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Woodin on the Continuum Problem: an overview and some objections

Luca Bellotti
Department of Philosophy, University of Pisa (Italy)
email: lucabellotti@quipo.it

ABSTRACT. I consider W. Hugh Woodin's approach to the Continuum Problem, giving a brief overview of it and discussing a few objections which can be raised against it. In the last few years, Woodin has developed a new approach to the problem, which has produced such results that although it is not yet solved, one could argue that there are at present clear symptoms that it could have a solution. Woodin's idea is to take an 'incremental' approach: we look for the relevant axioms in turn for the structures $H(\omega)$, $H(\omega_1)$, $H(\omega_2)$. Any 'complete' axiomatization (in a sense to be defined, taking into account the unavoidable Gödelian incompleteness) for the latter structure yields a solution to the Continuum Problem. The main result obtained by Woodin is that *there are* such axiomatizations, but any one of them must imply the *falsity* of the Continuum Hypothesis. Moreover, Woodin's work has led him to the following conjecture: *every* theory obtained by adding to ZFC an axiom which (a) is compatible with the existence of large cardinals, and (b) makes the properties of sets with hereditary cardinality at most $\aleph_1$ invariant under forcing, implies that the Continuum Hypothesis is false. I summarize Woodin's main results and then discuss some objections which have been raised to his approach, chiefly about generic absoluteness as a criterion to choose theories.

KEYWORDS: Continuum Hypothesis, Woodin. MSC2000: 03E50.

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1. Introduction

Cantor's Continuum Hypothesis, the object of Hilbert's first problem, is one of the most famous problems formally undecidable on the basis of the Zermelo–Fraenkel axioms of set theory with the Axiom of Choice (ZFC). In the last few years, W. Hugh Woodin ((Woodin 2000), (Woodin 2004), (Woodin 1999), (Woodin 2001)) has developed a new approach to the problem, which has produced a much deeper understanding of the question, such that although it is not yet solved, one could argue that there are at present clear symptoms that it *could* have a solution.

In this purely expository note, I shall summarize Woodin's main results and then describe and discuss some objections which have been raised to his approach, following quite closely and faithfully Woodin's and Dehornoy's accounts ((Woodin 2000), (Woodin 2004), (Woodin 2001), (Dehornoy 2003)) and a few discussions of the topic which can be found at present in the literature ((Dehornoy 2003), (Foreman 2003), (Larson 2003), (Shelah 2003), (Steel 2004), (Steel 2003)). I refer the reader to Woodin's own recent surveys of his work ((Woodin 2000), (Woodin 2004), (Woodin 2001)) for the definitions of some basic notions, for further motivation and details which I was compelled to omit here, and for the bibliography. I always give detailed page references to my sources below; the definitions and theorems are formulated exactly in the same form as in the sources. There is nothing new in this note, except perhaps in the discussion of the objections, and I do not claim any originality. My aim is only to give a quick summary of Woodin's main results and to hint at some delicate points which should be the object of further discussion.

2. The Continuum Problem

The Continuum Hypothesis (CH) is the following statement: if $X \subseteq \mathbb{R}$ is an uncountable set, then there exists a bijection $\pi : X \to \mathbb{R}$.

It is well known that Gödel proved that if ZFC is consistent, so is ZFC+CH, and Cohen proved that if ZFC is consistent, so is ZFC+$\neg$CH. Moreover, it is known that no known large cardinal axiom can solve the continuum problem. On the basis of these results, it has been maintained authoritatively (e.g., by Sol Feferman; see (Feferman 2000)) that the continuum problem is inherently vague, that it has in principle no solution. But it is well known that there are axioms for second order number theory which solve all the interesting problems for that theory, and solve them essentially transcending ZFC. One could wonder whether, roughly speaking, some extension of these axioms to more complicated sets could solve the continuum problem. This is, basically, Woodin's starting point (Woodin 2001, 569).

Woodin observes (Woodin 2004, 4) that the next natural structure to consider, after the structure $\langle P(\omega), \omega, +, \cdot, \in \rangle$, is $\langle P(\omega_1), \omega_1, +, \cdot, \in \rangle$, which is equivalent to the structure of all sets of hereditary cardinality less than $\aleph_2$. His approach is based precisely on the idea that the solution to the continuum problem comes from an understanding of $H(\omega_2)$, the set of all sets whose transitive closure has cardinality less than $\aleph_2$, because CH is expressible as a statement about this structure: more precisely, there exists a sentence ϕ_{CH} such that "$\langle H(\omega_2), \in\rangle \models \phi_{CH}$" is equivalent to CH. Thus, Woodin's idea is to take an 'incremental' approach (see, e.g., (Woodin 2001, 569)): we look for the relevant axioms in turn for the structures $H(\omega)$, $H(\omega_1)$, $H(\omega_2)$. Any complete (reasonably complete, modulo the unavoidable Gödelian incompleteness) axiomatization for the latter structure yields a solution to the continuum problem. The main result obtained by Woodin is that *there are* such theories, but what is remarkable is that any theory which is strongly canonical (in a sense that will be specified below) must imply the *falsity* of CH; what is more, there can be no strongly canonical theory for the structure $\langle P(\mathbb{R}), \mathbb{R}, +, \cdot, \in\rangle$, traditionally considered the next step after second order arithmetic (Woodin 2001, 570).

3. A summary of Woodin's results

We now give a quick overview of Woodin's results. We take as a basis the reformulation of Dehornoy (Dehornoy 2003), which can lead to certain simplifications, which in fact have been adopted by Woodin in his latest expositions (e.g. (Woodin 2004)).

First, let us consider Woodin's main result. It can be expressed by the following conjecture: every set theory which is compatible with the existence of large cardinals, and makes the properties of sets with hereditary cardinality at most $\aleph_1$ invariant under forcing, implies that the Continuum Hypothesis is false (Dehornoy 2003, 1). Let us call this conjecture *Woodin's Conjecture*. This conjecture is to be considered under the hypothesis that there are arbitrarily large Woodin cardinals. A precise formulation of the conjecture is the following (Steel 2003): let T be an axiomatizable theory extending ZFC such that the axioms of T going beyond ZFC are Σ^m_n for some m, n; suppose T is true in some set generic extension of V, and suppose that if G, H are set generic over V, and T is true in V[G] and V[H], then $\langle H(\omega_2), \in\rangle^{V[G]} \equiv \langle H(\omega_2), \in\rangle^{V[H]}$; then, in any set generic extension of V where T is true, CH is false.

The problem which conceptually constitutes the starting point of Woodin's research can be stated in the following way (Dehornoy 2003, 4): find an axiomatic framework, ZFC or ZFC completed with axioms which are compatible with the existence of large cardinals, providing a sufficiently complete description of a certain structure $\langle H, \in\rangle$, and making its properties invariant under forcing. We consider a sentence ϕ as *established* 'when it is necessarily true in every coherent framework that neutralizes the action of forcing until the level of ϕ' (Dehornoy 2003, 5). To use a physical analogy, the sentence is left as soon as temperature is sufficiently lowered to avoid that the 'thermic agitation' of forcing makes it impossible to distinguish between the sentence and its negation (ibid.).

A sentence is defined as Ω-*valid* iff it is satisfied in every model of ZFC of the type "$\langle V_\alpha, \in\rangle$ computed in an arbitrary generic extension of V" (Dehornoy 2003, 10). This definition is the main point of Dehornoy's reformulation (later adopted by Woodin himself). It allows the outright elimination of Ω^*-logic, which was defined by Woodin (e.g., (Woodin 2001, 689)) as follows: suppose that there exists a proper class of Woodin cardinals and that ϕ is a sentence; then $ZFC \vdash_{\Omega^*} \phi$ iff for all ordinals α and for all complete Boolean algebras $\mathbb{B}$, if $V_\alpha^{\mathbb{B}} \models ZFC$, then $V_\alpha^{\mathbb{B}} \models \phi$.

We need a few preliminary definitions in order to define the corresponding notion of provability, Ω-*provability* (see, e.g., (Woodin 2001, 683–685)). We say that a set $A \subseteq \mathbb{R}^n$ is *universally Baire* iff for every continuous function $F : \Omega \to \mathbb{R}^n$ where Ω is a compact Hausdorff space, the preimage of A by F has the property of Baire in Ω, i.e., it is open in that space modulo a meager set. Woodin observes that if $A \subseteq \mathbb{R}$ is universally Baire and V[G] is a set generic extension of V, then the set A has a canonical interpretation as a set $A_G \subseteq \mathbb{R}^{V[G]}$. If A is universally Baire, and M is a transitive set which models ZFC, we say that M is *A-closed* iff for each partial order $\mathbb{P} \in$ M, if $G \subseteq \mathbb{P}$ is V-generic, then, in V[G], we have $A_G \cap M[G] \in M[G]$. Now we can define Ω-provability. Suppose that there exists a proper class of Woodin cardinals and that ϕ is a sentence. Then $ZFC \vdash_\Omega \phi$ iff there is a universally Baire set $A \subseteq \mathbb{R}$ such that $\langle M, \in\rangle \models \phi$ for every countable transitive A-closed set M such that $\langle M, \in\rangle \models ZFC$. A very important property of the strong logic we have just defined is its invariance under forcing: if there is a proper class of Woodin cardinals and ϕ is a sentence, for each complete Boolean algebra $\mathbb{B}$, $ZFC \vdash_\Omega \phi$ iff $V^{\mathbb{B}} \models$ "$ZFC \vdash_\Omega \phi$".

If there exists a proper class of Woodin cardinals, then every Ω-provable sentence is Ω-valid; thus we obtain *soundness* for Ω-logic. In this setting, the Ω *Conjecture*, for Π_2 sentences, is the following statement: every Ω-valid sentence is Ω-provable; it expresses *completeness* for Ω-logic. In other words, the conjecture is (roughly) that all Π_2 sentences irrefutable by means of forcing have a proof in the universally Baire sets. We say that A is an Ω-*complete axiom* for a structure $\langle H, \in\rangle$ iff for every sentence ϕ, exactly one of the sentences $A \to$ "$\langle H, \in\rangle \models \phi$", $A \to$ "$\langle H, \in\rangle \models \neg\phi$" is Ω-provable. It is not difficult to show (Dehornoy 2003, 11) that if the conjecture is true, then the theory ZFC+A is a solution to the problem stated above for a structure iff A is an Ω-complete axiom for that structure.

The main result of Woodin can be stated as follows (Woodin 2001, 688): suppose that there exists a proper class of Woodin cardinals, a cardinal κ, and a sentence Ψ such that $V_\kappa \models ZFC + \Psi$, and for each sentence ϕ either $ZFC + \Psi \vdash_\Omega$ "$\langle H(\omega_2), \in\rangle \models \phi$" or $ZFC + \Psi \vdash_\Omega$ "$\langle H(\omega_2), \in\rangle \models \neg\phi$"; then *CH is false*. Similarly, Woodin's Conjecture (see above) can be rephrased in the following theorem (Dehornoy 2003, 14): if the Ω Conjecture is true, then *every* theory obtained by adding to ZFC an axiom which is compatible with the existence of large cardinals and makes the properties of $\langle H(\omega_2), \in\rangle$ invariant under forcing implies that *CH is false*. Thus, if the Ω Conjecture is true, *the negation of CH is established*, in the sense explained above.

Next, let us consider a more specific result of Woodin. He presents an Ω-consistent extension of ZFC which univocally determines the theory of $\langle H(\omega_2), \in\rangle$ with respect to forcing, and shows how it fixes the power of the continuum.

Woodin considers structures $\langle H(\omega_2), I_{NS}, X, \in\rangle$, where X is a universally Baire subset of $\mathbb{R}$, and I_{NS} is the *nonstationary ideal*, i.e. the σ-ideal of all sets $A \subseteq \omega_1$ such that $\omega_1 \backslash A$ contains a closed unbounded set. In order to deal with these structures the definition of Ω-consistency is suitably modified (see (Woodin 2001, 687)).

Axiom $(*)$ is the following assertion: there is a proper class of Woodin cardinals, and for each projective subset X of $\mathbb{R}$, for each Π_2 sentence ϕ, if the theory $ZFC +$ "$\langle H(\omega_2), I_{NS}, X, \in\rangle \models \phi$" is Ω-consistent, then $\langle H(\omega_2), I_{NS}, X, \in\rangle \models \phi$. As Woodin remarks (Woodin 2001, 687), this axiom is a sort of maximality principle; it is analogous to algebraic closure for a field. If there exists a proper class of Woodin cardinals and there is an inaccessible which is a limit of Woodin cardinals, then ZFC+axiom$(*)$ is Ω-consistent; moreover, if there exists a proper class of Woodin cardinals, then ZFC+axiom$(*)$ is Ω-complete.

Now, there is a Π_2 sentence, ψ_{AC}, regarding stationary, co–stationary subsets of ω_1, which, if it is true in the structure $\langle H(\omega_2), \in\rangle$, implies that $2^{\aleph_0} = \aleph_2$. We will not define this sentence here (see (Woodin 2001, 688)). What is relevant is that since the sentence "$\langle H(\omega_2), \in\rangle \models \psi_{AC}$" is Ω-consistent, we have that the axiom $(*)$ implies $2^{\aleph_0} = \aleph_2$.

Axiom $(*)$ gives exactly an extension of ZFC which univocally determines the theory of the powerset of ω_1 with respect to forcing. What is still lacking is a proof of its consistency with large cardinal axioms. If the Ω Conjecture holds, then the axiom is consistent with all large cardinal axioms. In fact, a further (rough) reformulation of the conjecture is that for Σ_2 sentences, Ω-consistency coincides with forceability (Larson 2003).

Finally, let us look briefly at the latest exposition to date of his main result by Woodin himself (Woodin 2004).

We say that $T \models_\Omega \phi$ iff for all complete Boolean algebras $\mathbb{B}$, for all ordinals α, if $V^{\mathbb{B}}_\alpha \models T$, then $V^{\mathbb{B}}_\alpha \models \phi$ (Woodin 2004, 5). This relation is the relation $T \vdash_{\Omega^*} \phi$ of Woodin's previous accounts ((Woodin 2000), (Woodin 2001)).

The definition of $T \vdash_\Omega \phi$ adopted by Woodin in this latest exposition (Woodin 2004, 6) is slightly different from the one he gave before (see above). We recall that

AD^+ is a certain variation of AD, the full Axiom of Determinacy (for space reasons, we do not give the definition here; see, e.g., (Woodin 2000, 26); roughly, AD^+ is such that $L(A,\mathbb{R}) \models AD^+$, for A universally Baire, is the correct generalization of $L(\mathbb{R}) \models AD$). We say that $T \vdash_\Omega \phi$ iff there exists a set $A \subseteq \mathbb{R}$ such that $L(A,\mathbb{R}) \models AD^+$, every set in $P(\mathbb{R}) \cap L(A,\mathbb{R})$ is universally Baire, and for all countable transitive A-closed sets M, for all ordinals $\alpha \in M$, if $M_\alpha \models T$ then $M_\alpha \models \phi$. Note that the present definition is given simply in ZFC.

In this formulation, the Ω Conjecture is the corresponding generalization of weak completeness (as usual, weak completeness regards logical consequence with respect to the empty theory) for *all* sentences (this is allowed by the new definition of the proof relation), under the existence of a proper class of Woodin cardinals. Woodin remarks (Woodin 2004, 8) that we cannot condition the conjecture on the existence of large cardinals which are much weaker than Woodin cardinals. According to Woodin (ibid.), the stronger conjecture about strong completeness is probably equivalent to the weak conjecture; in any case, the methods adopted at present in the search of a proof for the Ω Conjecture would solve also the stronger conjecture, if they have success. Quite naturally, we say that ϕ is Ω_{ZFC}-provable iff $ZFC \vdash_\Omega \phi$; similarly for validity. For this notion (as usual) consistency is unprovability of the negation, and satisfiability is non validity of the negation. Woodin defines a sentence ψ *good* iff it is an Ω-complete axiom in Dehornoy's terms (see above). Woodin's main result is that if there is a proper class of Woodin cardinals and ψ is a good sentence, then $ZFC + \psi \vdash_\Omega \neg$CH (Woodin 2004, 10).

This is indeed a remarkable result: we have 'an argument that CH is false based not on a specific choice of an axiom, but rather based simply on a completeness property the axiom is required to have' (Woodin 2004, 10). Of course, the Ω Conjecture settles in the positive the corresponding problem of finding a good sentence (in this case called by Woodin 'weakly good', ibid.) for satisfiability and validity.

As Steel points out (see (Steel 2003)), the point of Woodin's conjecture is that there is no generically absolute complete theory of $\langle H(\omega_2), \in\rangle$ which is consistent with all large cardinal hypotheses and encompasses CH. A possible route for a proof of this conjecture is to use the Ω Conjecture, which implies that indeed there is no generically absolute complete theory of $\langle V_{\omega+2}, \in\rangle$ consistent with all large cardinals. Woodin's results prove Woodin's Conjecture for a large part of the hierarchy of large cardinal hypotheses; it is not known whether one can extend the results to the entire hierarchy.

4. A few possible objections to Woodin's approach

We now present and briefly discuss a few objections which have been raised, or could be raised, to Woodin's treatment of the continuum problem. They could demonstrate that, as opposed to perhaps more famous and important unsolved mathematical problems, CH has a peculiar philosophical interest, in that no envisaged solution seems to be really satisfying without a parallel inquiry into the reasons why we ought to be disposed (or not) to consider certain results by full right *as a solution*.

First, Woodin admits (Woodin 2000, 48) that we cannot exclude a solution to the continuum problem on the basis of natural structural axioms not too far from the known large cardinal hypotheses. Steel ((Steel 2003), (Steel 2004)) envisages exactly this possibility. It is possible—he argues—that some natural extension of our current large cardinal hypotheses solves the continuum problem. We cannot exclude that in the future large cardinals will be discovered which are *not* preserved by forcing of smaller cardinality (admittedly, they would be very different from the large cardinals we know at present). In that case, our current situation would resemble that of someone who does not know measurable cardinals, thinks that every large cardinal is consistent with V=L, and hence believes that large cardinals will be forever useless for the measure problem of projective sets. Steel remarks that even if such a solution in terms of exotic large cardinals exists, it is not probable that it could occur in the short term.

In his discussion of Woodin's results ((Steel 2003), (Steel 2004)), Steel presents also an approach only partially close to Woodin's view. The basic point of this approach is that all large cardinals (also those which will be discovered in the future) should have the property of being preserved under forcing of smaller cardinality. The relevant metamathematical evidence that a large cardinal axiom is 'complete' is given by generic absoluteness theorems, whose best example is the following theorem of Woodin: if there are sufficient large cardinal axioms, then any two generic extensions of V satisfying CH satisfy exactly the same Σ^2_1 sentences. According to Steel (ibid.), the best chances for deciding CH should come from conditional generic absoluteness theorems of this kind for all levels of the Σ^m_n hierarchy and beyond. There could be conditionally generically absolute theories, consistent with all large cardinal axioms, some with CH and some with its negation. They would be intertranslatable, but in that case there would be an ambiguity in the language of set theory. But, if the Ω Conjecture is true, a solution to the continuum problem along this line is simply *impossible*. The conjecture says, roughly, that every Ω-valid (equivalently, Ω^*-provable) sentence is Ω-provable, and thus implies that every Ω^*-complete theory is Ω-complete, at any Σ^m_n level. But Woodin has proved that no axiomatizable, Ω-consistent theory is Ω-complete for all Σ^2_3 sentences (see (Woodin 2000, 24, 33–35)). So, according to Steel, the Ω Conjecture is a fundamental open problem.

If the conjecture is true, conditional generic absoluteness (alias Ω^*-completeness) coincides with Ω-completeness, and hence (by the above theorem) cannot be taken as a standard in the adoption or rejection of a theory. This is the reason why Steel disagrees (ibid.) on Woodin's point that a proof of the conjecture would be evidence in favor of the negation of CH, since by the above result Ω-completeness is not the right standard to choose theories. In this sense, according to Steel (ibid.), a proof of the conjecture would show that we are rather *far* from a solution to the continuum problem.

I point out that, in any case, this approach in terms of generic conditional absoluteness theorems 'all the way up' seems pretty far from the expectations (of Gödel and many other set theorists) of a *univocal* solution to the continuum problem. It is true that, as Steel remarks, any two (sequences of) theories giving the envisaged solution would be interpretable in each other by means of forcing, but this will hardly be enough for those who want to know whether CH is true or false in the real, *unique* universe of sets. In this sense, one should explain why this sort of inescapable 'ambiguity' in the language of set theory would be harmless, making no real difference, and not a symptom of a decisive incompleteness.

Foreman (Foreman 2003) maintains that a completely different approach to the continuum problem should be thoroughly pursued, in view of its advantages. It is his own approach in terms of *generic* large cardinals, which are defined by means of elementary embeddings of the universe into suitable transitive subclasses M of generic extensions of the universe, $V[G]$ (the difference with respect to large cardinal axioms is that the subclass and the embedding are defined in a generic extension of the universe). This approach gives results in the direction of a positive solution to GCH. Considering the classical problem of evaluating the different axiomatic extensions of ZF, and in particular the almost unanimous rejection of V=L, he concludes that 'considerations of completeness and absoluteness are secondary when considering axioms. The main criterion is what the axioms *say*' (Foreman 2003, 24). In this sense, according to Foreman, Woodin's 'solution' is defective: its combinatorial content (at least with respect to the present knowledge of set theorists) amounts just to a suitable restriction of the forcing axiom Martin's Maximum (see (Woodin 2000, 22)). Hence the 'solution' is nothing but 'a very sophisticated utilitarian argument, based more on the desirability of generic absoluteness than on what the content of the theory is' (ibid., 29). Foreman's approach, on the contrary, is much more on the line of ordinary large cardinal extensions, and consequently has stronger 'intrinsic' justifications (beyond 'extrinsic' ones).

It seems to me that the force of Foreman's objection comes from the fact that the definitions of generic large cardinals are really close to the ones of large cardinals, while Woodin himself admits that the abstract definition of large cardinal axiom which turns out in his own approach in connection with the Ω Conjecture is too general (see (Woodin 2001, 689)). On the other hand, Woodin considers Foreman's approach 'somewhat premature' (Woodin 2000, 47), since we still have no information on the consistency of some of the proposed axioms, and we do not obtain (as opposed to the case of Woodin's $(*)$ axiom) *first* order axioms over $H(\omega_2)$.

Shelah (Shelah 2003) raises a more radical objection to the overall approach. Discussing the beliefs of 'the California school of set theory' (Shelah 2003, 211), he argues as follows. Woodin thinks that $AD^{L(\mathbb{R})}$ is true, and that the axiom $(*)$ will be similarly accepted as true. Shelah, on the contrary, thinks that the right analogy is between the present status of $AD^{L(\mathbb{R})}$ and the axiom $(*)$ on the one hand, and the earlier status of GCH and V=L on the other, in the sense that in a certain period the latter were more informative than any alternative, and at present they are not marginalized, but only not preferred. More deeply, Shelah definitely rejects the position of the California school, essentially for three reasons. First, he does not think that 'the fact that a statement solves everything really nicely, even deeply, even being the best semi–axiom (if there is such a thing, which [he] doubt[s]) is a sufficient reason to say it is a "true axiom"' (Shelah 2003, 212). In particular, he does not see this as a compelling reason to consider the statement as *true*. Second, to consider certain semi–axioms as the best ones to adopt depends on the kind of problems in which one is interested: in the case of the California school, Descriptive Set Theory has a prominent role, but other kinds of problems can suggest different semi–axioms. Finally, even for Descriptive Set Theory the adoption of those axioms could be discussed. In this sense, one of Shelah's 'dreams' is just to find universes in which $AD^{L(\mathbb{R})}$ fails, with a different but interesting theory of the projective sets. In general, Shelah's attitude could be described as a sort of moderate formalism: he explicitly rejects extreme formalism, but he does not agree with Platonism. He thinks that we have many possible set theories, all conforming to ZFC, so that the phrase 'a universe of ZFC' works more like 'a human being', or 'a human being of some fixed nationality', than like 'the Sun'. This does not mean that all set theories are equal: some are definitely more interesting than others.

This moderate formalist attitude has to face, in its turn, some objections. Woodin remarks (Woodin 2000, 46) that the plausibility of the view that the entire bulk of independence results is the solution to CH (he calls it 'Ω^*-formalism') depends crucially on the Ω Conjecture. Suppose that the conjecture fails severely, in the sense that the set of theorems of ZFC in Ω^*-logic, which is Π_2 definable in V, reveals itself (under plausible assumptions) as recursively equivalent to the set of all Π_2 sentences which are true in V. In this case—Woodin argues—accepting Ω^*-formalism does not entail any loss in terms of complexity, and this position is compatible with a substantially unambiguous conception of the transfinite, although this conception *fails* to solve the continuum problem. But if the conjecture is true, then the position is substantially a formalist refusal of the transfinite above $H(c^+)$. In its extreme form, this is a refusal of the very possibility of any determinate conception of the uncountable sets. The trouble with the latter position is that the hierarchy of large cardinal axioms as calibrated by Ω-logic (see (Woodin 2001, 689)) is something well-determined, and this fact does not depend on the truth or falsity of the Ω Conjecture. Woodin underscores (ibid.) that there is no known theorem to object to this argument. This definition of the large cardinal hierarchy makes unavoidable reference to uncountable sets: this is, according to Woodin, 'a glimpse into the realm of the uncountable' (ibid.). True, it is not yet clear what is its scope and whether it will yield a solution of CH.

Dehornoy (Dehornoy 2003, 4) reports a further possible objection to the adoption of the criterion of conditional generic absoluteness to choose theories. The satisfaction of invariance under forcing is a strong argument in favor of the axiomatic system which achieves it. But it is possible to object that the variability due to forcing is the symptom of some weakness in our ability to conceive (or 'perceive') sets. The requirement of invariance under forcing is, in this sense, a restriction of our observation to those parts of the universe which do not suffer that sort of 'blurring'. But we have no reason to suppose that the solution of the continuum problem can be found in those fragments.

This objection seems to me rather strong, and I do not know of any really convincing defense against it. The obvious appeal to Gödel's famous argument in favor of the acceptance of those axioms which, though lacking intrinsic necessity, are so powerful and effective to deserve acceptance to the same degree of a robust physical theory, seems rather weak in this context. While in the case of Projective Determinacy we have implications from large cardinals on the one side, and the whole bulk of results of Descriptive Set Theory on the other, here we assume invariance under forcing as the criterion for the choice of a theory apparently just in view of its desirability, not on the basis of some really independent reason. This strategy has undeniably a rather *ad hoc* flavor. It is true that Woodin's results are really remarkable, showing (modulo the Ω Conjecture) that the symmetry induced by forcing breaks always in the direction of the negation of CH. But one could still wonder what exactly this shows (beyond giving a very interesting clue) with respect to the continuum problem in the 'real' universe of sets.

Finally, I underscore a philosophical problem which underlies any possible defense of Woodin's approach in terms of its efficacy. It is the same problem which affects many current arguments in favor of the adoption of Projective Determinacy: it is certainly a priori problematic to identify desirability or effectiveness (in the sense of efficacy) and truth, and this identification seems essentially what is going on here. Of course, in view of its many ramifications for the epistemology of mathematics in general, this is not the place to deal with this problem, but it is useful to keep it in mind when discussing Woodin's and other approaches to the continuum problem.

5. Woodin's view

Woodin thinks that his work maybe is not a route towards a solution of the continuum problem, but it yields 'convincing evidence that *there is* a solution' (Woodin 2001, 690). I conclude with a quotation which neatly expresses Woodin's attitude: '[P]erhaps there is after all a viable notion of (mathematical) truth for the transfinite, the vision is simply obscured by independence but not destroyed. I would go further and conjecture that fundamental questions such as that of Cantor's Continuum Hypothesis are solvable, or at the very least, in the case of the Continuum Hypothesis, claim that the theorem asserting otherwise has yet to be proved' (Woodin 2000, 1).

REFERENCES

Dehornoy, P. (2003). Progrès récents sur l'hypothèse du continu (d'après Woodin). *Séminaire Bourbaki*, 915.

Feferman, S. (2000). Why the programs for new axioms need to be questioned. *Bulletin of Symbolic Logic*, 6:401–413.

Foreman, M. (2003). Has the continuum hypothesis been settled? In *Logic Colloquium 2003 (Helsinki)*. To appear; slides at http: //www.math.helsinki.fi/ logic/LC2003/presentations.

Larson, P. (2003). Continuum hypothesis. *FOM*. See http: //www.cs.nyu.edu/pipermail/fom/ (May 2003).

Shelah, S. (2003). Logical dreams. *Bulletin of the American Mathematical Society*, 40:203–228.

Steel, J. (2003). Continuum hypothesis. *FOM*. See http: //www.cs.nyu.edu/pipermail/fom/ (May 2003).

Steel, J. (2004). Generic absoluteness and the continuum problem. To appear; slides at http: //math.berkeley.edu/˜steel/talks/Lectures.html.

Woodin, W. H. (1999). *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*. De Gruyter, Berlin.

Woodin, W. H. (2000). The continuum hypothesis. In *Logic Colloquium 2000 (Paris)*. To appear.

Woodin, W. H. (2001). The continuum hypothesis, I-II. *Notices Amer. Math. Soc.*, 48:567–576 and 681–690.

Woodin, W. H. (2004). Set theory after Russell. In *One Hundred Years of Russell's Paradox*. De Gruyter, Berlin. G. Link (ed.), pp. 29–47.