Pure Decrementing Service M/G/1 Queue with Multiple Adaptive Vacations

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Abstract: In this paper, an M/G/1 queue model with the multiple adaptive vacations and pure decrementing service policy is studied based on the classical M/G/1 queueing model. The PGF of stationary queue length is derived by using an embedded Markov chain method. The LST of stationary waiting time is also given according to the independence between waiting time and arrival process. The average probabilities of various system states are obtained. Some special cases are given to show the general properties of the new model.

Keywords: Embedded Markov chain, multiple adaptive vacations, pure decrementing service, service cycle.

1. Introduction

Some M/G/1 queues with various vacation policies have been well studied. The early work focus on exhaustive service policy where the system takes vacations if and only if the system becomes empty. The details can be found in [2],[4],[6]. However, multifarious vacation policies with nonexhaustive service have important application value in computer system and communication networks. Boxma [1] and Takagi [8] introduced several vacation policies with nonexhaustive service to investigate the performance analysis of polling system. In this kind of models, the system can take a vacation even if there are customers waiting in line. Fuhrmann and Cooper [3] and Shanathikumar [7] proved the stochastic decomposition rule of vacation queue with a general vacation policy. Takagi [9] gave the detailed analysis for several M/G/1 vacation queues. Recently, the monograph [10] dealt with GI/M/1 vacation queues and multiserver vacation queues besides various M/G/1 vacation models.

To adapt diversified application background, some new vacation policies were presented. Tian and Zhang [10] studied a class of M/G/1 queues with exhaustive service and multiple adaptive vacations. Zhang and Tian [11] investigated a discrete time Geom/G/1 queue with multiple adaptive vacations. Multiple adaptive vacations is a synthetical vacation policy and some simple vacation policies are its special cases. This paper studied a class of queue models with nonexhaustive service and multiple adaptive vacations–an M/G/1 queue with multiple adaptive vacations and pure decrementing service policy. Using embedded Markov chain method, we derived the transform formulae for distributions of the number of customers in the system and waiting time in steady state, and obtained their stochastic decomposition structure. The M/G/1 queues with pure decrementing service policy and multiple vacation or single vacation in Takagi [9] are two special cases of this paper.
2. Model Description

In a classical M/G/1 queueing system, we integrated pure decrementing service rule and multiple adaptive vacations policy [9]. Once the service period starts, the server will keep on working until that the number of customers in system is one less than the number of customers at service start instant, then the server will take vacations. If there is a customer waiting for service at a vacation completion instant, the server will start a new service period. If there is no customer at a vacation completion instant, the server will ask for taking $H$ vacations consecutively according to the assistant workload completed at present. $H$ is a random variable with positive integers, here

$$P(H = j) = c_j, j \geq 1; \quad H(z) = \sum_{j=1}^{\infty} c_j z^j.$$ 

Each consecutive vacation time $V_k (k=1,2,\cdots,H)$ is iid random variable. It includes two cases:

(1) For a natural number $k(1 \leq k \leq H)$, if there are customers arrived during the $k$th vacation, the vacation period will be stopped in advance at the $k$th vacation completion instant. The system will enter a new service period until that the number of customers in system is one less than the number of customers at service start instant, then the server will take vacations.

(2) If there is still no customer arrived during these $H$ vacations, and there is no customer arrived at the $H$th vacation completion instant, the system will enter an idle period to wait for a new customer arrived. While there is a customer coming in idle period, the server will enter the service period immediately until there is no waiting customer in system, then take vacations again at the service completion instant. The system will repeat the process above.

The basic assumptions of the new model are given as follows: Customers follow Poisson arrival process with rate $\lambda (> 0)$, which means that any neighbor inter-arrival times series $\{\tau_i, i \geq 1\}$ follow independent and identically negative exponential distribution $F(t) = 1 - e^{-\lambda t}, \quad t \geq 0$.

Service times series $\{\chi_i, i \geq 1\}$ follow independent and identically general distribution $B(t), \quad t \geq 0$. The means of service time, second moment and LST (Laplace Stieltjes Transform) are respectively denoted by

$$0 < 1/\mu = \int_{0}^{\infty} dB(t), \quad b^{(2)} = \int_{0}^{\infty} t^2 dB(t), \quad B^*(s) = \int_{0}^{\infty} e^{-st} dB(t).$$

Vacation time $V$ is a nonnegative random variable, following general distribution $V(x)$, first moment, second moment, and LST are $E(V), \quad E(V^2), \quad v^*(s)$ respectively. If there is only one server in system, system capacity is infinity, interarrival time, service time and vacation time are mutually independent, this model is denoted by M/G/1 (PD, MAV), where PD and MAV represent Pure Decrementing service system and Multiple Adaptive Vacations, respectively. Therefore, run mechanics in pure decrementing service queueing system are shown as Figure 1.

Where $(S)$ represents the server who is being busy, $(V^((J)))$ represents the vacation state after $J$ vacations, and $(V^((H))+I)$ represents the system’s idle state after $H$ vacations and busy periods.
3. Analysis of System Capability Indices

Let $L_v$ represent stationary queue length at the customer departure instant, $Q^{(s)}_b$ represent the number of customers in system at the $n$th vacation completion instant, its PGF (Probability Generating Function) is $Q^{(n)}_b(z)$. When permitting service period to be zero, $Q^{(n)}_b$ is the number of customers in system at next service period start instant. The number of customers in system, denoted by $L_v$, follows an identical distribution by analysis for the new model in the service order FCFS (First Come First Served) and LCFS (Last Come First Served). Therefore, we assume that this model is in LCFS order in order to solve it conveniently.

3.1. The Number of Customers in System at a Service Start Instant

According to multiple adaptive vacations and pure decrementing service policy, stages of vacations, $J$, is random variable at a service completion instant,

$$J = \min\{H, k : V_1 + \cdots + V_{k-1} < T < V_1 + \cdots + V_k\}$$

Define two events

$A_I = \{ \text{next service period starts with the first customer arrived during idle period on condition that there is no customer at a service completion instant}\},$

$A_v = \{ \text{next service period starts with the first customer arrived during a vacation on condition that there is no customer at a service completion instant}\},$

we have

$$P(A_I) = Q^{(s)}_b(0)H(\nu^*(\lambda)), \quad P(A_v) = Q^{(s)}_b(0)(1 - H(\nu^*(\lambda)))$$

According to pure decrementing service order, when permitting service period can be zero, $Q^{(s)}_b$ is the number of customers in system at next service period start instant. If $Q^{(s)}_b$ is greater than zero, service period starts immediately and keep on working until that the number of customers in system is one less than the number of customers at service start instant, then the server will take vacations. We have

$$Q^{(n+1)}_b = Q^{(n)}_b - 1 + \text{the number of customers arrived during a vacation.}$$

If $Q^{(n)}_b = 0$ next vacation starts immediately and has two cases:

1. If there are customers arrived during the $k$th ($1 \leq k \leq H$) vacation, a service period starts with the ending of the $k$th vacation. Then the number of customers in system at the service period start instant, $Q^{(n+1)}_b$, is equal to the number of customers arrived during a vacation.
(2) If there is no customer arrived during the $H$th vacation, a idle period starts with the ending of vacation waiting for a new customer coming, then $Q_h^{(n+1)}$ is equal to 1. Therefore by utilizing the argument of Takagi [9]

$$Q_h^{(n+1)}(z) = \frac{Q_h^{(n)}(z) - Q_h^{(n)}(0)}{z} \nu^*(\lambda(1-z)) + Q_h^{(n)}(0)H(\nu^*(\lambda))z$$

$$+ Q_h^{(n)}(0)(1 - H(\nu^*(\lambda)))\nu^*(\lambda(1-z)).$$  \hspace{1cm} (1)

If the system is in steady state, the PGF of Equation (1) has no relation with $n$. So we can obtain function of $Q_h(z)$,

$$Q_h(z) = \frac{Q_h(0)((1-z)(1-H(\nu^*(\lambda))))\nu^*(\lambda(1-z)) - H(\nu^*(\lambda))z^2}{\nu^*(\lambda(1-z))-z}.$$  \hspace{1cm} (2)

Let $Q_0(1) = 1$,

$$Q_h(0) = \frac{1-\lambda E(V)}{1+H(\nu^*(\lambda))(1-\lambda E(V))}.$$  \hspace{1cm} (3)

According to Foster rule, we can prove that if $\rho = \lambda / \mu < 1$ and $\lambda E(V) < 1$, the system can reach steady state.

3.2. Stationary Queue Length and Waiting Time

**Theorem 3.1** If $\rho = \lambda / \mu < 1$ and $\lambda E(V) < 1$, stationary queue length $L_v$ in M/G/1(PD, MAV) queue can be decomposed into the sum of three independent random variables:

$$L_v = L + L_d + L_r,$$

Where $L$ is stationary queue length in classical M/G/1 queue, with its PGF given in [9, 10]. The PGF of additional queue length, $L_d$ and $L_r$, is given by

$$L_d(z) = \frac{1 - \nu^*(\lambda(1-z)) + H(\nu^*(\lambda))\nu^*(\lambda(1-z)) - H(\nu^*(\lambda))z}{H(\nu^*(\lambda)) + \lambda E(V)(1-H(\nu^*(\lambda)))(1-z)},$$

$$L_r(z) = \frac{(1-\lambda E(V))(1-z)}{\nu^*(\lambda(1-z))-z}.$$  \hspace{1cm} (4)

**Proof**: Because of stochastic decomposition results of Fuhrmann and Cooper [3], we can write

$$L_v(z) = \chi(z)L_v(M/G/1;z),$$  \hspace{1cm} (5)

where $L_v(M/G/1;z) = (1-\rho)(1-z)B'(\lambda(1-z))/(B'(\lambda(1-z))-z)$ is the PGF of the number of customers present in the system at a random point of the classical M/G/1 queue and $\chi(z) = (Q_v(z) - Q_v(0))/((1-Q_v(0))z)$ is the PGF of number of customers in the system at a random point of time during the vacation period [9]. Substituting Equations (2) and (3) into Equation (5), we have

$$L_v(z) = L(z)L_d(z)L_r(z).$$  \hspace{1cm} (6)

Therefore, Theorem 3.1 has been proved.
Simplifying $L_d(z)$ in Theorem 3.1, $L_d$ will be equal to zero with probability
\[ \frac{H(v^*(\lambda))}{H(v^*(\lambda)) + \lambda E(V)(1 - H(v^*(\lambda)))}, \]
equal to the number of customers during a vacation with probability
\[ \frac{\lambda E(V)(1 - H(v^*(\lambda)))}{H(v^*(\lambda)) + \lambda E(V)(1 - H(v^*(\lambda)))}, \]

$L_r$ is the number of customers in system at a vacation start instant.

The average number of customers at steady state for M/G/1 (PD, MAV) queue is
\[ E(L_v) = \rho + \frac{\lambda^2 b^{(2)}}{2(1 - \rho)} + \frac{\lambda^2 E(V^2)(1 - H(v^*(\lambda)))}{2(H(v^*(\lambda)) + \lambda E(V)(1 - H(v^*(\lambda)))}, \]

Corollary 3.2 If $\rho = \lambda/\mu < 1$ and $\lambda E(V) < 1$, stationary waiting time $W_v$ in M/G/1 (PD, MAV) queue with can be decomposed into the sum of three independent random variables:
\[ W_v = W + W_d + W_r, \]

Where $W$ is stationary waiting time in classical M/G/1 queue, with its LST given in [9], [10]. LST of additional delay $W_d$ and $W_r$ are given by
\[ W_d(s) = \frac{\hat{\lambda}(1 - H(v^*(\lambda))(1 - v^*(s))) + H(v^*(\lambda))s}{(H(v^*(\lambda)) + \lambda E(V)(1 - H(v^*(\lambda)))s}, \]
\[ W_r(s) = \frac{(1 - \lambda E(V))s}{s - \hat{\lambda}(1 - v^*(s))}. \]

The proof of Corollary 3.2 is evident, where $L_v(z) = W_v^*(\hat{\lambda}(1 - z))B^*(\hat{\lambda}(1 - z))$ and (6) are used, or this follows directly from the result of Kleinrock [5], therefore the LST of stationary waiting time is obtained.

3.3. Analysis of Service Cycle

According to stages of consecutive vacations, we have
\[ P(J \geq 1) = 1, \]
\[ P(J \geq j) = P(H \geq j)P(V_1 + V_2 + \cdots + V_{j-1}) = (v^*(\lambda))^{j-1}\sum_{k=j}^\infty c_k, \quad j \geq 2, \]
So the PGF of $J$ is
\[ J(z) = 1 - \frac{1 - z}{1 - v^*(\lambda)z}(1 - H(v^*(\lambda)z)). \]
\[
W_G^*(s) = \frac{1-(1-H(\nu(\lambda))(1-\lambda E(V)))}{1+H(\nu(\lambda))(1-\lambda E(V))} \nu^*(s) + \frac{1-\lambda E(V)}{1+H(\nu(\lambda))(1-\lambda E(V))} \times \left\{ \frac{1-\nu^*(s)}{1-\nu(\lambda)\nu^*(s)} \right\}.
\]

Therefore, average vacations length is
\[
E(V_G) = \frac{1-(1-H(\nu(\lambda))(1-\lambda E(V)))}{1+H(\nu(\lambda))(1-\lambda E(V))} E(V) + \frac{1-\lambda E(V)}{1+H(\nu(\lambda))(1-\lambda E(V))} \frac{1-H(\nu(\lambda))}{1-\nu(\lambda)} E(V).
\]

In M/G/1 (PD, MAV) queue, the server may be in idle state as usual. If there are customers in system at a vacation start instant, idle period will be zero after a vacation completion. If there is no customer in system at a vacation start instant, and there is still no customer at the \(Jth\) vacation completion instant. The time when the server is in idle is an interarrival time (Assuming interarrival time follows a nonnegative exponential distribution), assuming that \(I_v\) represents the time that the server stays in idle state, then
\[
E(I_v) = \frac{1}{\lambda} \frac{(1-\lambda E(V))H(\nu(\lambda))}{1+H(\nu(\lambda))(1-\lambda E(V))}.
\]

According to pure decrementing service order, we know that the new model’s service period is identical to busy period for classical M/G/1 queueing system. Then the LST of service period satisfies
\[
S_p^*(s) = B^*(s+\lambda(1-S_p^*(s)))
\]

and average service period length is
\[
E(S_p) = \frac{1}{\mu-\lambda}.
\]

Let \(C\) be an intermediate time between two service period start instant, denoted by a service cycle. Then average service cycle, \(E(C)\), is
\[
E(C) = \frac{1-\nu(\lambda)+\nu^*(\lambda)(1-\lambda E(V))(1-H(\nu(\lambda)))}{(1+H(\nu(\lambda))(1-\lambda E(V))(1-\nu^*(\lambda)))} E(V) + \frac{1}{\lambda} \frac{(1-\lambda E(V))H(\nu(\lambda))}{1+H(\nu(\lambda))(1-\lambda E(V))} + \frac{1}{\mu-\lambda}.
\]

Let \(p_b\), \(p_v\) and \(p_i\) be stationary probabilities of server’s being busy, on vacation, and idle, respectively. We have
\[ P_b = \frac{1}{E(C)(\mu - \lambda)}, \]
\[ p_r = \frac{E(V)(1 - v^*(\lambda) + v^*(\lambda)(1 - \lambda E(V))(1 - H(v^*(\lambda))))}{E(C)(1 + H(v^*(\lambda))(1 - \lambda E(V)))(1 - v^*(\lambda))}, \]
\[ p_f = \frac{(1 - \lambda E(V))H(v^*(\lambda))}{\lambda E(C)(1 + H(v^*(\lambda))(1 - \lambda E(V)))}. \]

4. Special Cases

In the following, we present some existing results appeared in the literature, which are special cases of the model.

Case 1. If \( H = \infty \), our model represents M/G/1 (PD, MV), Equation (4) and (7) reduces to the special case stated in [9],[10].

Case 2. If \( H = 1 \), our model represents M/G/1 (PD, SV), Equation (4) and (7) reduces to the special case stated in [9],[10].

Case 3. If \( H \) follows a geometric distribution, the model corresponds to M/G/1 (PD, GV), namely \( P_g(H = i) = pq^{i-1}, i = 1,2,\ldots, p > 0, q > 0, p + q = 1 \), then
\[ H(z) = \frac{pz}{1 - qz} \]

substituting \( H(v^*(\lambda)) = pv^*(\lambda)/(1 - qv^*(\lambda)) \) into Equation (4) and (7), then the PGF of additional queue length and LST of additional delay are respectively given by
\[ L_d(z) = \frac{(1 - qv^*(\lambda))(1 - v^*(\lambda(1 - z))) + pv^*(\lambda)(v\lambda(1 - z) - z)}{pv^*(\lambda) + \lambda E(V)(1 - v^*(\lambda)(1 - z))}, \]
\[ L_r(z) = \frac{(1 - \lambda E(V))(1 - z)}{v^*(\lambda(1 - z)) - z}; \]
\[ W_d^*(s) = \frac{\lambda(1 - v^*(\lambda))(1 - v^*(\lambda(s))) + pv^*(\lambda)s}{pv^*(\lambda) + \lambda E(V)(1 - v^*(\lambda)s)}, \]
\[ W_r^*(z) = \frac{(1 - \lambda E(V))s}{s - \lambda(1 - v^*(s))}. \]

The above examples show that \( H \) follows to different distributions, and we can obtain different pure decrementing service queue model with vacations. It’s a general model including many queue models, and develops the classical M/G/1 queue extensively.

5. Numerical Results

In this section we present some numerical examples that provide insight on the system behavior. Using equation in section 3, we can numerically compare performance indices of the systems in three special cases of our model. Here we assume that service time and vacation time follow exponential distribution, \( i.e. \, S \) follow exponential distribution with mean \( 1/\mu \), \( V \) follow exponential distribution with mean \( E(V) \).
In Figure 2, 3, suppose $E(V) = 2$, $\mu = 0.9$, traffic intensity $\rho$ ranges from 0.1 to 0.5. We give figures of mean queue length $E(L_v)$ and mean vacation probability $p_v$ as traffic intensity $\rho$ ranging from 0.1 to 0.5, respectively. In two figures, we observe that mean queue length increase when $\rho$ increases, and can compare performance indices of three models in the same condition.
In Figure 4, suppose service time of M/G/1 (PD, GV) follow exponential distribution, \( \mu \) is equal to 0.9, 0.5 and 0.3, respectively. We give three figures of mean queue length \( E(L_q) \) as traffic intensity \( \rho \) ranging from 0.1 to 0.5, and observe that mean queue length increases when \( \rho \) increases. In Figure 5, suppose \( \mu = 0.9 \) in M/G/1 (PD, GV) queue, we give three figures of system state, and observe that the change trend as traffic intensity \( \rho \) ranging from 0.1 to 0.5.

6. Conclusion

We present a detail description on pure decrementing service queueing system with multiple adaptive vacations. We gave out the PGF of stationary queue length by using an embedded Markov chain method, the LST of customers waiting time, analysis of decomposition for additional queue length and delay. At last we work out the average time during various periods, probabilities of server's being busy, vacation and idle, respectively, and show many existing queueing systems are special cases of our queueing model. The new conclusions in this paper can be extensively used in computer network, and it is very useful to solve network flow.

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