Two integrable lattice hierarchies and their respective Darboux transformations

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A B S T R A C T

Two integrable lattice hierarchies associated with two discrete matrix spectral problems are derived, and the infinitely many conservation laws of the first integrable model is obtained. Moreover, the Darboux transformations based on different Darboux matrixes \((2.3.4)\) and \((3.2.4)\) for the above-mentioned integrable hierarchies are established with the help of two different gauge transformations of lax pairs in this paper. As an application, the explicit solutions of the two integrable hierarchies are presented.

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1. Introduction

Nonlinear integrable lattice equations \([1–9]\) have explicit solutions with perfect mathematical and physical properties. Therefore, to obtain the exact solutions for the integrable lattice equations is always one of the most fundamental and difficult topics. There are several new approaches of the solution methods which include a transformed rational function method for constructing traveling wave solutions \([10]\) and a multiple exp-function method for presenting multiple wave solutions \([11,12]\). Recently, the linear superposition principle was used to construct linear-subspaces of solutions to Hirota bilinear equations \([13]\). The Darboux transformation (DT) \([14–25]\) is one of the powerful methods in finding explicit solutions of soliton equation from a trivial seed. Essentially, the DT is a special gauge transformation of solutions of Lax pairs. The key for constructing DT is to expose a kind of covariant properties that corresponding spectral problems possess. However, it is difficult to solve the lax pairs of Lattice soliton equations if the potential functions are not trivial. So, only while the isospectral matrix is suitable given, the DT is an effect way to solve the lattice equations.

In the past decades, there has been significant progress in the development of DT. A kind of Darboux transformation for a Lax integrable system in \(2n\)-dimensions was established in \([26]\). The binary DT was discussed in \([27]\). The connection between the DT and some modern operator based approaches to quantum mechanics is also outlined in \([28]\). DTs are set up for the \((2+1)\)-dimensional integrable lattice equation \([29]\) and \((1+1)\)-dimensional systems. The DT of integrable lattice equations associated with \(3 \times 3\) matrix spectral problems are derived in \([30,31]\). In this paper, we would like to give the exact solutions of two integrable lattice equations by the DT method based on two different Darboux matrixes.

This paper is organized as follows. In Section 2, we derive the generalized Lotka–Volterra lattice equation \([32,33]\) and its DT. In Section 3, we deduce another integrable lattice equation \([34]\) and its DT. In Section 4, some conclusions and remarks are given.
2. The generalized Lotka–Volterra equation and its Darboux transformation

2.1. The generalized Lotka–Volterra equation and its Hamiltonian structure

We first introduce the following isospectral problem

\[ E\varphi_n = U_n(u_n, \lambda)\varphi_n, \quad U_n(u_n, \lambda) = \begin{pmatrix} -v_n & \lambda v_n \\ -1 & \lambda \end{pmatrix}, \tag{2.1.1} \]

where \( \varphi_n = (\varphi_1, \varphi_0)^T, \lambda_i = 0. \)

To get new nonlinear Lax integrable lattice equations, we solve the stationary discrete zero curvature equation

\[ (E\Gamma_n)U_n - U_n\Gamma_n = 0, \tag{2.1.2} \]

with

\[ \Gamma_n = \begin{pmatrix} a_n & \lambda b_n \\ c_n & -a_n \end{pmatrix}. \]

Then (2.1.2) becomes

\[ \begin{align*}
-\nu_n(a_{n+1} - a_n) - \lambda b_{n+1} - \lambda v_n c_n &= 0, \\
\lambda v_n(a_{n+1} + a_n) + (\lambda^2 - \lambda \frac{w_n}{\tau})b_{n+1} + \lambda v_n b_n &= 0, \\
-\nu_n c_n + \frac{\lambda}{\tau} c_n + a_{n+1} + a_n - \lambda c_n &= 0, \\
\lambda (v_n c_{n+1} - a_{n+1} + b_n + a_n) + \frac{w_n}{\tau}(a_{n+1} - a_n) &= 0.
\end{align*} \tag{2.1.3} \]

Taking \( a_n^0 = -\frac{1}{2}, b_n^0 = 0, c_n^0 = 0, \) and substituting the expanding expressions

\[ a_n = \sum_{m=0}^{\infty} a_n^{(m)}/\tau^m, \quad b_n = \sum_{m=0}^{\infty} b_n^{(m)}/\tau^m, \quad c_n = \sum_{m=0}^{\infty} c_n^{(m)}/\tau^m, \tag{2.1.4} \]

into (2.1.3), we obtain the following recursion relations

\[ \begin{align*}
-\nu_n(a_{n+1}^{(m)} - a_n^{(m)}) - b_{n+1}^{(m+1)} - \nu_n c_n^{(m+1)} &= 0, \\
\nu_n(a_{n+1}^{(m)} + a_n^{(m)}) + b_{n+1}^{(m+1)} - \nu_n b_n^{(m+1)} + \nu_n b_n^{(m)} &= 0, \\
a_{n+1}^{(m)} + \nu_n a_n^{(m)} - c_n^{(m+1)} + \frac{w_n}{\tau} c_n^{(m)} &= 0, \\
\nu_n(a_{n+1}^{(m+1)} - a_n^{(m+1)}) + \frac{w_n}{\tau}(a_{n+1}^{(m)} - a_n^{(m)} + b_n^{(m+1)} + a_n^{(m+1)}) &= 0.
\end{align*} \tag{2.1.5} \]

which are all difference polynomials in \( U_n \) with respect to the lattice variable \( n \). Under the initial-value conditions of selecting zero constants for the inverse operation of the difference operator \( Dj(n) = (E - 1)f(n) \) in computing \( a_n^{(m)}, m \geq 1 \), the recursion relations (2.1.5) uniquely determine \( a_n^{(m)}, b_n^{(m)}, c_n^{(m)}, m \geq 1 \) and the first few quantities are given by

\[ a_n^{(1)} = -\nu_{n-1}, \quad b_n^{(1)} = \nu_n, \quad c_n^{(1)} = -1, \]
\[ a_n^{(2)} = -\nu_{n-1} - \frac{\nu_n}{\tau} + w_{n-1}, b_n^{(2)} = \nu_n^2 + w_n, c_n^{(2)} = -\nu_{n-1} - \frac{w_n}{\tau}, \ldots \]

Now we define

\[ (\lambda^m \Gamma_n)_+ = \sum_{i=0}^{\infty} \frac{a_n^{(i)}/\tau^i}{c_n^{(i)}/\tau^i} b_n^{(i)}/\tau^i, \quad m \geq 0 \tag{2.1.6} \]

and \( V_n^{(m)} = (\lambda^{2m} \Gamma_n)_+ + \Delta_n^{(m)} \), where

\[ \Delta_n^{(m)} = \begin{pmatrix} -c_n^{(m+1)} & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.1.7} \]

We introduce the following auxiliary spectral problem

\[ \varphi_{n+1} = V_n^{(m)} \varphi_n. \tag{2.1.8} \]

It is easy to conclude that

\[ E(V_n^{(m)})U_n - U_n V_n^{(m)} = \begin{pmatrix} (b_n^{(m+1)} + \nu_n c_n^{(m+1)}) & -\lambda(b_n^{(m+1)} + \nu_n c_n^{(m+1)}) \\ 0 & \frac{w_n}{\tau}(a_n^{(m)} - a_n^{(m)}) \end{pmatrix}. \tag{2.1.9} \]
Then the compatibility conditions of (2.1.1) and (2.1.8) are
\[ U_{nn} = E(V_n^{(m)})U_n - U_nV_n^{(m)}, \quad m \geq 0, \]  
(2.1.10)
which give rise to the following hierarchy of lattice soliton equations
\[
\begin{cases}
  v_{nn} = -\left( v_n a^{(m+1)}_{n+1} + b^{(m+1)}_{n+1} \right), \\
  w_{nn} = W_n(c^{(m)}_{n+1} - d^{(m)}_{n+1}) - \frac{w_n}{v_n} (a^{(m)}_{n+1} - a^{(m)}_n).
\end{cases}
\]
(2.1.11)
are discrete spectral zero curvature representation of (2.1.10), and the discrete spectral problem (2.1.1) and (2.1.8) constitute the Lax pair of (2.1.11), which are Lax integrable nonlinear lattice equations. When \( m = 1 \), (2.1.11) become
\[
\begin{cases}
  v_{nn} = \frac{v_n w_{n+1} - w_n v_{n+1}}{v_{n+1}}, \\
  w_{nn} = \frac{w_n}{v_n} (v_{n-1} - v_n).
\end{cases}
\]
(2.1.12)
Interestingly, when \( v_n = v_n, s_n = \frac{w_n}{v_n} \) (2.1.12) becomes the well-known Lotka–Volterra lattice equation \([32,33]\)
\[ r_{nt} = r_n(s_{n+1} - s_n), \quad s_{nt} = s_n(r_{n+1} - r_n). \]
(2.1.13)
At the same time, when we set \( t_1 \to t \), the time part of the Lax pairs of the system (2.1.8) is given by
\[
V_n^{(1)} = \left( \begin{array}{cc}
-\frac{1}{\lambda} & \frac{w_n}{v_n} \\
1 & \frac{1}{\lambda} + v_{n-1}
\end{array} \right).
\]
(2.1.14)
To establish the Hamiltonian structures for (2.1.11), we define \( R_n = \Gamma_n U_n^{-1} \) and \( \langle M, N \rangle = \text{tr}(MN) \), where \( M \) and \( N \) are the same order square matrices, and we have
\[
\begin{aligned}
\left\langle R_n, \frac{\partial U_n}{\partial \lambda} \right\rangle &= -c_n, \\
\left\langle R_n, \frac{\partial U_n}{\partial v_n} \right\rangle &= \frac{a_{n+1} + a_n}{v_n} \lambda c_n - \frac{\lambda c_n}{v_n}, \quad \left\langle R_n, \frac{\partial U_n}{\partial w_n} \right\rangle &= \frac{1}{w_n} (\lambda c_n - a_n).
\end{aligned}
\]
(2.1.15)
By virtue of the discrete trace identity
\[
\frac{\delta}{\delta U_n} \sum_{m \neq j} \left\langle R_n, \frac{\partial U_n}{\partial \lambda} \right\rangle = \left( \lambda - \epsilon \right) \left( \frac{a_{n+1} + a_n}{v_n} - \frac{\lambda c_n}{v_n} \right),
\]
(2.1.16)
we have
\[
\frac{\delta}{\delta U_n} \sum_{m \neq j} c^{(m)}_n = \lambda - \epsilon \left( \frac{a_{n+1} + a_n}{v_n} - \frac{\lambda c_n}{v_n} \right).
\]
(2.1.17)
The substitution of (2.1.4) into (2.1.16) and comparing the coefficients of \( \lambda^m \) in both sides of the resulting equations yield
\[
\frac{\delta}{\delta U_n} \sum_{m \neq j} c^{(m)}_n = \left( \lambda - \epsilon \right) \left( \frac{a_{n+1} + a_n}{v_n} - \frac{\lambda c_n}{v_n} \right).
\]
(2.1.18)
To fix the constant \( \epsilon \), setting \( m = 0 \) in above equations gives us \( \epsilon = 0 \), which guarantees (2.1.11) another expression reading as
\[
U_{nn} = \left( \begin{array}{c}
\frac{v_n}{w_n} \\
1
\end{array} \right)_n,
\]
(2.1.19)
here
\[
J = \left( \begin{array}{cc}
0 & -v_n (1 - E) w_n \\
-w_n (1 - E) w_n & 0
\end{array} \right), \quad H_n^{(m)} = \sum_{m=2}^{m} c^{(m-1)}_n - a^{(m)}_n, \quad m > 0.
\]
(2.1.19)
Now we show that in the Hamiltonian systems (2.1.18), it is not difficult to verify that \( J \) is a Hamilton operator.

2.2. The conservation laws of (2.1.12)

In this section, we would like to derive the conservation laws of the lattice soliton Eq. (2.1.12). In a simple and direct way from (2.1.1), we have
\[
\begin{aligned}
\phi_{n+1}^1 &= -v_n \phi_n^1 + \lambda v_n \phi_n^2, \\
\phi_{n+1}^2 &= \phi_n^1 + \left( \lambda - \frac{w_n}{v_n} \right) \phi_n^2.
\end{aligned}
\]
(2.2.1)
and
\[
\frac{\phi_{n+1}^1}{\phi_n^1} = -v_n + \lambda v_n \frac{\phi_n^2}{\phi_n^1}, \quad \frac{\phi_{n+1}^2}{\phi_n^2} = -\frac{\phi_n^1}{\phi_n^2} + \lambda - \frac{W_n}{v_n}.
\] (2.2.2)

Assume that
\[
\theta_n = \frac{\phi_n^2}{\phi_n^1}
\] (2.2.3)
from (2.2.2), we have
\[
\theta_{n+1}(-v_n + \lambda v_n \theta_n) = -1 + \left( \lambda - \frac{W_n}{v_n} \right) \theta_n.
\] (2.2.4)

A direct calculation gives
\[
[\ln(-v_n + \lambda v_n \theta_n)]_t = (E - 1) \left( -\frac{1}{2} \lambda + \frac{W_n}{v_n} + \lambda v_{n-1} \theta_n \right).
\] (2.2.5)

Then, expanding \( \theta_n \) in the power series of \( 1/\lambda \)
\[
\theta_n = \sum_{j=1}^{\infty} \frac{\theta_n^{(j)}}{\lambda^j}.
\] (2.2.6)
Substituting (2.2.6) into (2.2.4), we obtain the following recursion relation
\[
\theta_n^{(1)} = 1,
\]
\[
\theta_n^{(j+1)} = \frac{W_n}{v_n} \theta_n^{(j)} - v_n \theta_n^{(j)} + v_n \sum_{k+i+j=1} \theta_n^{(k)} \theta_n^{(i)}+ \sum_{k+i+j=1} \theta_n^{(k)} \theta_n^{(i)}, \quad j \geq 1.
\] (2.2.7)
At this moment, we have
\[
\theta_n^{(2)} = \frac{W_n}{v_n},
\]
\[
\theta_n^{(3)} = W_n + \frac{W_n^2}{v_n}, \ldots
\] (2.2.8)
Substituting (2.2.7) into (2.2.5), we obtain
\[
\left[ \ln(-v_n + \sum_{j=1}^{\infty} \frac{\theta_n^{(j)}}{\lambda^j}) \right]_t = (E - 1) \left( \frac{W_n}{v_n} + v_{n-1} \sum_{j=1}^{\infty} \frac{\theta_n^{(j)}}{\lambda^j} \right).
\] (2.2.9)
Equating the power of \( 1/\lambda \) in (2.2.9), we can get an infinite number of conservation laws for the lattice soliton Eq. (2.1.12).

The first two conservation laws are pointed out as follows
\[
\ln \left[ \frac{W_n}{v_n} \right]_t = (E - 1) \theta_{n-1} \frac{W_n}{v_n},
\]
\[
\ln \left[ \frac{W_n}{v_n} \right]_t = (E - 1) \theta_{n-1} \left( \frac{W_n}{v_n} \right), \ldots
\] (2.2.10)

Similarly, we can get the conservation laws of other lattice equation in the hierarchy (2.1.11).

2.3. The Darboux transformation of (2.1.12)

In this section, a DT for (2.1.12) will be established to get its exact solutions. For this purpose, we introduce the gauge transformation
\[
\tilde{\phi}_n = T_n\phi_n,
\] (2.3.1)
which can transform two spectral problems (2.1.1) and (2.1.8) into
\[
\tilde{\phi}_{n+1} = \tilde{U}_n \tilde{\phi}_n, \quad \tilde{\phi}_n = \tilde{V}_n \tilde{\phi}_n,
\] (2.3.2)
with
\[
\tilde{U}_n = T_{n+1}U_nT_n^{-1}, \quad \tilde{V}_n = (T_m + T_n V_n)T_n^{-1}.
\] (2.3.3)
Set \( y_n = (y_n^0, y_n^1) \), \( z_n = (z_n^0, z_n^1) \) are two basic solutions of (2.1.1) and (2.1.8) by use of \( (y, z) \). We define the transformation matrix

\[
T_n = \begin{pmatrix}
\lambda + t_{11}(n) & \lambda t_{12}(n) \\
-1 & 2\lambda + t_{22}(n)
\end{pmatrix},
\]

(2.3.4)

with

\[
t_{11}(n) = \frac{\lambda_1 \lambda_2 (\sigma_2(n) - \sigma_1(n))}{\lambda_1 \sigma_1(n) - \lambda_2 \sigma_2(n)},
\]

\[
t_{12}(n) = \frac{\lambda_2 - \lambda_1}{\lambda_1 \sigma_1(n) - \lambda_2 \sigma_2(n)},
\]

\[
t_{22}(n) = 1 - 2\lambda_1 \sigma_1(n) / \sigma_1(n) = 1 - 2\lambda_2 \sigma_2(n) / \sigma_2(n)
\]

and

\[
\sigma_i(n) = \frac{y_n^2(\lambda_i)}{y_n^2(\lambda_i) - \gamma_i^2(\lambda_i)} \quad (i = 1, 2),
\]

(2.3.6)

here \( \lambda_i, \gamma_i (i = 1, 2) \) are the proper parameters and the terms in (2.3.5) and (2.3.6) are not zero. From (2.1.1) and (2.3.6), we gives

\[
\sigma_i(n + 1) = \frac{\mu_i(n)}{v_i(n)}, \quad i = 1, 2,
\]

(2.3.7)

here

\[
\mu_i(n) = -1 + \left( \lambda_i - \frac{w_n}{v_n} \right) \sigma_i(n),
\]

\[
v_i(n) = -v_n + \lambda_i v_n \sigma_i(n).
\]

We assume that \( \lambda_1 \) and \( \lambda_2 \) are two roots of \( \det T_n(\lambda) = 0 \). When \( \lambda = \lambda_i, \) \( i = 1, 2 \), it is not difficult to have

\[
\det T_n = 2\lambda^2 + (2t_{11}(n) + t_{12}(n) + t_{22}(n))\lambda + t_{11}(n)t_{22}(n).
\]

(2.3.9)

From Eq. (2.3.4), we have

\[
t_{11}(n) + \lambda t_{12}(n)\sigma_1(n) = 0,
\]

\[-1 + (2\lambda_i + t_{22}(n))\sigma_i(n) = 0
\]

(2.3.10)

from (2.3.5) and (2.3.7), we have

\[
t_{11}(n + 1) = \frac{\lambda_1 \lambda_2 (\mu_2(n) v_1(n) - \mu_1(n) v_2(n))}{\lambda_1 \mu_1(n) v_2(n) - \lambda_2 \mu_2(n) v_1(n)},
\]

\[
t_{12}(n + 1) = \frac{(\lambda_2 - \lambda_1) \mu_1(n) v_2(n) - \lambda_2 \mu_2(n) v_1(n)}{\lambda_1 \mu_1(n) v_2(n) - \lambda_2 \mu_2(n) v_1(n)},
\]

\[
t_{22}(n + 1) = \frac{v_1(n) - 2\lambda_1 \mu_1(n)}{\mu_1(n)} = \frac{v_2(n) - 2\lambda_2 \mu_2(n)}{\mu_2(n)}.
\]

(2.3.11)

Through direct but tedious calculations, from (2.3.5) and (2.3.11), we obtain the relations reading as

\[
2v_n t_{11}(n + 1) - (v_n + t_{12}(n + 1)) t_{22}(n) = 0,
\]

\[
v_n - t_{22}(n + 1) + t_{11}(n) = \frac{\mu_n t_{22}(n + 1)}{v_n t_{22}(n)}.
\]

Hence, we obtain the following assertions from all of the above statements:

**Proposition 1.** The matrix \( \bar{U}_n = T_{n+1} U_n T_n^{-1} \) has the same form as matrix \( U_n \) that is

\[
\bar{U}_n = \begin{pmatrix}
-\bar{v}_n & \lambda \bar{v}_n \\
-1 & \lambda - \frac{\bar{w}_n}{v_n}
\end{pmatrix}
\]

(2.3.12)

in which the transformation formulae between old and new potentials are defined by

\[
\bar{v}_n = \frac{v_n + t_{12}(n + 1)}{2}, \quad \bar{w}_n = \frac{(v_n + t_{12}(n + 1))w_n t_{22}(n + 1)}{2v_n t_{22}(n)}.
\]

(2.3.13)
The transformation \((\phi_n; \nu_n, w_n \rightarrow \phi_n; \nu_n, w_n)\) is called a Darboux transformation (DT) of the spectral problem (2.1.1), the transformation (2.3.13) is called a Bäcklund transformation (BT).

**Proof.** Let \(T_n^{-1} = T_n/det T_n\), we can suppose
\[
T_{n+1}U_nT_n = (det T_n)P_n,
\]
with
\[
P_n = \begin{pmatrix}
  p_{11}^0 & p_{12}^0 + p_{12}^0
p_{21}^0 & p_{22}^0 + p_{22}^0
\end{pmatrix}.
\]
That is
\[
T_{n+1}U_n = P_n T_n,
\]
where \(p_{ij}^0\) \((i, j = 1, 2; l = 0, 1)\) are undetermined functions independent of \(\lambda\). By comparing the coefficients of \(\lambda^l\) \((i = 0, 1, 2)\) in both sides of (2.3.15), we obtain
\[
p_{11}^0 = -\frac{\nu_n + t_{12}(n + 1)}{2} = -\tilde{v}_n, \quad p_{12}^0 = 0, \quad p_{12}^0 = \frac{\nu_n + t_{12}(n + 1)}{2} = \tilde{v}_n, \quad p_{21}^0 = -1, \quad p_{22}^0 = -\frac{w_n t_{22}(n + 1)}{\nu_n t_{22}(n)} = \frac{\tilde{w}_n}{\nu_n}, \quad p_{22}^0 = 1.
\]
Thus we complete the proof. □

**Proposition 2.** Under the gauge transformation (2.3.13), the matrix \(\tilde{V}_n\) defined by (2.1.14) has the same form as \(V_n\), that is
\[
\tilde{V}_n = \begin{pmatrix}
  \frac{-1}{2} + \frac{w_n}{\nu_n} & \tilde{v}_n - \frac{1}{2} \lambda & \tilde{v}_n - \frac{1}{2} \lambda \\
  -1 & 1 & 1
\end{pmatrix}.
\]

**Proof.** Let \(T_n^{-1} = T_n/det T_n\). Taking (2.3.5) and (2.3.6) into account yields
\[
\begin{align*}
  & t_{11}(n) + \lambda \sigma_1(n) t_{12}(n) + \lambda t_{12}(n) \sigma_1(n) = 0, \\
  & 2 \lambda \sigma_2(n) + \sigma_2(n) t_{22}(n) + t_{22}(n) \sigma_2(n) = 0, \\
  & \sigma_1(n) = -1 + \left( \lambda - \nu_{n-1} - \frac{w_n}{\nu_n} \right) \sigma_1(n) - \lambda \nu_{n-1} \sigma_1(n)^2.
\end{align*}
\]
Based on the above results, we can suppose
\[
(T_n + U_n \tilde{V}_n)T_n = (det T_n)G_n,
\]
with
\[
G_n = \begin{pmatrix}
  \tilde{g}_{11}^1 + g_{11}^0 & \tilde{g}_{12}^1 + g_{12}^0 \\
  \tilde{g}_{21}^0 + g_{21}^0 & \tilde{g}_{22}^0 + g_{22}^0
\end{pmatrix}.
\]
That is
\[
T_{n+1}U_n \tilde{V}_n = G_n T_n,
\]
where \(g_{ij}^0\) \((i, j = 1, 2; l = 0, 1)\) are undetermined functions independent of \(\lambda\). By comparing the coefficients of \(\lambda^l\) \((i = 0, 1, 2)\) in both sides of (2.3.15), we have
\[
g_{11}^1 = -\frac{1}{2}, \quad g_{11}^0 = \frac{\tilde{w}_n}{\nu_n}, \quad g_{12}^0 = 0, \quad g_{12}^1 = \frac{\nu_{n-1} + b_n}{2} = \tilde{v}_{n-1}, \quad g_{21}^0 = -1, \quad g_{22}^1 = -\frac{1}{2}, \quad g_{22}^0 = \frac{\nu_{n-1} + b_n}{2} = \tilde{v}_{n-1}.
\]
The proof is thus completed. □

From the above propositions we come to the following Proposition.

**Proposition 4.** Every solution \(r_n, s_n\) of (2.1.12) is mapped into a new solution \(\tilde{r}_n, \tilde{s}_n\) under the BT (2.3.13). Homoplastically, noticing a fact that other equations are based on the same eigenvalue problem (2.1.1), we can construct a homeotypic Darboux matrix for these equations.

**Proposition 5.** The transformation \((\phi_n; \nu_n, w_n \rightarrow \phi_n; \nu_n, w_n)\) is the Darboux transformation of the spectral problem (2.1.1), under the transformation (2.3.13), the solution \(\nu_n, w_n\) are mapped into the new solution \(\tilde{v}_n, \tilde{w}_n\).

From (2.1.12), substituting the trivial solution \(\nu_n = w_n = 1\) into (2.1.1) and (2.1.8) leads to
\[
\begin{align*}
  & v_n = \frac{1}{2}, \quad w_n = \frac{1}{2}, \quad t_{11}(n) = \frac{1}{2}, \quad t_{12}(n) = -\frac{1}{2}, \quad \sigma_1(n) = -1, \quad \sigma_2(n) = 0,
  & t_{22}(n) = 0, \quad \nu_{n-1} = 1, \quad \nu_n = 1, \quad \tilde{v}_n = 1, \quad \tilde{w}_n = 1.
\end{align*}
\]
\[ \phi_{n+1} = \begin{pmatrix} -1 & \lambda \\ -1 & \lambda - 1 \end{pmatrix} \phi_n, \quad \phi_n = \begin{pmatrix} -\frac{1}{2} \lambda + 1 & \lambda \\ -1 & \frac{1}{2} \lambda + 1 \end{pmatrix} \phi_n, \]

(3.20)

its basic solutions can be chosen as

\[ y_n = \tau_1 \exp \left( \frac{2 + \sqrt{\lambda^2 - 4 \lambda} t}{2} \right) \left( \frac{\lambda - \sqrt{\lambda^2 - 4 \lambda}}{2} \right). \]

\[ z_n = \tau_2 \exp \left( \frac{2 - \sqrt{\lambda^2 - 4 \lambda} t}{2} \right) \left( \frac{\lambda + \sqrt{\lambda^2 - 4 \lambda}}{2} \right). \]

(3.21)

in which

\[ \tau_1 = \frac{\lambda_i - 2 + \sqrt{\lambda_i^2 - 4 \lambda_i}}{2}, \quad \tau_2 = \frac{\lambda_i - 2 - \sqrt{\lambda_i^2 - 4 \lambda_i}}{2}. \]

Substituting (3.21) into (3.6), we obtain

\[ \sigma_i(n) = \frac{\omega_i e^{i t} - \gamma_i}{\omega_i e^{i t} \lambda_i \mu_i(n) - \gamma_i \lambda_i \mu_i(n)} \quad (i = 1, 2), \]

where

\[ \omega_i = \frac{\lambda_i^2 - 4 \lambda_i}{2} + (\lambda_i - 2) \sqrt{\lambda_i^2 - 4 \lambda_i}, \quad i = 1, 2, \]

\[ \gamma_i = \sqrt{\lambda_i^2 - 4 \lambda_i}, \quad i = 1, 2, \]

therefore, from (3.1) and (3.13), the new solutions are given as follows

\[ \hat{v}_n = \frac{1 + t_{12}(n + 1)}{2} + \frac{1}{2} \left( \frac{\lambda_i \mu_i(n)}{\lambda_i \mu_i(n) - \lambda_i \mu_i(n)} \right), \]

\[ \tilde{w}_n = \frac{(1 + t_{12}(n + 1)) v_{22}(n + 1)}{2 t_{22}(n)} = \left( 1 + \frac{\lambda_i \mu_i(n)}{\lambda_i \mu_i(n) - \lambda_i \mu_i(n)} \right) \left( \frac{v_i(n)}{\mu_i(n)} \right) \left( \frac{1 - 2 \lambda_i \sigma_i(n)}{\sigma_i(n)} \right). \]

(3.22)

Starting from the explicit solitons (3.22), we apply the Darboux transformation (3.13) once again, then new solitons of (2.12) are obtained. This process can be done continually. Therefore, we can obtain many explicit solitons for the lattice Eq. (2.11).

3. The second integrable lattice hierarchy and its Darboux transformation

3.1. Another new completely integrable Hamiltonian system in the Liouville sense

Let us recall the following discrete matrix isospectral problem [34]

\[ E \phi_n = U_n(u_n, \lambda) \phi_n, \quad U_n(u_n, \lambda) = \begin{pmatrix} \lambda & \lambda n \\ r & r + s + 1 \end{pmatrix}, \quad \phi_n = \begin{pmatrix} \phi_n^1 \\ \phi_n^2 \end{pmatrix}, \quad U_n = \begin{pmatrix} r & r + s + 1 \\ s & s + 1 \end{pmatrix}. \]

(3.1.1)

Noting

\[ \Gamma_n = \begin{pmatrix} a_n & \lambda b_n \\ c_n & -a_n \end{pmatrix}. \]

Solving the stationary discrete zero curvature equation for \( \Gamma_n \)

\[ (E \Gamma_n) U_n - U_n \Gamma_n = 0 \]

(3.1.2)

gives rise to

\[ \begin{cases} \lambda (a_{n+1} - a_n) + \lambda r_n b_{n+1} - \lambda (1 + \frac{b_n}{r_n}) c_n = 0, \\
\lambda ((1 + \frac{b_n}{r_n}) (a_{n+1} + a_n) + \lambda (r_n + s_n + 1)) b_{n+1} - \lambda^2 b_n = 0, \\
(\lambda + \frac{b_n}{r_n}) c_{n+1} - \lambda r_n b_n - (r_n + s_n + 1) (a_{n+1} - a_n) = 0. \end{cases} \]

(3.1.3)
Let
\[ a_n = \sum_{m=0}^{\infty} a_n^{(m)} x^{-m}, \quad b_n = \sum_{m=0}^{\infty} b_n^{(m)} x^{-m}, \quad c_n = \sum_{m=0}^{\infty} c_n^{(m)} x^{-m}. \] (3.14)

From (3.13), we have the following initial value
\[ b_n^{(0)} = c_n^{(0)} = 0, \quad a_n^{(0)} - a_n^{(0)} = \left(1 + \frac{s_n}{r_n}\right) c_n^{(0)} - r_n b_n^{(0)} \]
and recurrence relation
\[
\begin{cases}
  a_n^{(m)} - a_n^{(m)} + r_n b_n^{(m+1)} - (1 + \frac{s_n}{r_n}) c_n^{(m+1)} = 0, \\
  \left(1 + \frac{s_n}{r_n}\right) a_n^{(m+1)} + (r_n + s_n + 1)b_n^{(m+1)} = b_n^{(m+1)} + (r_n + s_n + 1)c_n^{(m+1)}, \\
  -r_n (a_n^{(m+1)} + a_n^{(m)}) - (r_n + s_n + 1)c_n^{(m+1)} = -c_n^{(m+1)}, \\
  -(r_n + s_n + 1) (a_n^{(m+1)} - a_n^{(m)}) = r_n b_n^{(m+1)} - \left(1 + \frac{s_n}{r_n}\right) c_n^{(m+1)}.
\end{cases}
\] (3.15)

Take \( a_0^{(0)} = \begin{cases} 1 & \text{if } j = 1, \quad b_0^{(0)} = c_0^{(0)} = 0, \quad a_0^{(j)} = 0, \quad b_0^{(j)} = c_0^{(j)} = 0, \quad j \geq 1 \end{cases} \). The first coefficients are given as follows:
\[ a_n^{(1)} = -\frac{r_n - (r_n + s_n)}{r_n}, \quad b_n^{(1)} = 1 + \frac{s_n}{r_n}, \quad c_n^{(1)} = r_n. \]
\[ a_n^{(2)} = s_n^2 + 2r_n s_n^2 + r_n s_n^2 - r_n s_n^2 - r_n s_n - s_n - 1. \]
\[ b_n^{(2)} = 1 - r_n + \frac{r_n s_n^2}{r_n} + \frac{s_n}{r_n}. \]
\[ c_n^{(2)} = r_n - r_n^2 - \frac{r_n s_n}{r_n}, \ldots \]

By this way, the recursion relation (3.15) determines uniquely \( a_j, b_j, c_j, j \geq 1 \). Denote
\[ \phi_n = V_n^{(m)} \phi_n = \sum_{j=0}^{m} \left( a_n^{(j)} x^{-m-j} b_n^{(j)} x^{-m-j} - c_n^{(j)} x^{-m-j} \right) \phi_n, \quad m \geq 0. \]

Direct calculation reads
\[
(\mathbb{E}V_n^{(m)}) U_n - U_n V_n^{(m)} = \begin{pmatrix} 0 & \beta b_n^{(m+1)} \\ -c_n^{(m+1)} & -r_n b_n^{(m+1)} - \left(1 + \frac{s_n}{r_n}\right) c_n^{(m+1)} \end{pmatrix}.
\] (3.17)

Then the discrete zero curvature equation admits the following positive hierarchy
\[ u_n^{(m)} = \left( \begin{array}{c} r_n \\ s_n \end{array} \right) = \left( \begin{array}{c} c_n^{(m+1)} \\ -r_n b_n^{(m+1)} - \frac{s_n}{r_n} c_n^{(m+1)} \end{array} \right). \] (3.18)

When \( m = 1 \), (3.18) reduces to
\[
\begin{cases}
  r_n = r_n^2 - r_n - 1 + \frac{r_n s_n}{r_n}, \\
  s_n = (r_n + r_n s_n) \left(1 + \frac{s_n}{r_n}\right) + r_n s_n \left(s_n - r_n r_n - 1\right).
\end{cases}
\] (3.19)

Accordingly, When \( m = 1 \), the t-part of Lax pairs for this equation is as follows
\[ \phi_n = V_n^{(1)} \phi_n = \begin{pmatrix} \frac{1}{2} \lambda - \frac{r_n - (r_n + s_n)}{r_n} \\ \frac{1}{2} \lambda + \frac{r_n + s_n + 1}{r_n} \end{pmatrix} \phi_n. \] (3.10)

From the recursion relations (3.15), we known that the hereditary recursion operator \( \Phi \) reads:
\[ \Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}, \]
where
\[ \Phi_{11} = r_n (1 + E) (1 - E)^{-1} \left( r_n E \frac{s_n}{r_n} - \frac{s_n}{r_n} E^{-1} - E^{-1} \right) + (r_n + s_n + 1) E^{-1}. \]
\( \Phi_{12} = -r_n(1 + E)(1 - E)^{-1}r_n E \frac{1}{r_n} \),

\( \Phi_{21} = r_n(1 + E)(1 - E)^{-1} \left( -r_n E \frac{s_n}{r_n^2} + \frac{s_n E^{-1}}{r_n} + E^{-1} \right) - (r_n + s_n + 1)r_n \frac{s_n}{r_n^2} + (r_n + s_n + 1) \frac{s_n E^{-1}}{r_n} \).

\( \Phi_{22} = (r_n + s_n + 1)r_n \frac{1}{r_n} + r_n(1 + E)(1 - E)^{-1}r_n \frac{1}{r_n} \).

To establish the Hamiltonian structure for system (3.1.8), we define \( V_n = \Gamma_n U_n^{-1} \) and \( \langle A, B \rangle = \text{Tr}(AB) \), where \( A \) and \( B \) are the same order square matrices. We have

\[
\left\langle V_n, \frac{\partial U_n}{\partial \lambda} \right\rangle = \left( r_n + s_n + 1 \right) \frac{\partial a_n}{\lambda} - r_n b_n + \left( r_n + s_n + 1 \right) c_n \left( 1 + \frac{s_n}{r_n} \right) + \frac{r_n(1 + \frac{s_n}{r_n}) a_n}{\lambda},
\]

\[
\left\langle V_n, \frac{\partial U_n}{\partial r_n} \right\rangle = b_{n+1} \frac{s_n c_n}{r_n^2}, \quad \left\langle V_n, \frac{\partial U_n}{\partial s_n} \right\rangle = c_n \frac{1}{r_n}.
\]

From the discrete trace identity

\[
\frac{\delta}{\delta u_n} \sum_{k \in \mathbb{Z}} \left( V_n, \frac{\partial U_n}{\partial \lambda} \right) = \left( \lambda^{-\varepsilon} \left( \frac{\partial}{\partial \lambda} \right)^\varepsilon \right) \left( V_n, \frac{\partial U_n}{\partial u_n} \right), \quad i = 1, 2. \tag{3.1.11}
\]

We have

\[
\left( \frac{\delta}{\delta u_n} \right) \sum_{k \in \mathbb{Z}} \frac{a_{n+1}}{\lambda}(k) = \lambda^{-\varepsilon} \left( \frac{\partial}{\partial \lambda} \right)^\varepsilon \left( b_{n+1} - \frac{s_n c_n}{r_n^2} \right).
\]

Comparison of the coefficient of \( \lambda^{-m-1} \) yields

\[
\left( \frac{\delta}{\delta u_n} \right) \sum_{k \in \mathbb{Z}} \frac{a_{n+1}}{\lambda}(k) = (\varepsilon - m) \left( b_{n+1}^{(m)} - \frac{s_n^{(m)} c_n^{(m)}}{r_n^2} \right).
\]

Taking \( m = 0 \) gives \( \varepsilon = 0 \). Thus,

\[
u_{m+1}^n = \left( \begin{array}{c} r_n \\ s_n \end{array} \right)_n = J \frac{\delta H_{n+1}^{(m+1)}}{\delta u_n} = J \left( b_{n+1}^{(m+1)} - \frac{s_n^{(m+1)} c_n^{(m+1)}}{r_n^2} \right), \quad m \geq 0. \tag{3.1.12}
\]

where

\[
J = \left( \begin{array}{cc} 0 & -E r_n \\ r_n E^{-1} & r_n E^{-1} \frac{1 - s_n}{r_n} \end{array} \right)
\]

and

\[
H_{n+1}^{(m+1)} = \sum_{k \in \mathbb{Z}} \frac{a_{n+1}}{m + 1}(k), \quad m \geq 0, \quad H_{n}^{(0)} = \sum_{k \in \mathbb{Z}} \ln(r_n + s_n + 1). \tag{3.1.13}
\]

So the system (3.1.8) can be written as

\[
u_{m+1}^n = \left( \begin{array}{c} r_n \\ s_n \end{array} \right)_n = J \Phi \left( b_{n+1}^{(m+1)} - \frac{s_n^{(m+1)} c_n^{(m+1)}}{r_n^2} \right), \quad m \geq 0. \tag{3.1.14}
\]

Let

\[
K = \Phi = \left( \begin{array}{cc} -r_n(1 + E)(1 - E)^{-1} r_n & r_n(1 + E)(1 - E)^{-1} r_n - (1 + s_n + r_n) r_n \\ r_n(1 + E)(1 - E)^{-1} r_n + (1 + s_n + r_n) r_n & -r_n(1 + E)(1 - E)^{-1} r_n \end{array} \right).
\]

It is easy to verify that \( K \) is a skew-symmetric operator. In fact, it is shown that the hierarchy possesses a bi-Hamiltonian structure and a hereditary recursion operator, which implies that there exist infinitely many common commuting symmetries and infinitely many conserved functionals.
3.2. The Darboux transformation of (3.1.9)

In what follows, we shall construct a DT of equation (3.1.9). We introduce the gauge transformation
\[ \hat{\phi}_n = T_n \phi_n, \]  
which can transform the Lax pairs (3.1.1) and (3.1.6) into
\[ \hat{\phi}_{n+1} = \hat{U}_n \hat{\phi}_n, \quad \hat{\phi}_n = \hat{V}_n \hat{\phi}_n, \]  
with
\[ \hat{U}_n = T_{n+1} U_n T_n^{-1}, \quad \hat{V}_n = (T_m + T_n V_n) T_n^{-1}. \]  
Set \( y_n = (y_n^1, y_n^2)^T \), \( z_n = (z_n^1, z_n^2)^T \) are two basic solutions of (3.1.1) and (3.1.6) by use of \( (y_n, z_n) \). We define the transformation matrix
\[ T_n = \begin{pmatrix} \lambda + t_{11}(n) & \lambda t_{12}(n) \\ t_{21}(n) & \lambda + t_{22}(n) \end{pmatrix}, \]  
with
\[ t_{11}(n) = \frac{\lambda_1 \lambda_2 (\sigma_1(n) - \sigma_2(n))}{\lambda_2 \sigma_2(n) - \lambda_1 \sigma_1(n)}, \quad t_{12}(n) = \frac{\lambda_1 - \lambda_2}{\lambda_2 \sigma_2(n) - \lambda_1 \sigma_1(n)}, \]  
\[ t_{21}(n) = \frac{\sigma_1(n) \sigma_2(n) (\lambda_1 - \lambda_2)}{\sigma_1(n) - \sigma_2(n)}, \quad t_{22}(n) = \frac{\lambda_2 \sigma_2(n) - \lambda_1 \sigma_1(n)}{\sigma_1(n) - \sigma_2(n)} \]  
and
\[ \sigma_i(n) = \frac{y_n^2(\lambda_i) - \gamma_i^2 z_n^2(\lambda_i)}{y_n^1(\lambda_i) - \gamma_i z_n^1(\lambda_i)}, \quad (i = 1, 2), \]  
here \( \lambda_i, \gamma_i \ (i = 1, 2) \) are the proper parameters and the terms in (3.2.5) and (3.2.6) are not zero. From (3.2.2) and (3.2.5), we gives
\[ \sigma_i(n + 1) = \frac{\mu_i(n)}{v_i(n)}, \quad i = 1, 2, \]  
here
\[ \mu_i(n) = r_n + (r_n + s_n + 1) \sigma_i(n), \]  
\[ v_i(n) = \lambda + \lambda (1 + \frac{s_n}{r_n}) \sigma_i(n). \]  
We assume that \( \lambda_1 \) and \( \lambda_2 \) are two roots of \( detT_n(\lambda) = 0 \). When \( \lambda = \lambda_i, \ i = 1, 2 \), it is not difficult to have
\[ detT_n = (\lambda - \lambda_1)(\lambda - \lambda_2). \]  
From Eq. (3.2.5), we have
\[ \dot{\lambda}_1 + t_{11}(n) + \dot{\lambda}_2 t_{12}(n) \sigma_i(n) = 0, \]  
\[ t_{21}(n) + \lambda \sigma_i(n) + t_{22}(n) \sigma_i(n) = 0 \]  
from (3.2.7) and (3.2.8), we have
\[ t_{11}(n + 1) = \frac{\dot{\lambda}_1 \dot{\lambda}_2 (\mu_1(n) v_2(n) - \mu_2(n) v_1(n))}{\dot{\lambda}_2 \mu_2(n) v_1(n) - \dot{\lambda}_1 \mu_1(n) v_2(n)}, \]  
\[ t_{12}(n + 1) = \frac{\dot{\lambda}_1 \mu_1(n) v_2(n) - \dot{\lambda}_2 \mu_2(n) v_2(n)}{\dot{\lambda}_2 \mu_2(n) v_1(n) - \dot{\lambda}_1 \mu_1(n) v_1(n)}, \]  
\[ t_{21}(n + 1) = \frac{\mu_1(n) \mu_2(n) (\lambda_2 - \lambda_1)}{\mu_2(n) v_1(n) - \mu_1(n) v_2(n)}, \]  
\[ t_{22}(n + 1) = \frac{\dot{\lambda}_1 \mu_1(n) v_2(n) - \dot{\lambda}_2 \mu_2(n) v_2(n)}{\mu_2(n) v_1(n) - \mu_1(n) v_2(n)}. \]  
Through direct but tedious calculations, from (3.2.6) and (3.2.11), we obtain the relations reading as
\[ t_{11}(n + 1) - t_{11}(n) + r_n t_{12}(n + 1) = \left( 1 + \frac{s_n}{r_n} - t_{12}(n) \right) t_{21}(n), \]  
\[ r_n t_{22}(n) t_{22}(n + 1) - (r_n + s_n + 1) t_{21}(n) t_{22}(n + 1) = (r_n + t_{21}(n + 1)) t_{11}(n). \]
Hence, we obtain the following assertions from all of the above statements:

**Proposition 6.** The matrix \( \hat{U}_n = T_{n+1} U_n T_n^{-1} \) has the same form as matrix \( U_n \) that is

\[
\hat{U}_n = \begin{pmatrix}
\frac{1}{T_n} \lambda & \frac{1}{T_n} \lambda + \frac{1}{T_n} \frac{s_n}{r_n} \\
\frac{1}{T_n} \tilde{r}_n + \tilde{s}_n + 1
\end{pmatrix},
\]

(3.2.12)
in which the transformation formulæ between old and new potentials are defined by

\[
\hat{r}_n = r_n + t_{21}(n + 1),
\]

\[
\hat{s}_n = (r_n + t_{21}(n + 1)) \left( \frac{s_n}{r_n} - t_{12}(n) \right).
\]

(3.2.13)

The transformation \((\hat{\phi}_n; \hat{r}_n, \hat{s}_n) \rightarrow (\phi_n; r_n, s_n)\) is called a Darboux transformation (DT) of the spectral problem (3.1.1), the transformation (3.2.13) is called a Bäcklund transformation (BT).

**Proof.** Let \( T_n^{-1} = T_n/detT_n \) and

\[
T_{n+1} U_n T_n^* = \begin{pmatrix}
\lambda_1(n) & \lambda_2(n) \\
\lambda_2(n) & \lambda_1(n)
\end{pmatrix}.
\]

It is easy to see that \( \lambda_1(n), \lambda_2(n), \lambda_2(\hat{\lambda}, n) \) and \( \lambda_2(\hat{\lambda}, n) \) are cubic-polynomials in \( \lambda \). Also, we can readily verify that \( \lambda(l; \lambda_1(n), \lambda_2(n)) = 0 \) (i, k, l = 1, 2). Based on the above results, we can suppose

\[
T_{n+1} U_n T_n^* = (detT_n)P_n,
\]

(3.2.14)
with

\[
P_n = \begin{pmatrix}
p_1(n) & p_1(n) \\
p_2(n) & p_2(n)
\end{pmatrix}.
\]

That is

\[
T_{n+1} U_n = P_n T_n,
\]

(3.2.15)
where \( p_i(n), (i, j = 1, 2; l = 0, 1) \) are undetermined functions independent of \( \lambda \). By comparing the coefficients of \( \lambda^l \) (\( i = 0, 1, 2 \)) in both sides of (3.2.15) and using (DT) (3.2.13), we obtain

\[
p_1(n) = 1, \quad p_2(n) = 1 + \frac{s_n}{r_n} - t_{12}(n) = 1 + \frac{s_n}{r_n}, \quad p_0(n) = r_n + t_{21}(n + 1) = \hat{r}_n,
\]

\[
p_2(n) = \left( 1 + \frac{s_n}{r_n} \right) t_{21}(n + 1) - (r_n + t_{21}(n + 1)) t_{12}(n) + r_n s_n + 1 = \hat{r}_n + \hat{s}_n + 1,
\]

\[
p_0(n) = 0, \quad p_0(n) = 0.
\]

Thus we complete the proof. □

**Proposition 7.** Under the gauge transformation (3.2.13), \( \hat{V}_n \) has the following form

**Proof.** Let \( T_n^{-1} = T_n/detT_n \) and

\[
\hat{V}_n = \begin{pmatrix}
\frac{1}{T_n} \lambda & \frac{1}{T_n} \lambda + \frac{1}{T_n} \frac{s_n}{r_n} \\
\frac{1}{T_n} \tilde{r}_n + \tilde{s}_n + 1
\end{pmatrix}.
\]

(3.2.16)

\[
(T_n + U_n V_n) T_n^* = \begin{pmatrix}
g_{11}(\lambda, n) & g_{12}(\lambda, n) \\
g_{21}(\lambda, n) & g_{22}(\lambda, n)
\end{pmatrix},
\]

it is easy to see that \( g_{11}(\lambda, n), g_{12}(\lambda, n), g_{21}(\lambda, n) \) and \( g_{22}(\lambda, n) \) are cubic-polynomials in \( \lambda \). Also, we can readily verify that \( g_{id}(\lambda, n) = 0 \) (i, k, l = 1, 2), respectively. Taking (3.2.6) and (3.2.10) into account yields

\[
t_{11}(n) + \lambda_1(\sigma_1(n)) t_{12}(n) + \lambda_1(\sigma_1(n)) t_{12}(n) = 0,
\]

\[
t_{21}(n) + \sigma_1(n) t_{22}(n) + \lambda_1(\sigma_1(n)) t_{22}(n) + \lambda_1(\sigma_1(n)) t_{12}(n) = 0,
\]

\[
\sigma_1(n) = r_{n-1} - \lambda_1(\sigma_1(n)) + \frac{2r_{n-1}(r_n + s_n)}{r_n} \sigma_1(n) - \lambda_1 \left( 1 + \frac{s_n}{r_n} \right) \sigma_1(n)^2.
\]

(3.2.17)
Based on the above results, we can suppose
\[(T_m + U_n V_n)T_n = (\det T_n)G_n, \tag{3.2.18}\]
with
\[G_n = \begin{pmatrix}
\hat{x}g^1_{11}(n) + g^0_{11}(n) & \hat{x}g^1_{12}(n) + g^0_{12}(n) \\
g^0_{21}(n) & \hat{x}g^1_{22}(n) + g^0_{22}(n)
\end{pmatrix}.
\]
That is
\[T_m + U_n V_n = G_n T_n, \tag{3.2.19}\]
where \[r_{ij}^l \quad (i, j = 1, 2; \quad l = 0, 1)\] are undetermined functions independent of \( \lambda \). By comparing the coefficients of \( \lambda^l \quad (i = 0, 1, 2) \) in both sides of (3.2.19) and using (DT) (3.2.13), we have
\[
g^1_{11}(n) = \frac{1}{2}, \quad g^1_{12}(n) = 1 + \frac{s_n}{r_n} - \sigma_{12}(n) = 1 + \frac{s_n}{r_n} - \frac{1}{2}, \quad g^1_{22}(n) = -\frac{1}{2},
\]
\[
g^0_{21}(n) = r_{n-1} + t_{21}(n) = \sigma_{n-1}, \quad g^0_{11}(n) = -(r_{n-1} + t_{21}(n)) \left(1 + \frac{s_n}{r_n} - t_{21}(n)\right) = -\frac{r_{n-1}(\sigma_n + s_n)}{r_n},
\]
\[
g^0_{12}(n) = 0, \quad g^0_{22}(n) = (r_{n-1} + t_{21}(n)) \left(1 + \frac{s_n}{r_n} - t_{21}(n)\right) = \frac{r_{n-1}(\sigma_n + s_n)}{r_n}.
\]
The proof is thus completed. \( \square \)

From the above propositions we come to the following proposition.

**Proposition 8.** Every solution \( r_n, s_n \) of (3.1.9) is mapped into a new solution \( \tilde{r}_n, \tilde{s}_n \) under the BT (3.2.13). Homoplastically, noticing a fact that other equations are based on the same eigenvalue problem (3.1.1), we can construct a homeotypic Darboux matrix for these equations.

**Proposition 9.** The transformation \((\tilde{\phi}_n; \tilde{r}_n, \tilde{s}_n \rightarrow \phi_n; r_n, s_n)\) is the Darboux transformation of the spectral problem (3.1.1) under the Bäcklund transformation
\[
\tilde{r}_n = r_n + t_{21}(n + 1),
\]
\[
\tilde{s}_n = (r_n + t_{21}(n + 1)) \left(\frac{s_n}{r_n} - t_{12}(n)\right).
\]
the solution \( r_n, s_n \) are mapped into the new solution \( \tilde{r}_n, \tilde{s}_n \).

From (3.2.2), substituting the trivial solution \( r_n = 1, \quad s_n = 0 \) into (3.1.1) and (3.1.10) leads to
\[
\phi_{n+1} = \frac{\lambda}{1} \phi_n, \quad \phi_n = \begin{pmatrix}
\frac{\lambda}{2} - 1 & \frac{\lambda}{2} - 1 \\
1 & -\frac{\lambda}{2} + 1
\end{pmatrix} \phi_n, \tag{3.2.20}
\]
its basic solutions can be chosen as
\[
y_n = \beta_1^n \exp\left(\frac{\sqrt{\lambda^2 + 4} + 4}{2} t\right) \left(\frac{\lambda}{2 + \sqrt{\lambda^2 + 4}}\right),
\]
\[
z_n = \beta_2^n \exp\left(-\frac{\sqrt{\lambda^2 + 4} + 4}{2} t\right) \left(\frac{\lambda}{2 - \sqrt{\lambda^2 + 4}}\right), \tag{3.2.21}
\]
in which
\[
\beta_1 = \frac{2 + \lambda + \sqrt{\lambda^2 + 4}}{2}, \quad \beta_2 = \frac{2 + \lambda - \sqrt{\lambda^2 + 4}}{2}.
\]
Substituting (3.2.21) into (3.2.6) we obtain
\[
\sigma_i(n) = \frac{\eta_i e^{i\lambda} \left(2 - \lambda_i + \sqrt{\lambda^2_i + 4}\right) - \gamma_i \left(2 - \lambda_i + \sqrt{\lambda^2 + 4}\right)}{2\eta_i e^{i\lambda} - 2\gamma_i}, \quad (i = 1, 2), \tag{3.2.22}
\]
where
\[
\eta_i = \frac{4 + 2\lambda + \lambda^2 + (2 + \lambda)\sqrt{\lambda^2_i + 4}}{2 \lambda_i}, \quad (i = 1, 2),
\]
\[
\gamma_i = \sqrt{\lambda^2_i + 4}, \quad (i = 1, 2),
\]
\[
\xi_i = \sqrt{\lambda^2_i + 4}, \quad (i = 1, 2),
\]
therefore, from (3.2.13) and (3.2.21), the new solutions are given as follows:

\[
\begin{align*}
\bar{r}_n &= 1 + \frac{(\lambda_1 - \lambda_2)(1 + 2\sigma_1(n))(1 + 2\sigma_2(n))}{\lambda_2(1 + 2\sigma_1(n))(1 + 2\sigma_2(n)) - \lambda_1(1 + 2\sigma_2(n))(1 + \sigma_1(n))}, \\
\bar{s}_n &= -\left(1 + \frac{(\lambda_1 - \lambda_2)(1 + 2\sigma_1(n))(1 + 2\sigma_2(n))}{\lambda_2(1 + 2\sigma_1(n))(1 + 2\sigma_2(n)) - \lambda_1(1 + 2\sigma_2(n))(1 + \sigma_1(n))}\right),
\end{align*}
\]  

(3.2.23)

Starting from the explicit solitons (3.2.22), we apply the Darboux transformation (3.2.13) once again, then new solitons of (3.1.9) are obtained. This process can be done continually. Therefore, we can obtain many explicit solitons for the lattice Eq. (3.1.8).

4. Conclusions and remarks

In [32,33], the Lotka–Volterra equation are discussed in some way. Here, we concentrating on the generalized Lotka–Volterra equation and its Darboux transformation based on Darboux matrices (2.3.4). In [34], two hierarchies of integrable positive and negative lattice equations in connection with the discrete isospectral problem (3.1.1) are derived. In this paper, the Darboux transformations based on Darboux matrices (3.2.4) for the positive integrable hierarchies are established with the help of gauge transformations of lax pairs. Maybe, further in-depth studies are being conducted with the two integrable lattice hierarchies mentioned in this paper.

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