In the following sections however we will discuss applications of Colombeau algebras to general relativity using the language of the simplified version.

4. The Schwarzschild and Kerr Spacetimes

In this section we review (linear and nonlinear) distributional treatments of the Schwarzschild and Kerr spacetimes. We use the language of the special Colombeau algebra although strictly speaking we should be using the special version of the theory of generalised tensor fields on manifolds, which we will introduce in §8. However the precise details will not be needed as we aim at presenting the main ideas and concepts in the most elementary way.

Balasin and Nachbagaauer considered rotating, charged, Kerr-Newman black-hole solutions in a number of papers ([4, 5, 6, 7, 9, 10]). The solutions considered have the feature that they are all examples of Kerr-Schild geometries. Such metrics may be written in the form

$$g_{ab} = \eta_{ab} + f k_a k_b \tag{19}$$

where η_{ab} is the flat Minkowski metric, f is an arbitrary function, $k_a = \eta_{ab} k^b$ and k^a is null and geodetic with respect to η_{ab} .

The simplest example of such a solution is the Schwarzschild solution which in the standard Minkowski coordinates has Kerr-Schild form given by

$$f = \frac{2m}{r}, \quad k^a = (1, x^i). \tag{20}$$

The expression for the Ricci tensor of a Kerr-Schild metric takes the surprisingly simple form

$$R_b^a = \frac{1}{2} \eta^{cd} \eta^{ea} [\partial_e \partial_c (f k_d k_b) + \partial_b \partial_c (f k_d k_e) + \partial_c \partial_d (f k_e k_b)]. \tag{21}$$

The energy momentum tensor is then given by Einstein's equations as

$$T_b^a = R_b^a - \frac{1}{2} \delta_b^a R. \tag{22}$$

One may then calculate the energy momentum tensor as follows. The Kerr-Schild form is regarded as being valid on the *whole* of Minkowski space with $\frac{2m}{r}$ replaced by some suitable regularised function. One now considers this as an element of \mathcal{G} and computes the components of T_b^a (in Minkowski coordinates) as elements of \mathcal{G} . Finally one can show that

$$T_0^0 \approx -m \delta^{(3)} \tag{23}$$

$$T_b^a \approx 0 \quad \text{otherwise.} \tag{24}$$

Different authors ([67, 39, 66]) using various regularization procedures have also assigned a distributional energy-momentum tensor to the Schwarzschild geometry.

These approaches have been compared using the language of the special algebra in [33].

The calculation of the energy-momentum tensor for the Kerr solution is significantly harder. Unlike the Schwarzschild case we cannot easily write the Kerr solution as the limit of a one parameter family of regular Kerr-Schild metrics. The problem arises because of the topology of the maximal analytic extension of the Kerr solution which leads to a branch singularity when the metric is written using the standard “flat” Kerr-Schild decomposition. Balasin [9] avoided this problem by considering metrics of generalised Kerr-Schild form. These are metrics which can be written

$$g_{ab} = \hat{g}_{ab} + f k_a k_b \quad (25)$$

where \hat{g}_{ab} is now a background metric, $k_a = \hat{g}_{ab} k^b$ and k^a is null and geodetic with respect to \hat{g}_{ab} (and also g_{ab} because of the form of the metric). One now has

$$R_b^a = \hat{R}_b^a + \hat{g}^{cd} \hat{g}^{ef} \hat{R}_{ceb}^a f k_d k_f + \frac{1}{2} \hat{g}^{cd} \hat{g}^{ea} [\partial_e \partial_c (f k_d k_b) + \partial_b \partial_c (f k_d k_e) + \partial_c \partial_d (f k_e k_b)] \quad (26)$$

where \hat{R}_{bcd}^a is the curvature of \hat{g}_{ab} .

We may write the Kerr metric in generalised Kerr-Schild form by taking the background metric to have the form

$$\hat{g}_{ab} dx^a dx^b = -dt^2 + \frac{\Sigma}{r^2 + a^2} dr^2 + \Sigma d\theta^2 + (r^2 + a^2) d\phi^2 \quad (27)$$

where $\Sigma = r^2 + a^2 \cos^2 \theta$ and $(t, r, \theta, \phi) \in \mathbb{R}^2 \times S^2$. Performing all the calculations in the special algebra one may compute $\sqrt{\hat{g}} R_b^a$ which is found to have the following associated distribution

$$\sqrt{\hat{g}} R_b^a \approx 2\pi \delta(\cos \theta) (-a \delta(u) \partial_u^a du_b + \partial_\theta^a d\theta_b + m \delta'(u) (\partial_t^a - (1/a) \partial_\phi^a) (dt_b + a d\phi_b)). \quad (28)$$

The above calculation was carried out by Balasin using the special algebra. This is probably the only practical way of doing the calculation given the topology of the manifold and the complexity of the metric. It suffers from the usual problem when using the special algebra of a non-canonical embedding. Rather than embed using a convolution, the embedding has been chosen to preserve the generalised Kerr-Schild form. However the embedding used is not unique within this class and it would be desirable to show that any reasonable embedding which preserved the form of the decomposition gave the same result. It is even less clear that a “natural regularisation” in some other coordinate system would give the same result. Nevertheless the calculation is an impressive example of the complicated calculations that can be performed in general relativity using the special algebra.

5. Ultrarelativistic Black Holes

In 1971 Aichelburg and Sexl [1] derived the ultrarelativistic limit of the Schwarzschild geometry. Below we give a description of the limit using the language of the special

algebra. We start by considering the Kerr-Schild form of the Schwarzschild metric written in double null coordinates $u = t - x$ and $w = t + x$,

$$ds^2 = dudw - dy^2 - dz^2 + \frac{2m}{r}k_a k_b dx^a dx^b \quad (29)$$

where in these coordinates $k^a = ((r - x)/r, (r + x)/r, y/r, z/r)$.

The Minkowski background enables us to have a well defined concept of boost and we may therefore boost the solution by velocity v along the x -axis. We therefore write

$$u = \sqrt{\frac{1+v}{1-v}}\bar{u} \quad w = \sqrt{\frac{1-v}{1+v}}\bar{w} \quad (30)$$

and to keep the energy of the “particle” finite we rescale the mass according to the special relativistic formula

$$m = (1 - v^2)^{\frac{1}{2}}p. \quad (31)$$

Substituting this into (29) gives a 1-parameter family of metrics depending on the boost velocity v . We are interested in the ultrarelativistic limit in which v reaches the speed of light (i.e. $v \rightarrow 1$), so we replace v by $1 - \epsilon$ and regard $\tilde{g}_{ab} := [g_{(\epsilon)ab}]$ as an element of the special algebra. It is readily shown that most of the terms in the perturbation $\frac{2m}{r}k_a k_b$ are associated to zero. The only surviving term is $\frac{2m}{r}k_0 k_0$. Using calculations very similar to Steinbauer [80] we find this is associated to $8p \ln \rho \delta(u)$ and hence $\tilde{g}_{ab} \approx g_{(0)ab}$ where

$$g_{(0)ab}dx^a dx^b = 8p \ln \rho \delta(u)du^2 + dudw - dy^2 - dz^2. \quad (32)$$

This metric describes a pp-wave and is flat everywhere except on the null plane $u = 0$ which contains the “particle”. This line element was first derived by Aichelburg and Sexl [1], who started with Schwarzschild written in isotropic coordinates and simultaneously boosted the solution and made a v -dependent coordinate transformation to compute the limiting metric.

It should be pointed out that this result is entirely consistent with the calculation of the energy-momentum tensor of the Schwarzschild solution given in the previous section. The ultrarelativistic limit of the latter in the (u, w, x, y) coordinate system is associated to

$$\delta(u)\delta^{(2)}(y, z)p_a p^b \quad \text{where } p_a = (1, 0, 0, 0), \quad (33)$$

which is just the energy-momentum tensor of \tilde{g}_{ab} . Indeed this observation was used by Balasin and Nachbagauer [6] to *derive* the ultrarelativistic limit of the Schwarzschild and Kerr geometries.

It is also possible to calculate the ultrarelativistic limit of the Reissner-Nordström solution. However in this case it is also necessary to rescale the charge according to the formula

$$e^2 = (1 - v^2)^{\frac{1}{2}}f^2, \quad (f \text{ a constant}) \quad (34)$$

in order to obtain a distributional limit at all as v tends to the speed of light. The limiting metric was found by Loustó and Sánchez [57] using the methods of [1] to be

$$ds^2 = \left\{ 8p \ln \rho + \frac{3\pi f^2}{2\rho} \right\} \delta(u) du^2 + dudw - dy^2 - dz^2. \quad (35)$$

This was confirmed using a calculation in \mathcal{G} by Steinbauer [79]. The solution obtained again represents a pp-wave and is flat everywhere except on the null plane $u = 0$. The ultrarelativistic limit of the electromagnetic energy-momentum tensor of Reissner-Nordström is also found to be compatible with the energy-momentum tensor of (35).

However the rescaling of the charge has the rather unexpected effect that while the electromagnetic field vanishes in the \mathcal{D}' -limit the ultrarelativistic energy-momentum tensor does not (cf [57]). Steinbauer in [80] rephrased this fact using association in \mathcal{G} i.e. showed that all the components of the electromagnetic field were associated to zero while the 00-component of the energy-momentum tensor was associated with a multiple of the delta function. However it must be stressed that regarded as elements of \mathcal{G} the electromagnetic field components are *non-zero* (even though they are *associated* to zero). Calculating the energy-momentum tensor of this field within \mathcal{G} gives the correct result. There is nothing unusual within Colombeau theory in having objects in \mathcal{G} which are associated to zero having products which are not associated to zero. This is simply a reflection of the fact that association does not respect multiplication in general.

The procedure of Aichelburg and Sexl has also been used to derive the ultrarelativistic limit of the Kerr metric by several authors see [56, 58, 23, 32]. In fact a number of different sources such as cosmic strings, domain walls and monopoles have been boosted to obtain ultrarelativistic spacetimes of impulsive pp-waves which in turn have been used to describe quantum scattering processes of highly energetic particles (see [86, 87] for an overview).

On the other hand Dray and 't Hooft [21] have generalised Penrose's "scissors and paste" method [68] (see also §6) to non-flat backgrounds and used it as an alternative way to derive the Aichelburg-Sexl geometry as well as more general gravitational shock waves. Using this method Dray and 't Hooft derived the spherical shock wave due to a massless particle moving at the speed of light along the horizon of a Schwarzschild black hole which was used to study the influence of matter, falling into the black hole on its Hawking-radiation. These ideas lie at the heart of 't Hooft's S-matrix approach to quantum gravity [85].

6. Geodesics for impulsive gravitational wave spacetimes

In the previous section we showed how the ultrarelativistic limit of the Schwarzschild solution lead to an impulsive gravitational wave spacetime. We now consider geodesics in such spacetimes with metric taking the form

$$ds^2 = f(x^A) \delta(u) du^2 + dudw - \delta_{AB} dx^A dx^B \quad (36)$$

where $A, B = 2, 3$ denote the transverse coordinates. These spacetimes have been constructed by Penrose using his vivid "scissors and paste" approach (see [69]). Geodesics for these impulsive gravitational wave spacetimes have been considered by Ferrari, Pendenza and Veneziano [22], Balasin [8] and Steinbauer [81]. As one might expect, one can regularise the geodesic equations, and show that the geodesics consist of broken and refracted straight lines.

A more rigorous derivation of these results using existence and uniqueness theorems within the Colombeau algebra has been given by Kunzinger and Steinbauer in [41]. More precisely they replaced the delta function in (36) by a generalised function D possessing a so called strict delta net $\{\rho_\epsilon\}$ as a representative i.e. they considered the generalised line element

$$\hat{d}s^2 = f(x^A)D(u)du^2 + dudw - \delta_{AB}dx^A dx^B \quad (37)$$

with $D = [\{\rho_\epsilon\}]$ and $\text{supp}(\rho_\epsilon) \rightarrow \{0\}$ and $\int \rho_\epsilon(x) dx \rightarrow 1$ as $\epsilon \rightarrow 0$ and $\|\rho_\epsilon\|_{L^1}$ uniformly bounded in ϵ . They were able to show that the geodesic as well as the geodesic deviation equation for the metric (37) may be solved uniquely in \mathcal{G} . Moreover these unique generalised solutions possess the physically expected associated distributions. Note that diffeomorphism invariance of these results is assured by diffeomorphism invariance of the class of strict delta nets. Note further that strictly speaking these calculations have been performed using the concept of generalised functions taking values in a manifold, cf §8, since geodesics are curves from an interval into spacetime.

In the literature impulsive pp-waves have frequently been described in different coordinates where the metric tensor is actually continuous (cf [69]). In the special case of a plane wave, $f(x, y) = x^2 - y^2$ and u_+ denoting the kink function,

$$ds^2 = (1 + u_+)^2 dX^2 + (1 - u_+)^2 dY^2 - dudV. \quad (38)$$

Clearly a transformation relating (38) and (36) cannot even be continuous, hence in addition to involving ill-defined products of distributions it changes the topological structure of the manifold. However, the two mathematically distinct spacetimes are equivalent from a physical point of view, i.e. the geodesics and the particle motion agree on a heuristic level (see [82]).

Using their results on the geodesic equation in \mathcal{G} , Kunzinger and Steinbauer in [42] succeeded in showing that the discontinuous change of coordinates is just the associated distributional map of a generalised coordinate transform. More precisely modelling the (distributional form of the) impulsive pp-wave metric in a diffeomorphism invariant way by the generalised metric (37) the latter may be subject to a generalised change of coordinates T . In either coordinates the associated distributional metric may be computed giving the distributional (respectively the continuous) form of the pp-wave metric. Physically speaking the two forms of the impulsive metric arise as the (distributional) limits of a sandwich wave in different coordinate systems. Hence impulsive pp-waves are indeed sensibly modelled by the generalised spacetime metric (37). However in the different coordinate systems a different distributional picture is obtained.

A similar situation arises in the case of impulsive spherical waves which have also been introduced in [69]. In this case the distributional form of the metric arises as an impulsive limit of type-N Robinson-Trautman solutions, which however due to the fact that the metric is quadratic in the profile function formally involves the square of the delta function. An explicit discontinuous coordinate transformation relating this form of the metric with the continuous form was given in [71]. A study of this situation using Colombeau methods relies on a better understanding of the geodesics of these spacetimes; a study of these has been initiated in [72].

7. Conical Singularities and Cosmic Strings

An important example of a two dimensional singularity is provided by the conical spacetime

$$ds^2 = dt^2 - dr^2 - A^2 r^2 d\phi^2 - dz^2. \quad (39)$$

where $A \neq 1$ and ϕ is a standard 2π -periodic angular coordinate. This spacetime is locally flat for $r \neq 0$ and heuristic arguments suggest that it has delta-function like curvature on the $r = 0$ axis. The corresponding energy-momentum tensor then describes a string with stress equal to the mass per unit length μ , where $\mu = 2\pi(1 - A)$. This is precisely the form of the energy-momentum tensor of a cosmic string in the weak field thin string limit. Such spacetimes are also important mathematically as they provide simple examples of quasi-regular singularities (i.e. singularities for which the components of the Riemann tensor measured in a parallelly propagated frame tend to a well defined limit, see e.g. Vickers [88] for further details).

Unfortunately such spacetimes are not gt-regular, so one cannot expect the curvature to be well-defined using conventional distribution theory. Furthermore Geroch and Traschen showed how it was possible to obtain different values of the mass per unit length by taking different regularisations. Because of this Clarke, Vickers and Wilson [15] chose to investigate the curvature of the cone using the “full” Colombeau algebra where one has a *canonical* embedding of distributions. See §8 below for further details.

The singular part of the curvature arises from the conical singularity in the 2-cone

$$ds^2 = dr^2 + A^2 r^2 d\phi^2 \quad (40)$$

and for simplicity we will describe the calculation of the curvature (density) of this metric. Because polar coordinates are not well defined at the origin one must first write the metric in a regular coordinate system which includes the origin. To make things simple we will choose to work in Cartesian coordinates but as will be shown below the final result is independent of the coordinates used. In Cartesian coordinates one has

$$g_{ab} = \frac{1}{2}(1 + A^2)\delta_{ab} + \frac{1}{2}(1 - A^2)h_{ab}, \quad (41)$$

where

$$h_{ab} = \begin{pmatrix} \frac{x^2 - y^2}{x^2 + y^2} & \frac{2xy}{x^2 + y^2} \\ \frac{2xy}{x^2 + y^2} & \frac{y^2 - x^2}{x^2 + y^2} \end{pmatrix}. \quad (42)$$

We now regard the metric as an element of $\mathcal{E}_M(\mathbb{R}^2)$ by taking the convolution of the components g_{ab} with an arbitrary molifier. To find these we need to calculate

$$\tilde{h}_\epsilon(x, y) = \frac{1}{\epsilon^2} \int_{\mathbb{R}^2} h(u+x, v+y) \Phi((u/\epsilon), (v/\epsilon)) du dv \quad (43)$$

where $h(x, y) := e^{2i\phi}$.

By expanding in circular harmonics, using the compactness of the support of the molifier and using the residue theorem one can obtain a Fourier series for \tilde{h} and hence for $\tilde{g}_{(\epsilon)ab}$. One may now estimate the curvature \tilde{R}_ϵ of $\tilde{g}_{(\epsilon)ab}$, and use the Gauss-Bonnet theorem to show that

$$[\tilde{R}\sqrt{\tilde{g}}] \approx 4\pi(1-A)\delta^{(2)}. \quad (44)$$

It is important to stress that the methods described here may be used to calculate the curvature of the full four dimensional cone, although one may no longer use the Gauss-Bonnet theorem to calculate the curvature and therefore requires more delicate estimates (see Wilson [93] for details). One then obtains the heuristically expected energy-momentum tensor. Furthermore it is not hard to modify the results to deal with a metric which is not exactly conical, but approaches one quadratically as $r \rightarrow 0$. A second point to note is that the result does not depend upon the coordinates in which the calculation is carried out so long as they are smoothly related to Cartesians. This may be shown by explicitly transforming to new coordinates $X = X(x, y)$ and $Y = Y(x, y)$ and doing the whole calculation in the new coordinates. Again more delicate estimates are required but one can show that the resulting curvature density transforms in exactly the same way as a delta-function (see Vickers and Wilson [90] for details).

The above calculation shows that the curvature of a 4-dimensional cone, when calculated in the full Colombeau algebra, has a curvature which is associated to a delta-function which gives a mass per unit length equal to the deficit angle as one would expect from the heuristic calculation. Note that this result does not say that the curvature is equal to a delta function, but simply that it is associated to a delta function. Thus if one works at the level of association this result shows that regularisations based upon smoothing convolutions give an unambiguous answer for the curvature of the cone. Indeed a number of authors have applied different regularisations and obtained the same result. Balasin and Nachbagauer [4] used a regularisation based on a family of smooth hyperboloidal surfaces converging to a cone. A number of authors including Marder [60] and Geroch and Traschen [25] have looked at ‘‘rounding off’’ the point of a cone with a spherical cap, and Louko and Sorkin [55] have looked at a regularisation based on a different coordinate system. Unfortunately things are not quite as straightforward as one would hope. It is also possible to choose regularisations which do not yield a mass per unit length equal to the deficit angle. An important example of this was given by Geroch and Traschen [25]. They first formed a regularisation sequence $\tilde{g}_{(\epsilon)ab}$ which gave the standard answer for the mass per unit length. They then introduced a second regularisation sequence $\hat{g}_{(\epsilon)ab}$ which was related to the first sequence by a conformal

factor

$$\hat{g}_{(\epsilon)ab} = \Omega^2 \tilde{g}_{(\epsilon)ab}, \quad (45)$$

where

$$\Omega = \exp(\lambda f(r/\epsilon)) \quad (46)$$

and f is a smooth function whose support is $[1/2, 1]$. They then found that the mass per unit length of the limiting spacetime was dependent on both λ and f :

$$\mu = 2\pi(1 - A) - 2\pi \int_{1/2}^1 \lambda^2 \sin(\gamma r) f'(r)^2 dr. \quad (47)$$

From the point of view of Colombeau theory this “bad” behaviour of the above regularisation is not unexpected since although $\Omega^2 \approx 1$, it is not equal to the unity function in $\mathcal{G}(\mathbb{R}^4)$ and hence \hat{g}_ϵ *does not* represent the conical spacetime in the full Colombeau algebra. Wilson [93] looked at general regularisations of conical spacetimes and gave a condition that ensured that a given regularisation sequence gave the standard result for the mass per unit length. He showed that provided that the difference between the connection one-forms of the regularised spacetime and the original spacetime diverged no faster than $1/r$ then the answer would be the standard result. Thus although it is possible to obtain regularisations which give a different mass per unit length, the geometry of these bad regularisations diverges strongly from that of a cone as one approaches the axis.

A significant generalisation of the curvature calculations for a conical spacetime was obtained by Wilson [94] who extended the results to a four dimensional time dependent cosmic string. He considered cylindrically symmetric pure radiation solutions of Einstein’s equations. These metrics may be written in the form

$$ds^2 = e^{2\gamma(t-r)}(dt^2 - dr^2) + r^2 d\phi^2 + dz^2 \quad (48)$$

and naively one would expect a delta-function contribution to the curvature due to the angular deficit of $2\pi(1 - e^{-\gamma(t)})$. However, because of the time dependence of the angular deficit, the singularity cannot be quasi-regular but must be stronger. In fact it is an example of an “intermediate singularity” and the components of the Riemann tensor have a limit in a special choice of frame. Wilson calculated the energy-momentum tensor of this spacetime by first writing the metric in null Cartesian coordinates and then proceeded to obtain a smooth metric by convolution, estimate the curvature of the smooth metric and show that it is associated to a distribution. This result shows that even in the radiating case the mass per unit length is given by the expected formula

$$\mu(u, z) \approx 2\pi(1 - e^{-\gamma(u)}). \quad (49)$$

Furthermore by applying the methods of Ashtekar et al. [3] one may show that this result agrees with the asymptotically measured mass per unit length. However unlike the case of the static string it is unclear whether the calculation is coordinate invariant. The (u, x, y, z) coordinates used are natural for investigating radiating systems but have the disadvantage that Minkowski space appears to be singular on the axis when written

in these coordinates. However the correct mass per unit length for a static cosmic string is obtained, and this vanishes for Minkowski space.

8. Nonlinear distributional geometry

An underlying principle of general relativity is that the measurement of physical quantities should be independent of the coordinate system used. Mathematically this is reflected in the fact that the theory is formulated in terms of tensor fields on manifolds. The minimum differentiability of the coordinate transformations and the dependence on the differential structure of the manifold is quite subtle, but at the very least the theory should be invariant under smooth diffeomorphisms.

The definition of the special algebra introduced in §2 may be generalised in a more or less straight forward manner to yield a special Colombeau algebra $\mathcal{G}(M)$ on a differentiable manifold M [20]. This setup was extended by Kunzinger and Steinbauer in [43] to a theory of generalised sections of vector bundles. Furthermore in [44] the study of (pseudo-)Riemannian geometry in the generalised setting was initiated and in [45] generalised connections and curvature in general principal and vector bundles were studied. However both geodesics and diffeomorphisms involve considering functions with values in a manifold and if one wishes to consider these objects in the generalised setting one is forced to consider manifold valued generalised functions (a concept which it is not possible to deal with using distributions). The study of such functions was first looked at in [40], where the space $\mathcal{G}[X, Y]$ of generalised functions defined on the manifold X and taking values in the manifold Y was defined. This work was extended to a functorial theory in [46] where several global characterisations of the notions of moderateness and negligibility for generalised functions from X to Y are given. These characterisations provide the key to proving that composition of generalised functions between manifolds can be carried out unrestrictedly. Using these ideas it is possible to consider generalised geodesics and flows as well as singular ODEs on manifolds (see [47] for details). An overview over this “nonlinear distributional geometry” can be found in [27, Ch. 3.2].

However in all these constructions the embedding of distributions and functions of finite differentiability is non-canonical; in addition to the dependence on the choice of a mollifier (see §2), any embedding into $\mathcal{G}(M)$ will not be diffeomorphism invariant. So how can this setting be of use in a diffeomorphism invariant theory like general relativity?

Firstly in some applications there will be a natural physical parameter, such as a coupling constant, that may be used to directly provide a representation of the singular objects involved in \mathcal{G} . Hence there will be no need for using any embedding at all.

Another possibility is to model a singular spacetime metric by a whole class of generalised metrics that is by definition diffeomorphism invariant. This has been done in case of the description of impulsive pp waves in \mathcal{G} (cf §6).

Finally one may regard the Colombeau algebra as an intermediate calculational

tool for obtaining distributional answers. One picks some coordinate system, embeds the components of the metric into the Colombeau algebra (in the given coordinates) and calculates the components of the curvature and energy-momentum tensor etc. One then tries to show that these objects are associated to distributions. Finally one repeats the entire calculation in some other (smoothly related) coordinate system and tries to show that the answer transforms in the expected way for a distribution.

Although such calculations can be useful in some situations it would be preferable to make the embedding into the algebra coordinate invariant. To explain how this can be done we first introduce the full Colombeau algebra $\mathcal{G}^e(\mathbb{R}^n)$ [16] which, unlike the simplified version, has a canonical embedding of distributions.

This is achieved by substituting the index set $(0, 1]$ by a suitable class of molifiers. More precisely we introduce the following grading on the space of all molifiers

$$\begin{aligned} \mathcal{A}_0 &:= \{\Phi \in \mathcal{D}(\mathbb{R}^n) : \int \Phi(x) dx = 1\} \\ \mathcal{A}_q &:= \{\Phi \in \mathcal{A}_0 : \int \Phi(x)x^\alpha dx = 0, 1 \leq |\alpha| \leq q\} \quad (q \in \mathbb{N}) \end{aligned}$$

and take the basic space to be

$$\mathcal{E}^e := \{f : \mathcal{A}_0 \times \mathbb{R}^n \rightarrow \mathbb{R}^n \mid f \text{ smooth in } x\}.$$

Now the respective spaces of moderate and negligible functions may be defined as follows (again $\Phi_\epsilon(x) := (1/\epsilon^n)\Phi(x/\epsilon)$).

8.1 Definition.

(i) *(Moderate functions)*

$$\begin{aligned} \mathcal{E}_M^e(\mathbb{R}^n) &:= \{f \in \mathcal{E}^e : \forall K \subset\subset \mathbb{R}^n \forall \alpha \in \mathbb{N}_0^n \exists p \in \mathbb{N}_0 \forall \Phi \in \mathcal{A}_p : \\ &\quad \sup_{x \in K} |D^\alpha f(\Phi_\epsilon, x)| = O(\epsilon^{-p}) \text{ as } \epsilon \rightarrow 0\} \end{aligned}$$

(ii) *(Negligible functions)*

$$\begin{aligned} \mathcal{N}^e(\mathbb{R}^n) &:= \{f \in \mathcal{E}(\Omega) : \forall K \subset\subset \Omega \forall \alpha \in \mathbb{N}_0^n \forall p \in \mathbb{N}_0 \exists q \forall \Phi \in \mathcal{A}_q : \\ &\quad \sup_{x \in K} |D^\alpha R(\Phi_\epsilon, x)| = O(\epsilon^p) \text{ as } \epsilon \rightarrow 0\} \end{aligned}$$

(iii) *(Full algebra)*

$$\mathcal{G}^e(\mathbb{R}^n) := \mathcal{E}_M^e(\mathbb{R}^n) / \mathcal{N}^e(\mathbb{R}^n).$$

Distributions are now simply embedded into \mathcal{G}^e by convolution with the molifiers i.e.

$$\iota(T) = [T * \Phi] \tag{50}$$

and one obtains a differential algebra of generalised functions on \mathbb{R}^n (or open subsets thereof) just as in the special version with the additional benefit of a canonical embedding of the space of distributions.

Unfortunately this construction cannot be generalised to the manifold setting in a simple way as the definition of the spaces \mathcal{A}_q is not invariant and moreover the embedding is not diffeomorphism invariant since convolution again depends on the linear

structure of \mathbb{R}^n . However an invariant embedding can be achieved by demanding that the mollifiers Φ transform in an appropriate way. Colombeau and Meril in [18] made the first decisive steps towards a diffeomorphism invariant full Colombeau algebra by weakening the moment conditions to only hold asymptotically (which makes them invariant) and enlarging \mathcal{A}_q to a space of bounded paths $\epsilon \mapsto \Phi^\epsilon \in \mathcal{D}(\mathbb{R}^n)$. A flaw in this construction was found and removed by Jelínek [37] who developed an improved version of the theory which involved some subtle changes of definitions and established a number of important technical results. These ideas were then fully developed and given a firm mathematical basis in two papers dealing with the foundations of nonlinear generalised functions [26] where the first diffeomorphism invariant full Colombeau algebra on (open sets of) \mathbb{R}^n was constructed.

For applications in general relativity it is moreover desirable to have a geometric and global version of the theory rather than simply giving transformation rules for the local theory. Such a construction was given in [28] (for an overview see [29]). Here we only mention that the key idea is to replace the scaled and unbounded paths $(1/\epsilon)\Phi^\epsilon$ which are employed in the definition of moderateness and negligibility in the local theory by *smoothing kernels* Φ which are C^∞ maps from $(0, 1] \times M$ to compactly supported n -forms on M . In this way one obtains a geometrically constructed full Colombeau algebra on a differential manifold M where the canonical embedding of distributions commutes with Lie derivatives.

However this theory still lacks a canonical embedding of distributional *tensors*. Although the work of [28] described above provides one with an invariant embedding of scalars, one cannot simply apply this to the components of a tensor and obtain an invariant embedding of the tensor. Embedding the components of a tensor and then transforming would in general give a different answer from transforming and then embedding since multiplication by a smooth function does not in general commute with the embedding. The problem really stems from the way the convolution integrates the components of the tensor at *different* points of the manifold. The solution to this is to introduce some additional structure which enables one to first transport the tensor fields to the same point p in M so that one can then do the integration in a meaningful way. Such a transport is naturally provided by specifying a background connection, and in keeping with the spirit of the full algebra this is made an argument of the generalised tensor field. Thus a generalised tensor field depends upon a background connection γ , a smoothing kernel Φ and the point p . This is currently work in progress but see [89] and [48] for a more detailed description of this approach.

This approach now provides a "nonlinear distributional geometry" plus a canonical and invariant embedding of distributional objects. For example one can canonically embed the conical metric of the previous section to obtain a generalised metric. One then finds that this metric has a curvature tensor which is associated to the Dirac distribution and this shows that in any coordinate system a conical spacetime has a generalised Einstein tensor which satisfies

$$\tilde{G}_{ab} \approx \tilde{T}_{ab} \tag{51}$$

where \tilde{T}_{ab} is the embedding into the Colombeau algebra of the energy-momentum tensor of a (thin) cosmic string.

9. Weak singularities and Generalised hyperbolicity

As discussed earlier the definition of a singularity in general relativity is different from other field theories where one has a background metric. In general relativity one detects the presence of singularities by showing that the spacetime is incomplete in some sense. In the standard approach to singularities (see e.g. Hawking and Ellis, [31]), a singularity is regarded as an obstruction to extending a geodesic. However this definition does not correspond very closely to one's physical intuition of a singularity. This led to a consideration of whether physical objects would be subjected to unbounded deformations as one approached the singularity and was formulated mathematically in terms of strong curvature conditions. Unfortunately it is hard to model the behaviour of real physical objects in a strong gravitational field and because of this Clarke [13] suggested that one should consider instead the behaviour of physical fields (for which one has a precise mathematical description) near the singularity. According to the philosophy of ‘‘generalised hyperbolicity’’ one should regard singularities as obstructions to the Cauchy development of these fields rather than an obstruction to the extension of geodesics. However even a mild singularity is an obstruction if one uses the standard theory of distributions. By considering solutions to the wave equation in the Colombeau algebra one finds a certain class of weak singularities which do not prevent the evolution of test fields (Vickers and Wilson, [92]).

In this section we will look at solving the wave equation on a spacetime with a locally bounded singular metric (such as the conical spacetime considered in §7). The standard Cauchy problem takes the form

$$\begin{aligned}\square u(t, x^\alpha) &= 0 \\ u(0, x^\alpha) &= v(x^\alpha) \\ \partial_t u(0, x^\alpha) &= w(x^\alpha)\end{aligned}\tag{52}$$

with initial data (v, w) lying in the Sobolev spaces $H^1(S) \times H^0(S)$ prescribed on the initial surface S , given by $t = 0$. Because of the form of the metric one would expect the solution (if it exists) to be defined as a distribution. This however will cause difficulty in interpreting

$$\square u = (-g)^{-1/2} \partial_a ((-g)^{1/2} g^{ab} \partial_b u)\tag{53}$$

as a distribution in the framework of classical distribution theory because the above equation has (non-constant) singular coefficients and involves ill-defined products. Again we overcome these difficulties by using the nonlinear generalised function theory of Colombeau. For an overview of the treatment of PDEs with singular coefficients, data and solutions in this setting see [64].

We first canonically embed the metric g_{ab} into the full Colombeau algebra $\mathcal{G}^e(M)$ by using a convolution integral (15) as in §7 to obtain a representative $(\tilde{g}_{(\epsilon)ab}) \in \mathcal{E}_M^e(M)$.

Since the initial data (v, w) does not have to be smooth, we must also embed it into the algebra as (V, W) represented again by convolution integrals denoted v_ϵ and w_ϵ respectively.

The generalised function wave operator acting on a generalised function U represented by u_ϵ may then be written as

$$\square^\epsilon u_\epsilon = (-g_\epsilon)^{-1/2} \partial_a ((-g_\epsilon)^{1/2} g_\epsilon^{ab} \partial_b u_\epsilon). \quad (54)$$

We would like to then be able to solve the Cauchy problem in the space $\mathcal{G}^\epsilon(M)$

$$\begin{aligned} \square U(t, x^\alpha) &= 0 \\ U(0, x^\alpha) &= V(x^\alpha) \\ \partial_t U(0, x^\alpha) &= W(x^\alpha) \end{aligned} \quad (55)$$

and obtain a solution $U \in \mathcal{G}^\epsilon(M)$ which is associated to a distribution. In practice one works with the equivalent problem in $\mathcal{E}_M^\epsilon(M)$

$$\begin{aligned} \square^\epsilon u_\epsilon(t, x^\alpha) &= f_\epsilon(t, x^\alpha) \\ u_\epsilon(0, x^\alpha) &= v_\epsilon(x^\alpha) \\ \partial_t u_\epsilon(0, x^\alpha) &= w_\epsilon(x^\alpha) \end{aligned} \quad (56)$$

where (f_ϵ) is negligible.

In [92] it is shown how to estimate solutions u_ϵ of (56) and its derivatives in terms of powers of ϵ , given the moderate and null bounds of f_ϵ , v_ϵ and w_ϵ , using a method of energy estimates following Hawking and Ellis [31] and Clarke [13]. The additional complication in this situation is that one needs more explicit bounds because one needs to know the precise way in which the constants depend upon ϵ . This is accomplished by working with function spaces defined using higher order energy estimates (related to the super-energy tensors of Senovilla [77]) for which one has bounds in terms of the covariant derivatives of the curvature. Using this method one can show that the Cauchy problem (55) has a unique solution $U \in \mathcal{G}^\epsilon(M)$ which is independent of the representation chosen for g_{ab} , V or W . In the case of a conical spacetime one can go further and show that this solution is actually associated to a distributional solution. Thus a conical spacetime satisfies the condition of “generalised hyperbolicity” as claimed. It seems likely that these techniques may then be used to show that a much wider class of spacetimes with weak singularities satisfy the generalised hyperbolicity condition. See ([62]) for some recent results in this direction.

10. Conclusion

In this review we have looked at the extent to which it is possible to use conventional distribution theory to look at solutions of Einstein’s equations. Although there is an important class of new solutions that can be obtained by going beyond the confines of C^{2-} metrics the largest class that one can work with that is “stable” is given by the gt-regular metrics. Such metrics can be used to describe solutions with singular

support of the curvature on a hypersurface but are unable to deal with singularities of higher codimension. To deal with these it is necessary to go beyond distributions and work with a theory of nonlinear generalised functions. We have shown that such an appropriate description of nonlinear generalised functions is given by the theory of Colombeau algebras.

The special theory provides a straightforward computational tool for calculating the distributional curvature of a number of singular metrics and throwing some light on the physical nature of the singularity. Furthermore it has also been possible to define generalised functions taking values in a manifold and this allows one to talk about generalised geodesics and generalised symmetries (see e.g. [2]) of a spacetime. Unfortunately due to lack of space we have had to omit from this review both this latter topic and the topic of impulsive pp-waves and ultrarelativistic black holes with non-vanishing cosmological constant (see e.g. [70]).

However the special algebra does not provide one with a canonical embedding of distributions so there is always a question about the extent to which the answer depends upon the particular embedding that is used. The full Colombeau algebra rectifies this problem and a global formulation that is independent of the coordinate system has been given. Although the details of the tensorial theory remain to be fully worked out this work provides the basis for a coordinate and embedding independent theory of generalised (pseudo-)Riemannian geometry which can be used to analyse a wide class of singular spacetimes. In particular it is possible to give a distributional interpretation to a many physically reasonable singularities. The remaining singularities can therefore be regarded as true gravitational singularities. An outstanding project is to consider the singularity theorems in this generalised setting and show that they predict true gravitational singularities rather than simply distributional singularities.

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