Subdivisions of graphs: A generalization of paths and cycles
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Abstract
One of the basic results in graph theory is Dirac’s theorem, that every graph of order $n \geq 3$ and minimum degree $\geq n/2$ is Hamiltonian. This may be restated as: if a graph of order $n$ and minimum degree $\geq n/2$ contains a cycle $C$ then it contains a spanning cycle, which is just a spanning subdivision of $C$. We show that the same conclusion is true if instead of $C$, we choose any graph $H$ such that every connected component of $H$ is non-trivial and contains at most one cycle. The degree bound can be improved to $(n - t)/2$ if $H$ has $t$ components that are trees.

We attempt a similar generalization of the Corrádi–Hajnal theorem that every graph of order $\geq 3k$ and minimum degree $\geq 2k$ contains $k$ disjoint cycles. Again, this may be restated as: every graph of order $\geq 3k$ and minimum degree $\geq 2k$ contains a subdivision of $kK_3$. We show that if $H$ is any graph of order $n$ with $k$ components, each of which is a cycle or a non-trivial tree, then every graph of order $\geq n$ and minimum degree $\geq n - k$ contains a subdivision of $H$.

Keywords: Spanning subdivision; Minimum degree condition; Unicyclic graphs

1. Introduction
The study of paths and cycles in a graph is an important topic in graph theory with many fundamental results and extensive literature. An excellent survey of this literature may be found in [1]. In this paper, we attempt to generalize some of these results by viewing a path as a subdivision of $K_2$ and a cycle as a subdivision of $K_3$. A subdivision of a graph $G$ is a graph obtained from $G$ by replacing some of the edges of $G$ by internally vertex-disjoint paths.

One of the basic results on paths and cycles is Dirac’s theorem [6] that every graph of order $n \geq 3$ and minimum degree $\geq n/2$ is Hamiltonian. This result has been generalized in several ways, some of which may be found in [1]. We consider another generalization in terms of subdivisions of graphs.

Dirac’s theorem may be restated as: if a graph $G$ of order $n$ and minimum degree $\geq n/2$ contains a cycle $C$, then it contains a spanning cycle, which is just a spanning subdivision of $C$. We show that the same conclusion is true, if instead of the cycle $C$, we consider any graph $H$ such that every connected component of $H$ is non-trivial and contains at most one cycle. In particular, if $G$ contains $k$ disjoint cycles, then $G$ has a 2-factor with exactly $k$ components. This special case has already been proved by Brandt et al. [4], with the weaker assumption that the sum of degrees of any two non-adjacent vertices in $G$ is $\geq n$. However, our result applies to graphs whose components may be trees or arbitrary graphs with exactly one cycle. The degree bound can be improved to $(n - t)/2$ where $t$ is the number of tree components of $H$. 
Another fundamental result on cycles in graphs, due to Corrádi and Hajnal [5], is that every graph of order \( \geq 3k \) and minimum degree \( \geq 2k \) contains \( k \) disjoint cycles. Enomoto [7] gave a simpler proof of the same result, with a weaker assumption that the sum of degrees of any two non-adjacent vertices is \( \geq 4k - 1 \). On the other hand, Brandt [3] showed that if \( H \) is any forest of order \( n \) with \( k \) components, then any graph of order \( \geq n \) and minimum degree \( \geq n - k \) contains \( H \). Schuster [8] combined both results and showed that if \( H \) is any forest of order \( n \) with \( m \) non-trivial components, and \( G \) is any graph of order \( \geq n + 3k \) and minimum degree \( \geq n - m + 2k \), then \( G \) contains \( H \) and \( k \) disjoint cycles that are also disjoint from \( H \).

We can view these results in terms of subdivisions of graphs. The Corrádi–Hajnal theorem may be restated as: if \( H \) is a graph containing \( k \) components, each of which is a 3-cycle, and \( G \) is any graph of order \( \geq |H| \) and minimum degree \( \geq |H| - k \), then \( G \) contains a subdivision of \( H \). Again, we show that the same conclusion is true if \( H \) is any graph, each of whose components is either a cycle or a non-trivial tree. Further, we show that \( G \) contains a subdivision of \( H \) such that only edges of \( H \) contained in a cycle are replaced by paths. We call such a subdivision a cyclic subdivision of \( H \). Thus our result generalizes Schuster’s since edges of \( H \) contained in a tree component are not subdivided in a cyclic subdivision of \( H \). Our proof follows that of Enomoto [7] and is simpler than Schuster’s.

All graphs considered are simple and finite. All terms that are not defined are standard and may be found in [2], for example. We say that a graph \( G \) contains a graph \( H \) if there is a subgraph of \( G \) isomorphic to \( H \). If \( G \) contains \( H \), and \( f \) is an isomorphism from \( H \) to a subgraph \( f(H) \) of \( G \), a vertex \( v \) of \( H \) is said to correspond to the vertex \( f(v) \) of \( G \). Similarly, \( G \) contains a (spanning) subdivision of \( H \) if there is a (spanning) subgraph of \( G \) isomorphic to some subdivision of \( H \).

If \( H \) is any subgraph of \( G \) and \( v \) a vertex in \( G \), then \( d(v, H) \) is the number of vertices of \( H \) that are adjacent to \( v \) in \( G \). If \( v \notin V(H) \), \( H + v \) is the subgraph of \( G \) obtained by adding the vertex \( v \) to \( H \) and all edges in \( G \) joining \( v \) to a vertex of \( H \). In all cases, the graph \( H \) will be understood from the context.

If \( G \) is any graph and \( S \) is either a vertex or an edge in \( G \), a subset of vertices or edges, or any subgraph of \( G \), then \( G - S \) is the subgraph of \( G \) obtained by deleting all vertices and edges in \( S \). If \( S \) is a subset of vertices of \( G \) then \( G[S] \) denotes the subgraph of \( G \) induced by \( S \). A bridge is an edge whose removal increases the number of connected components in the graph. A connected graph is said to be unicyclic if it has exactly one cycle. Note that any unicyclic graph is obtained by adding an edge to a tree.

In the next section, we prove the generalization of Dirac’s theorem, while in Section 3 we prove the generalization of Corrádi–Hajnal theorem. We conclude in Section 4 with some remarks indicating further possible generalizations.

### 2. Hamiltonian cycles

**Lemma 2.1.** Let \( H \) be any unicyclic graph of order \( k \). Any graph \( G \) with \( \delta(G) \geq k - 1 \) contains a cyclic subdivision of \( H \).

**Proof.** Let the vertices of \( H \) be enumerated as \( v_1, v_2, \ldots, v_k \) such that \( v_1, v_2, \ldots, v_l \) is the cycle in \( H \) and every vertex \( v_i \), for \( l < i \leq k \), is adjacent to exactly one vertex \( v_j \) with \( j < i \). Since \( \delta(G) \geq k - 1 \geq l - 1 \), \( G \) contains a cycle of length \( \geq l \). Let \( C = u_1, u_2, \ldots, u_m \) be a shortest cycle in \( G \) such that \( m \geq l \). Note that \( d(v, C) < l \) for all vertices \( v \in V(C) \), and \( d(v, C) \leq l \) for all vertices \( v \in V(G) \setminus V(C) \), otherwise we can find a shorter cycle in \( G \) of length \( \geq l \). Now we choose the vertices \( u_{m+1}, \ldots, u_{m+k-l} \) corresponding to the vertices \( v_{l+1}, \ldots, v_k \) to construct a cyclic subdivision of \( H \) in \( G \). If \( v_i \), for \( l < i \leq k \), is adjacent to \( v_j \) with \( j < i \), choose \( u_{m+i-1} \) to be a vertex adjacent to \( u_{m+j-l} \) that is different from \( u_r \), for \( 1 \leq r < m + i - l \). Such a vertex must exist since \( u_{m+j-l} \) is adjacent to at most \( i - 2 \leq k - 2 \) vertices in \( \{u_1, u_2, \ldots, u_{m+i-l-1}\} \). This gives the cyclic subdivision of \( H \) in \( G \). \( \square \)

**Theorem 2.1.** Let \( H \) be a graph, contained in a graph \( G \), such that every connected component of \( H \) is either a non-trivial tree or unicyclic. Let \( t \) be the number of tree components of \( H \). If \( |G| = n \) and \( \delta(G) \geq (n - t)/2 \) then \( G \) contains a spanning subdivision of \( H \).

**Proof.** Since \( H \) is contained in \( G \), \( G \) contains a subdivision of \( H \). Let \( H^s \) be a subgraph of \( G \) isomorphic to some subdivision of \( H \) such that \( |H^s| \) is maximum. We will assume that \( H^s \) is not a spanning subgraph of \( G \), derive
various properties of $H^s$, and show that we can find a larger subdivision of $H$ in $G$, thus contradicting the maximality of $|H^s|$.

Let $|H^s| = n^s < n$ and let $G^s = G - H^s$. No vertex in $G^s$ can be adjacent to any leaf vertex of $H^s$, otherwise $G$ contains a subdivision of $H$ of order $> n^s$. Similarly, no vertex in $G^s$ can be adjacent to both endpoints of any edge in $H^s$. So, for any vertex $v$ in $G^s$, $d(v, H^s) < |H^s|/2$ for any unicyclic component $H^s_j$ of $H^s$ and $d(v, H^s) < (|H^s_j| - 1)/2$ for any tree component $H^s_j$ of $H^s$.

**Claim 2.1.** The vertices of $H^s$ that have a neighbour in $G^s$ must have degree $2$ in $H^s$ and must be contained in a cycle of $H^s$.

**Proof.** Suppose there exists a vertex $x$ in a component of $H^s$, say $H^s_1$, such that it is adjacent to a vertex $p$ in $G^s$, and either it has degree $\geq 3$ in $H^s_1$ or it is not contained in the cycle of $H^s_1$. At least one edge $e$ in $H^s_1$, incident to $x$, must be a bridge. Let $y$ be the other end of $e$. At least one component of $H^s_1 - e$ is a tree and we choose $x$ and $e$ such that the order of the smallest tree component $T$ in $H^s_1 - e$ is as small as possible. No vertex of $T$ can be adjacent to a vertex in $G^s$, otherwise $T$ is non-trivial and we can choose a vertex and an edge in $T$ as $x$ and $e$. This implies that $y$ is in $T$. Let $|H^s_1| = n_1$ and $|T| = t_1$.

For any vertex $v$ in $G^s$, if $H^s_1$ is a tree then $d(v, H^s_1) \leq (n_1 - t_1)/2$ and if $H^s_1$ is unicyclic then $d(v, H^s_1) \leq (n_1 - t_1 + 1)/2$. In either case, $d(v, H^s) \leq (n^s - t_1 - t + 1)/2$. This implies that $\delta(G^s) \geq n_1$, which implies that $G^s$ contains a subdivision of $H^s_1$ of order $> n_1$. Replacing $H^s_1$ by this tree in $H^s$ gives a subdivision of $H$ in $G$ of order $> n^s$, a contradiction. $\Box$

**Claim 2.2.** If the number of tree components of $H$ is larger than zero (i.e. $t > 0$) then $H^s$ is a spanning subdivision of $H$ in $G$.

**Proof.** Suppose that $H^s_1$ is a tree component of $H^s$ with $|H^s_1| = n_1 > 1$. No vertex of $G^s$ can be adjacent to any vertex in $H^s_1$, by Claim 2.1. Thus $d(v, H^s) \leq (n^s - n_1 - (t - 1))/2$ for every vertex $v$ in $G^s$ and therefore $\delta(G) \geq (n - n^s) + n_1 - 1)/2$. Hence $\delta(G) \geq n_1$, which implies that $G^s$ contains a subdivision of $H^s_1$ of order $> n_1$. Replacing $H^s_1$ by this tree in $H^s$ gives a subdivision of $H$ in $G$ of order $> n^s$, a contradiction. $\Box$

Since $d(v, H^s) \leq (n^s)/2$ for any vertex $v$ in $G^s$, by Claim 2.2 we may assume that $\delta(G) \geq \frac{n}{2}$, so $\delta(G^s) \geq |G^s|/2$. This implies that $|G^s| > 1$ and $G^s$ has a Hamiltonian path. For any edge $uv$ in $H^s$, at least one of $u, v$ has no neighbour in $G^s$, otherwise we can replace the edge by a path of length $> 1$, with all internal vertices in $G^s$, to get a larger subdivision of $H$ in $G$. Since $\delta(G) \geq n/2$, either $u$ or $v$ has $\geq n/2$ neighbours in $H^s$. Therefore $n^s \geq n/2 + 1$, $|G^s| \leq n/2 - 1$ and every vertex in $G^s$ has at least two neighbours in $H^s$.

**Claim 2.3.** $\delta(G^s) \geq (|G^s| + 2)/2$.

**Proof.** Suppose there is a vertex $v$ in $G^s$ such that $d(v, G^s) \leq (|G^s| + 1)/2$. Then $d(v, H^s) \geq (n^s - 1)/2$, which implies that $d(v, H^s_1) = (|H^s_1|)/2$ for all components of $H^s$ except perhaps for one, say $H^s_1$, and $d(v, H^s_1) \geq (|H^s_1| - 1)/2$. Then, by Claim 2.1, every component apart from $H^s_1$ is an even cycle and $v$ is adjacent to every other vertex in the cycle. $H^s_1$ is a unicyclic graph with at most one vertex not contained in the cycle. Note that if $|H^s_1|$ is even then $H^s_1$ must also be an even cycle.

Let $u$ be any neighbour of $v$ in $G^s$. If $u$ has a neighbour $y$ in a component of $H^s$ that is an even cycle, then since $v$ is adjacent to every other vertex of the cycle, there exists a neighbour $x \neq y$ of $v$ at distance at most 2 from $y$ in the cycle. Replacing the path of length at most 2 between $x$ and $y$ in the cycle by the path $x, v, u, y$ gives a larger subdivision of $H$ in $G$.

If $u$ has no such neighbour, then $|H^s_1|$ must be odd and $u$ has all its neighbours in $H^s$ in the component $H^s_1$. Since $u$ has at least two neighbours in $H^s$, $|H^s_1| > 3$ otherwise $u$ is adjacent to both endpoints of an edge in $H^s_1$. $H^s_1$ contains exactly one pair of adjacent vertices $p, q$ such that $v$ is not adjacent to both $p$ and $q$. $H^s_1 - \{p, q\}$ is a path of odd order such that $v$ is adjacent to every other vertex in the path including the endpoints of the path. Since $u$ cannot be adjacent to both $p$ and $q$, $u$ has a neighbour $y$ in $H^s_1 - \{p, q\}$, and there exists a neighbour $x \neq y$ of $v$ in $H^s_1 - \{p, q\}$ at distance
at most 2 from y. Replacing the path of length at most 2 between x and y in $H_1 - \{p, q\}$ by the path x, v, u, y gives a larger subdivision of $H$ in $G$.

Therefore $d(v, H^s) \leq (n^s - 2)/2$ for every vertex $v$ in $G^s$ and hence $\delta(G^s) \geq (|G^s| + 2)/2$. □

Claim 2.4. There exist two distinct vertices $x$, $y$ in a component of $H^s$ such that they are, respectively, adjacent to two distinct vertices $p$, $q$ in $G^s$.

**Proof.** Let $p$ be a vertex of minimum degree in $G^s$ and $q$ be any other vertex in $G^s$. Let $y$ be a neighbour of $q$ in a component, say $H_1^s$, of $H^s$ and let $|H_1^s| = n_1$. If $p$ has at least 2 neighbours in $H_1^s$, we choose $x$ to be a neighbour of $p$ in $H_1^s$ that is different from $y$. If $p$ has at most one neighbour in $H_1^s$, then $d(p, H^s) \leq (n^s - n_1)/2 + 1$, and

\[d(p, G^s) = \delta(G^s) \geq (|G^s| + n_1 - 2)/2.\]

Since $|G^s| > \delta(G^s)$, we get $\delta(G^s) \geq n_1 - 1$. If $\delta(G^s) \geq n_1$, then $G^s$ contains a subdivision of $H_1^s$ of order $\geq n_1$, by Lemma 2.1. Replacing $H_1^s$ by this subgraph in $H^s$, we get a larger subdivision of $H$ in $G$. Therefore $\delta(G^s) = n_1 - 1$, $|G^s| = n_1$ and $(n^s - n_1)/2 \geq d(p, H^s - H_1^s) \geq n/2 - n_1 = (n^s - n_1)/2$. Hence, equality holds and every component of $H^s$ other than $H_1^s$ is an even cycle and $p$ is adjacent to every other vertex of these cycles. Also $p$ is adjacent to exactly one vertex in $H_1^s$.

If $q$ has at least two neighbours in $H_1^s$, we choose $x$ to be the neighbour of $p$ in $H_1^s$ and $y$ to be the neighbour of $q$ different from $x$. If $q$ has only one neighbour in $H_1^s$, then since $d(q, H^s) > 1$, we choose $y$ to be a neighbour of $q$ in a component of $H^s$ other than $H_1^s$, say $H_2^s$. Since $p$ has at least 2 neighbours in every component of $H^s$ other than $H_1^s$, we can choose $x$ to be a neighbour of $p$ in $H_2^s$ that is different from $y$. Thus in all cases we can find two distinct vertices $x$, $y$ in some component of $H^s$ that are, respectively, adjacent to distinct vertices $p$, $q$ in $G^s$. □

Choose vertices $x$, $y$ satisfying Claim 2.4 such that the length of a shortest path between $x$ and $y$ in $H_1^s$ is minimum. Let $p$, $q$ be the vertices in $G^s$ adjacent to $x$, $y$, respectively, and let $H_1^s$ be the component of $H^s$ containing $x$, $y$. Let $x, v_1, \ldots, v_l, y$ be a shortest path between $x$ and $y$ in $H_1^s$. Note that $l \geq 1$ and none of the vertices $v_1, 1 \leq i \leq l$ can have a neighbour in $G^s$, otherwise it contradicts the choice of the vertices $x$, $y$. Let $T_1$ and $T_2$ be the components of $H_1^s - \{xv_1, v_lv\}$ and assume, without loss of generality, that $v_1 \in V(T_1)$. Both $T_1$ and $T_2$ are trees by Claim 2.1 and let $|T_i| = t_i$, for $i = 1, 2$.

Claim 2.5. $\delta(G^s) \geq t_1 + 1$.

**Proof.** Let $v$ be a vertex of minimum degree in $G^s$. No vertex in $G^s$ can be adjacent to any vertex in $T_1$. Therefore,

\[d(v, H^s) \leq (n^s - t_1 + 1)/2 \quad \text{and} \quad d(v, G^s) = \delta(G^s) \geq (n - n^s + t_1 - 1)/2,\]

which implies $\delta(G^s) \geq t_1$. If $\delta(G^s) = t_1$, then $n - n^s = t_1 + 1$. Since $d(v, H^s - H_1^s) \leq (n^s - t_1 + t_2)/2$, $d(v, H_1^s) = d(v, T_1) \geq (n - n^s + t_1 + t_2)/2 - t_1 = (t_2 + t_1)/2$.

This implies that $T_2$ is a path of odd order between $x$ and $y$ and $v$ is adjacent to every other vertex in the path starting from $x$. Without loss of generality, $v \neq p$, and let $z$ be the vertex at distance 2 from $x$ in $T_2$. Replace the path of length two in $T_2$ from $x$ to $z$ by a path from $p$ to $v$ in $G^s$, and add the edges $xp, vz$. This gives a larger subdivision of $H$ in $G$. □

We now complete the proof of Theorem 2.1. Let the vertices of $T_1$ be enumerated as $v_1, v_2, \ldots, v_{t_1}$ such that $v_i$ is adjacent to exactly one vertex $v_j$ with $j < i$ for all $i > 1$. Note that $v_1, v_2, \ldots, v_l$ are the vertices of $T_1$ in the path from $x$ to $y$ in $H_1^s$. By Claim 2.3, $\delta(G^s) \geq (|G^s| + 2)/2$ and hence $G^s$ is pancutected [9], that is, there is a path of length $i$ between any two vertices in $G^s$, for all $2 \leq i < |G^s|$. By Claim 2.5, $|G^s| > t_1 + 1 > l + 1$ and hence there is a path $p, u_1, u_2, \ldots, u_l, q$ of length $l + 1$ between $p$ and $q$ in $G^s$.

We will show that there exists a subgraph of $G^s - \{p, q\}$ isomorphic to $T_1$ such that vertex $v_i$ corresponds to vertex $u_i$ for $1 \leq i \leq l$. If vertex $v_i$, for $l < i \leq t_1$, is adjacent to the vertex $v_j$ with $j < i$ in $T_1$, we choose the corresponding vertex $u_i$ in $G^s$ to be a neighbour of $u_j$ in $G^s$ that is different from $p, q$ and $u_m$, for $1 \leq m < i$. We can always find such a vertex since $\delta(G^s) \geq t_1 + 1$ by Claim 2.5, and $u_j$ has at most $i \leq t_1$ neighbours in $\{p, q, u_1, u_2, \ldots, u_{i-1}\}$. Replace the tree $T_1$ by this tree in $H_1^s$ and add the edges $xp, ut_1, u_1q$ and $qy$. This gives a subdivision of $H$ in $G$ of order $> n^s$, a contradiction.

Thus the largest subdivision of $H$ contained in $G$ must be a spanning subdivision. This completes the proof of the theorem. □
3. Disjoint cycles

**Theorem 3.1.** Let $H$ be any graph of order $n$ with $k$ connected components, each of which is either a non-trivial tree or a cycle. Let $G$ be any graph such that $|G| \geq n$ and $\delta(G) \geq n - k$. Then $G$ contains a cyclic subdivision of $H$.

**Proof.** Suppose there exists a counterexample. Choose graphs $H$ and $G$ satisfying the hypothesis of the theorem but $G$ does not contain a cyclic subdivision of $H$. Choose $H$ such that the number of edges in $H$ is minimum. If $H$ is a forest then $G$ contains $H$, by the theorem of Brandt [3]. Therefore at least one component, say $H_1$, of $H$ is a cycle. Let $e$ be any edge in $H_1$. Then $H - e$ contains fewer edges than $H$ and by the minimality of $H$, $G$ contains a cyclic subdivision of $H - e$. This implies that there is a subgraph $H^*$ of $G$ isomorphic to a cyclic subdivision of $H - e$. Let $G^* = G - H^*$ be the component of $H^*$ isomorphic to a cyclic subdivision of $H_j$, for all $j \neq i$. Let $\beta(H^*) = \sum_{j \neq i} |E(G[V(H^*_j)])|$. Choose $H^*$ such that

1. $|H^*|$ is as small as possible.
2. Subject to condition (1), $\beta(H^*)$ is as large as possible.
3. Subject to conditions (1) and (2), the length of a longest path in $G^*$ is as large as possible.

From the previous discussion, any cycle component of $H$ is a missing component and $H$ contains at least one such component. Before proving the main theorem, we prove some properties of missing components.

**Lemma 3.1.** Let $H_i$ be a missing component of $H$ and let $H^*_i$ be any component of $H^*$. Then for any vertex $v$ in $G^*$, $d(v, H^*_i) \leq |H_j|$ and for every vertex $u$ in $H^*_j$, $d(u, H^*_j) < |H_j|$. Further, if $d(v, H^*_j) = |H_j|$, then $G[V(H^*_j)]$ is a clique of order $|H_j|$.

**Proof.** Suppose $|H^*_j| = |H_j|$ and $d(v, H^*_j) = |H_j|$. Since $v$ is adjacent to all vertices in $H^*_j$, we can replace any vertex in $H^*_j$ by $v$. Since $\beta(H^*)$ is maximum, $G[V(H^*_j)]$ must be a clique of order $|H_j|$.

Suppose $H^*_j$ is a cycle of length $> |H_j|$. If $d(v, H^*_j) \geq |H_j|$, then $H^*_j + v$ contains a cycle of length $< |H^*_j|$ but $\geq |H_j|$. This contradicts the choice of $H^*$. The same argument holds if $d(u, H^*_j) \geq |H_j|$. ⊙

**Lemma 3.2.** Let $H_i$ be a missing component of $H$. Let $u, v$ be any two vertices in $G^*$ such that $d(u, G^*) + d(v, G^*) < 2(|H_i| - 1)$. Then there exists a component $H^*_j$ of $H^*$ such that $d(u, H^*_j) + d(v, H^*_j) \geq 2|H_j| - 1$. Further, $G[V(H^*_j)]$ is a clique of order $|H_j|$ and there exists a neighbour $w$ of $v$ in $H^*_j$ such that $G[V(H^*_j)] + u - w$ is a clique of order $|H_j|$.

**Proof.** Suppose there is no such component $H^*_j$. Then $d(u, H^*) + d(v, H^*) \leq \sum_{j \neq i} 2(|H_j| - 1) = 2(n - |H_i| - k + 1)$. Therefore $d(u, G) + d(v, G) < 2(n - |H_i| - k + 1) + 2(|H_i| - 1) = 2(n - k)$, contradicting $\delta(G) \geq n - k$.

Since at least one of $u, v$ has $|H_j|$ neighbours in $H^*_j$, $G[V(H^*_j)]$ is a clique of order $|H_j|$, by Lemma 3.1. Since $|H_j| > 1$, both $u$ and $v$ have at least one neighbour in $H^*_j$.

If $d(u, H^*_j) = |H_j|$, we can choose $w$ to be any neighbour of $v$. If $d(u, H^*_j) = |H_j| - 1$, then $d(v, H^*_j) = |H_j|$ and we can choose $w$ to be the vertex in $H^*_j$ that is not adjacent to $u$. Note that $G[V(H^*_j)] + u - w$ is a clique of order $|H_j|$. ⊙

**Lemma 3.3.** $K_2$ cannot be a missing component of $H$.

**Proof.** If $K_2$ is a missing component, then $G^*$ contains at least two isolated vertices $u, v$. By Lemma 3.2, there exists a component $H^*_j$ of $H^*$ and a neighbour $w$ of $v$ in $H^*_j$ such that $G[V(H^*_j)] + u - w$ is a clique of order $|H_j|$. The edge $uw$ forms a $K_2$ and thus $G$ contains a cyclic subdivision of $H$, a contradiction. ⊙

**Lemma 3.4.** Let $H_i$ be a missing component of $H$ that is a cycle. Then $G^*$ has a Hamiltonian path.
Proof. Let $P$ be a longest path in $G^s$ and $u, v$ be the endpoints of $P$. Suppose there is a vertex $x$ in $G^s - P$. If $d(v, G^s) \geq |H_i| - 1$ then there is a cycle of length $\geq |H_i|$ in $G^s$ and hence $G$ contains a cyclic subdivision of $H$, a contradiction. If $d(x, G^s) < |H_i|$, by Lemma 3.2, there exists a neighbour $w$ of $v$ in some component $H_j$ of $G^s$ such that $G[V(H^*_j)] + x - w$ is a clique of order $|H_j|$. This contradicts the choice of $H^*$ since $G^s + w - x$ contains a path longer than $P$.

Therefore $d(x, G^s) \geq |H_i|$ for all vertices $x$ in $G^s - P$. If $d(x, P) \geq |H_i| - 1$, $G^s$ contains a cycle of length $\geq |H_i|$, and $G$ contains a cyclic subdivision of $H$, a contradiction. Therefore $\delta(G^s - P) \geq 2$. Let $Q$ be a longest path in $G^s - P$ and let $x$ and $y$ be the endpoints of $Q$. We must have $d(x, Q), d(y, Q) < |H_i| - 1$ and since $d(x, G^s), d(y, G^s) \geq |H_i|$, we have $d(x, P), d(y, P) \geq 2$. Let $p$ be a neighbour of $x$ or $y$ in $P$ that is nearest to $v$ in $P$. Without loss of generality, $p$ is a neighbour of $x$. Let $q$ be the neighbour of $y$ in $P$ that is farthest from $v$ in $P$. Since $d(y, G^s) \geq |H_i|$ and $d(y, Q) \leq |Q|$, we have $d(y, P) \geq |H_i| - |Q| + 1$, and the subpath of $P$ between $p$ and $q$ contains at least $|H_i| - |Q| + 1$ vertices. This path, together with $Q$ and the edges $xp, yq$ forms a cycle of length $\geq |H_i|$ in $G^s$, and $G$ contains a cyclic subdivision of $H$, a contradiction. Hence $P$ contains all the vertices of $G^s$ and is a Hamiltonian path. □

We now come to the proof of the main theorem. We consider two different cases, depending on whether a missing component has order 3 or order $\geq 4$. Note that if a missing component has order 3, it is sufficient to consider the case when it is $K_3$, since $K_{1,2}$ is a subgraph of any cycle.

Case 1: Suppose there exists a missing component $H_i$ of $H$ that is a cycle of length 3. This part of the proof is essentially the same as Enomoto’s [7]. By Lemma 3.4, $G^s$ contains a Hamiltonian path. Note that in this case $G^s$ itself must be a path of length $\geq 2$, since if it contains a cycle, $G$ contains a cyclic subdivision of $H$. Let $u, v$ be the endpoints of the Hamiltonian path. Both $u$ and $v$ have degree 1 in $G^s$, and by Lemmas 3.1 and 3.2, there exists a component $H^*_j$ of $H^s$ such that $d(u, H^*_j) \leq |H_j|, d(v, H^*_j) \leq |H_j|$ and $d(u, H^*_j) + d(v, H^*_j) \geq 2|H_j| - 1$. Further, Lemma 3.2 implies that $H^*_j$ is a clique of order $|H_j|$. We may assume, without loss of generality, that $d(v, H^*_j) = |H_j|$ and $d(u, H^*_j) \geq |H_j| - 1$. Let $x$ be a vertex of $H^*_j$ such that $u$ is adjacent to all vertices in $H^*_j - x$.

Claim 3.1. No vertex of $G^s$ other than $u, v$ is adjacent to any vertex in $H^*_j$.

Proof. If a vertex $w$ in $G^s$, different from $u$ and $v$, is adjacent to $x$, then $H_j^* + u - x$ contains $H_j$ and $G^s + x - u$ contains a cycle, and therefore $G$ contains a cyclic subdivision of $H$. Similarly, if $w$ is adjacent to a vertex $y \neq x$ in $H^*_j$, then $G^s + y - v$ contains a cycle and $H^*_j + v - y$ contains $H_j$, contradicting the fact that $G$ does not contain a cyclic subdivision of $H$. □

Let $w$ be any vertex in $G^s$ other than $u, v$. By Claim 3.1, $d(w, G^s) + d(w, H^*_j) = 2 < |H_i| + |H_j| - 2$, since $|H_i| = 3$ and $|H_j| \geq 2$. Also, for all $z \in \{u, v, x\}, d(z, G^s) + d(z, H^*_j) \leq |H_j| + 1 = |H_i| + |H_j| - 2$.

Claim 3.2. There exists a component $H^*_m$ of $H^s - H^*_j$ such that $2(d(w, H^*_m) + d(x, H^*_m)) + d(u, H^*_m) + d(v, H^*_m) \geq 6|H_m| - 5$.

Proof. If there is no such component, then $2(d(w, H^s - H^*_j) + d(x, H^s - H^*_j)) + d(u, H^s - H^*_j) + d(v, H^s - H^*_j) \leq 6(n - |H_i| - |H_j| - k + 2)$. Therefore $2(d(w, G) + d(x, G)) + d(u, G) + d(v, G) < 6(n - |H_i| - |H_j| - k + 2) + 6(|H_i| + |H_j| - 2) = 6(n - k)$, contradicting $\delta(G) \geq n - k$. Hence there exists such a component $H^*_m$. □

Claim 3.3. $G[V(H^*_m)]$ is a clique of order $|H_m| > 2$.

Proof. By Claim 3.2, either $d(w, H^*_m) + d(x, H^*_m) \geq 2|H_m| - 1$ or $d(u, H^*_m) + d(v, H^*_m) \geq 2|H_m| - 1$. This implies that at least one of $u, v, w, x$ is adjacent to $|H_m|$ vertices in $H^*_m$. If any vertex $z \in \{u, v, w\}$ is adjacent to $|H_m|$ vertices in $H^*_m$, $G[V(H^*_m)]$ is a clique of order $|H_m|$, by Lemma 3.1. Further, $H^*_m + z$ contains a 3-cycle and therefore $G$ contains a cyclic subdivision of $H - H_m$ of order $\leq |G| - |H_m|$. This implies that $H_m$ is a missing component of $H$.

Similarly, if $x$ is adjacent to $|H_m|$ vertices in $H^*_m$, since $G[V(H^*_j)] + u - x$ is a clique of order $|H_j|$ we can apply the same argument as above after interchanging the vertices $u$ and $x$. Since $H_m$ is a missing component of $H$, by Lemma 3.3, $H_m$ is not a $K_2$. Thus $G[V(H^*_m)]$ is a clique of order $|H_m| > 2$. □
Let $H_m^s$ be the component of $H^s$ satisfying Claims 3.2 and 3.3. By Claim 3.2, either $d(u, H_m^s) + d(v, H_m^s) \geq 2|H_m| - 1$ or $d(w, H_m^s) + d(x, H_m^s) \geq 2|H_m| - 1$. We consider the two cases separately.

**Case 1.1:** Suppose $d(u, H_m^s) + d(v, H_m^s) \geq 2|H_m| - 1$. Then at least one of $u, v$, without loss of generality say $v$, is adjacent to $|H_m|$ vertices in $H_m^s$. At most one vertex in $H_m^s$ is not adjacent to $u$. Since $d(w, H_m^s) + d(x, H_m^s) \geq 2|H_m| - 2$ and $|H_m| > 3$, $w$ has a neighbour $y$ in $H_m^s$. If $y$ is adjacent to $u$, then $H_m^s + v + y$ contains a cycle. If $y$ is not adjacent to $u$, then $H_m^s + u - y$ contains $H_m$ and $G^s + y - u$ contains a cycle. In either case, $G$ contains a cyclic subdivision of $H$, a contradiction.

**Case 1.2:** Suppose $d(w, H_m^s) + d(x, H_m^s) \geq 2|H_m| - 1$. We consider subcases based on which of the vertices $w, x$ are adjacent to $|H_m|$ vertices in $H_m^s$.

**Case 1.2.1:** Suppose $d(x, H_m^s) = |H_m|$ and $d(w, H_m^s) = |H_m| - 1$. Then $d(u, H_m^s) + d(v, H_m^s) \geq 2|H_m| - 3$ and at least one of $u, v$, without loss of generality $v$, has $|H_m| - 1$ neighbours in $H_m^s$. Since $|H_m| > 2$, $w$ and $v$ have a common neighbour $y$ in $H_m^s$. Then $H_m^s + w - y$ contains $H_m$ and $G^s + w - y$ contains a cycle, hence $G$ contains a cyclic subdivision of $H$.

**Case 1.2.2:** If both $d(x, H_m^s), d(w, H_m^s) = |H_m|$ and $d(u, H_m^s) + d(v, H_m^s) \geq 2|H_m| - 5 \geq 1$. Without loss of generality, $v$ has a neighbour $y$ in $H_m^s$. Then $G^s + v - y$ contains a cycle, $H_m^s + v - x$ contains $H_j$ and $H_m^s + x - y$ contains $H_m$. Thus $G$ contains a cyclic subdivision of $H$.

**Case 1.3:** Suppose $d(x, H_m^s) = |H_m| - 1$ and $d(w, H_m^s) = |H_m|$. Let $y$ be the vertex of $H_m^s$ that is not adjacent to $x$. Suppose $y$ is adjacent to either $u$ or $v$, without loss of generality $u$, then $G^s + y - u$ contains a cycle, $H_j + u - x$ contains $H_j$ and $H_m^s + x - y$ contains $H_m$. Therefore both $d(u, H_m^s)$ and $d(v, H_m^s)$ are $|H_m| - 1$, and since $d(u, H_m^s) + d(v, H_m^s) \geq 2|H_m| - 3$, $u$ must be adjacent to a vertex $z \neq y$ in $H_m^s$. Since $v$ is adjacent to all vertices in $H_j^s$, $G[\{v, x, z\}]$ is a 3-cycle, $H_j^s + u - x$ contains $H_j$ and $H_m^s + w - z$ contains $H_m$. Thus $G$ contains a cyclic subdivision of $H$.

This completes the proof for the case when the missing component has order 3.

**Case 2:** Suppose the missing component $H_j$ of $H$ is a cycle of length $> 3$. By Lemma 3.4, $G^s$ has a Hamiltonian path. Let $u, v$ be the endpoints of the Hamiltonian path in $G^s$ and let $y$ and $x$ be the neighbours of $u$ and $v$, respectively, in the path. Note that $x \neq y$ since the path has order $>|H_j|\geq 4$. We have $d(u, G^s), d(v, G^s) < |H_j| - 1$ and $d(x, G^s), d(y, G^s) \leq |H_j| - 1$, otherwise $G^s$ contains a cycle of length $>|H_j|$ and $G$ contains a cyclic subdivision of $H$.

**Claim 3.4.** There exists a component $H_j^s$ of $H^s$ such that $d(u, H_j^s) + d(v, H_j^s) + d(x, H_j^s) + d(y, H_j^s) \geq 4|H_j| - 3$. Further, $H_j^s$ is a clique of order $>|H_j|$ and $H_j$ is a missing component of $H$.

**Proof.** If there is no such component, then $d(v, H_j^s) + d(u, H_j^s) + d(x, H_j^s) + d(y, H_j^s) \leq 4(n - |H_j| - k + 1)$, which implies $d(v, G) + d(u, G) + d(x, G) + d(y, G) < 4(n - k)$, a contradiction. Since at least one of $u, v, x, y$ is adjacent to at least $|H_j|$ vertices in $H_j^s$, $G[V(H_j^s)]$ must be a clique of order $>|H_j|$, by Lemma 3.1.

Either $u$ and $y$ or $v$ and $x$ have a common neighbour $z$ in $H_j^s$, otherwise $2|H_j| \geq 4|H_j| - 3$, a contradiction. Since $G^s + z$ contains a cycle of length $>|H_j|$ excluding either $u$ or $v$, $G$ contains a cyclic subdivision of $H - H_j$ of order $\leq |G| - |H_j|$. Hence $H_j$ is a missing component of $H$. □

If the component $H_j^s$ satisfying Claim 3.4 has order $\leq 3$, we can apply Case 1 of the proof to this missing component. Suppose $|H_j^s| \geq 4$.

Since $d(u, H_j^s) + d(v, H_j^s) \geq 2|H_j| - 3$, we may assume, without loss of generality, that $d(v, H_j^s) \geq |H_j| - 1$. Vertices $u$ and $y$ have a common neighbour $z$ in $H_j^s$, since $4|H_j| - 3 > 3|H_j|$. Since $G[V(H_j^s)]$ is a clique and $v$ is adjacent to at least 3 vertices in $H_j^s$, $G[V(H_j^s)] + v - z$ contains $H_j$ and $G^s + z - v$ contains a cycle of order $> |H_j|$. Thus $G$ contains a cyclic subdivision of $H$.

Therefore, in all cases, we can find a cyclic subdivision of $H$ in $G$, contradicting the fact that $G$ and $H$ are a counterexample to the theorem. This completes the proof of the theorem. □

**4. Remarks**

It is possible to generalize many of these results. The minimum degree condition in Theorem 3.1 can be replaced by an Ore-type condition on the sum of degrees of non-adjacent vertices, as in [4,7]. Further, Theorem 3.1 is true for graphs $H$ containing arbitrary unicicyclic components. These will be presented separately.
Another possible generalization is to consider similar questions for other types of graphs $H$, perhaps complete graphs. It is not difficult to show that the complete bipartite graph $K_{n/2,n/2}$ does not contain a spanning subdivision of $K_5$. However, we do not know any other examples. It is possible that for every fixed graph $H$, there exists an integer $f(H)$ such that every graph of order $n \geq f(H)$ and minimum degree $\geq (n + 1)/2$ contains a spanning subdivision of $H$.

It would be interesting to see if other results on paths and cycles can be generalized in a similar way.

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References