GEOMETRIC STRUCTURE IN SMOOTH DUAL AND LOCAL LANGLANDS CONJECTURE

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This expository note is based on the Takagi lectures given by the second-named author in November, 2012.

Topics in the lectures:

#1. Review of the LL (Local Langlands) conjecture.
#2. Statement of the ABPS(Aubert-Baum-Plymen-Solleveld) conjecture.
#3. Brief indication of the proof that for any connected split reductive p-adic group G both ABPS and LL are valid throughout the principal series of G.

Class field theory, a subject to which Professor Teiji Takagi made important and fundamental contributions, is a basic point in all three topics.

Notation. Let K be a field, and n a positive integer.
$K^\times$ is the multiplicative group of all non-zero elements of K.

$$K^\times := K - \{0\}$$

$M(n,K)$ is the K vector space of all $n \times n$ matrices with entries in K.

$GL(n,K)$ is the group of all $n \times n$ invertible matrices with entries in K.

$$GL(n,K) := \{[a_{ij}] \mid 1 \leq i,j \leq n \text{ and } a_{ij} \in K \text{ and } det[a_{ij}] \neq 0\}$$

$SL(n,K) \subset GL(n,K)$ is all $[a_{ij}] \in GL(n,K)$ with $det[a_{ij}] = 1$.

$SO(n,K)$ is the subgroup of $SL(n,K)$ consisting of all $[a_{ij}] \in SL(n,K)$ such that ${}^t[a_{ij}] = [a_{ij}]^{-1}$, where ${}^t[a_{ij}]$ is the transpose of $[a_{ij}]$.

$Sp(2n,K)$ is the subgroup of $SL(2n,K)$ consisting of all $[a_{ij}] \in SL(2n,K)$ with ${}^t[a_{ij}]J[a_{ij}] = J$ where J is the $2n \times 2n$ matrix

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

I_n is the $n \times n$ identity matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$K^\times$ injects into $GL(n,K)$ $\quad \lambda \mapsto \lambda I_n \qquad PGL(n,K) := GL(n,K)/K^\times$.

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1. p-adic Fields

Fix a prime p. $p \in \{2, 3, 5, 7, 11, 13, 17, \ldots\}$. $\mathbb{Q}_p$ denotes the field of p-adic numbers. To construct $\mathbb{Q}_p$, for $n \neq 0$, $n \in \mathbb{Z}$, let $\mathrm{ord}_p(n)$ to be the largest $r \in \{0, 1, 2, 3, \ldots\}$ such that p^r divides n.

$$\mathrm{ord}_p(n) := \text{largest } r \in \{0, 1, 2, 3, \ldots\} \text{ such that } n \equiv 0 \ (p^r)$$

For $\frac{n}{m} \in \mathbb{Q}$, define $\| \frac{n}{m} \|_p$ by

$$\| \frac{n}{m} \|_p := \begin{cases} p^{\mathrm{ord}_p(m) - \mathrm{ord}_p(n)} & \text{if } \frac{n}{m} \neq 0 \\ \\ 0 & \text{if } \frac{n}{m} = 0 \end{cases}$$

For $x, y \in \mathbb{Q}$, set $\delta_p(x, y) = \| (x - y) \|_p$.

$$\delta_p(x, y) = \| (x - y) \|_p \qquad\qquad x, y \in \mathbb{Q}$$

Then δ_p is a metric on $\mathbb{Q}$ and $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ using the metric δ_p. $\mathbb{Q}_p$ is a locally compact totally disconnected topological field. $\mathbb{Q}$ (topologized by the metric δ_p) is a dense subfield.

$$\mathbb{Q} \subset \mathbb{Q}_p$$

Any $x \in \mathbb{Q}_p$ is uniquely of the form :

$$x = a_n p^n + a_{n+1} p^{n+1} + a_{n+2} p^{n+2} + \cdots$$

where

$$n \in \mathbb{Z} \qquad \text{and} \qquad a_j \in \{0, 1, 2, \ldots, p-1\} \qquad \text{and} \qquad a_n \neq 0$$

For example

$$-1 = (p-1)p^0 + (p-1)p^1 + (p-1)p^2 + (p-1)p^3 + \cdots$$

Note that for any $x \in \mathbb{Q}_p$, the "pole" $a_n p^n + a_{n+1} p^{n+1} + \cdots + a_{-1} p^{-1}$ has at most finitely many non-zero terms.

For $x \in \mathbb{Q}_p^\times$ with $x = a_n p^n + a_{n+1} p^{n+1} + a_{n+2} p^{n+2} + \cdots$, the valuation of x, denoted $val(x)$, is the smallest $j \in \mathbb{Z}$ with $a_j \neq 0$.

$$val\colon \mathbb{Q}_p^\times \longrightarrow \mathbb{Z}$$

is a homomorphism of topological groups where $\mathbb{Z}$ has the discrete topology and $\mathbb{Q}_p^\times$ is topologized as a subspace of $\mathbb{Q}_p$.
Set

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p^\times \mid val(x) \geq 0\} \cup \{0\}$$

Then:

- $\mathbb{Z}_p$ is a compact subring of $\mathbb{Q}_p$ — and is the unique maximal compact subring of $\mathbb{Q}_p$.
- As a topological ring $\mathbb{Z}_p$ is isomorphic to the p-adic completion of $\mathbb{Z}$.

$$\mathbb{Z}_p \cong \varprojlim_{\infty \leftarrow n} (\mathbb{Z}/p^n\mathbb{Z})$$

- $\mathbb{Z}_p$ is a local ring whose unique maximal ideal $\mathcal{J}_p$ is :

$$\mathcal{J}_p = \{x \in \mathbb{Q}_p^\times \mid val(x) \geq 1\} \cup \{0\}$$

- The quotient $\mathbb{Z}_p/\mathcal{J}_p$ is the finite field with p elements.

$$\mathbb{Z}_p/\mathcal{J}_p \cong \mathbb{Z}/p\mathbb{Z}$$

$\mathbb{Z}_p$ is the *integers* of $\mathbb{Q}_p$ and $\mathbb{Z}_p/\mathcal{J}_p$ is the *residue field* of $\mathbb{Q}_p$.

Definition. A *p-adic field* is a field F which is a finite extension of $\mathbb{Q}_p$.

$$F \supset \mathbb{Q}_p \qquad \dim_{\mathbb{Q}_p} F \ < \ \infty$$

Any p-adic field F is a locally compact totally disconnected topological field.

The valuation $val\colon \mathbb{Q}_p^\times \to \mathbb{Z}$ of $\mathbb{Q}_p$ extends to give the non-normalized valuation

$$F^\times \longrightarrow \mathbb{Q}$$

This is a group homomorphism from $F^\times$ to the additive group of rational numbers. The image of this homomorphism has a smallest positive element s. Dividing by s then gives the normalized valuation of F

$$val_F\colon F^\times \longrightarrow \mathbb{Z}$$

val_F is a surjective homomorphism of topological groups where $\mathbb{Z}$ has the discrete topology and $F^\times$ is topologized as a subspace of F.

The *integers* of F, denoted $\mathcal{Z}_F$, is defined by :

$$\mathcal{Z}_F := \{x \in F^\times \mid val_F(x) \geq 0\} \ \cup \ \{0\}$$

Then:

- $\mathcal{Z}_F$ is a compact subring of F — and is the unique maximal compact subring of F.
- $\mathcal{Z}_F$ is a local ring whose unique maximal ideal $\mathcal{J}_F$ is :

$$\mathcal{J}_F = \{x \in F^\times \mid val_F(x) \geq 1\} \ \cup \ \{0\}$$

- The quotient $\mathcal{Z}_F/\mathcal{J}_F$ is a finite field which is an extension of the field with p elements.

$$\mathcal{Z}_F/\mathcal{J}_F \supset \mathbb{Z}/p\mathbb{Z}$$

$\mathcal{Z}_F/\mathcal{J}_F$ is the *residue field* of F.

2. The Weil Group

Let F be a p-adic field. $\overline{F}$ denotes the algebraic closure of F. $Gal(\overline{F}\,|F)$ is the Galois group of $\overline{F}$ over F i.e. $Gal(\overline{F}|F)$ is the group consisting of all automorphisms φ of the field $\overline{F}$ such that

$$\varphi(x) = x \qquad \forall x \in F$$

With its natural topology $Gal(\overline{F}\,|F)$ is a compact totally disconnected topological group and comes equipped with a continuous surjection onto the pro-finite completion of $\mathbb{Z}$.

$$Gal(\overline{F}\,|F) \twoheadrightarrow \widehat{\mathbb{Z}} := \varprojlim_{\infty \leftarrow n} (\mathbb{Z}/n\mathbb{Z})$$

The kernel of the continuous surjection $Gal(\overline{F}\,|F) \twoheadrightarrow \widehat{\mathbb{Z}}$ is denoted $\mathcal{I}_F$ and is the *inertia group* of F. Consider the short exact sequence of locally compact totally disconnected topological groups

$$1 \longrightarrow \mathcal{I}_F \hookrightarrow Gal(\overline{F}\,|F) \twoheadrightarrow \widehat{\mathbb{Z}} \longrightarrow 0$$

$\mathbb{Z}$ injects into $\widehat{\mathbb{Z}}$ via the inclusion $\mathbb{Z} \subset \widehat{\mathbb{Z}}$. Within $Gal(\overline{F}|F)$ let $\mathcal{W}_F^{\diamond}$ be the pre-image of $\mathbb{Z}$ so that there is the short exact sequence of topological groups

$$1 \longrightarrow \mathcal{I}_F \hookrightarrow \mathcal{W}_F^{\diamond} \twoheadrightarrow \mathbb{Z} \longrightarrow 0$$

Note that in this short exact sequence $\mathcal{I}_F$ and $\mathcal{W}_F^{\diamond}$ are topologized as subspaces of $Gal(\overline{F}\,|F)$. $\mathbb{Z}$ is topologized as a subspace of $\widehat{\mathbb{Z}}$, so in this short exact sequence $\mathbb{Z}$ does not have the discrete topology.

Minimally enlarge the collection of open sets in $\mathcal{W}_F^{\diamond}$ so that with this new topology the map $\mathcal{W}_F^{\diamond} \twoheadrightarrow \mathbb{Z}$ is continuous where $\mathbb{Z}$ now has the discrete topology. $\mathcal{W}_F^{\diamond}$ with this new topology is the *Weil group of* F [14] and is denoted $\mathcal{W}_F$. $\mathcal{W}_F$ is a topological group and there is the short exact sequence of topological groups.

$$1 \longrightarrow \mathcal{I}_F \hookrightarrow \mathcal{W}_F \twoheadrightarrow \mathbb{Z} \longrightarrow 0$$

in which $\mathbb{Z}$ has the discrete topology. $\mathcal{I}_F$ is an open (and closed) subgroup of $\mathcal{W}_F$. The topology that $\mathcal{I}_F$ receives as a subspace of $\mathcal{W}_F$ is the same as its topology as a subspace of $Gal(\overline{F}\,|F)$.
Denote the continuous map $\mathcal{W}_F \twoheadrightarrow \mathbb{Z}$ by $\varepsilon\colon \mathcal{W}_F \to \mathbb{Z}$.

3. Local Class Field Theory

As above, F is a p-adic field and $\mathcal{W}_F$ is the Weil group of F. $\mathcal{W}_F^{der}$ is the derived group of $\mathcal{W}_F$, i.e. $\mathcal{W}_F^{der}$ is the closure (in $\mathcal{W}_F$) of the commutator subgroup. The abelianization of $\mathcal{W}_F$, denoted $\mathcal{W}_F^{ab}$, is the topological group which is the quotient $\mathcal{W}_{\mathcal{F}}/\mathcal{W}_F^{der}$ with the quotient topology.

$$\mathcal{W}_F^{ab} := \mathcal{W}_{\mathcal{F}}/\mathcal{W}_F^{der}$$

Local class field theory [18] asserts that there is a canonically defined surjective homomorphism of topological groups

$$\alpha_F\colon \mathcal{W}_F \longrightarrow F^{\times}$$

with commutativity in the diagram

$$\begin{array}{ccc} \mathcal{W}_F & \xrightarrow{\alpha_F} & F^{\times} \\ {\scriptstyle \varepsilon}\downarrow & & \downarrow{\scriptstyle val_F} \\ \mathbb{Z} & \xrightarrow[I_{\mathbb{Z}}]{} & \mathbb{Z} \end{array}$$

($I_{\mathbb{Z}}$ = the identity map of $\mathbb{Z}$) such that the map α_F gives from $\mathcal{W}_F^{ab}$ to $F^{\times}$ is an isomorphism of topological groups.

$$\alpha_F\colon \mathcal{W}_F^{ab} \cong F^{\times}$$

α_F is the *Artin reciprocity map.*

4. Reductive p-adic Groups

If k is any field and V is a finite dimensional k vector space, there are the polynomial functions $V \to k$. If k' is an extension of k, then any polynomial function $V \to k$ extends canonically to give a polynomial function $k' \otimes_k V \to k'$.

As above, F is a p-adic field and $\overline{F}$ is the algebraic closure of F.
A subgroup $G \subset GL(n, F)$ is *algebraic* if there exist polynomial functions $P_1, P_2, \dots, P_r$

$$P_j \colon M(n, F) \longrightarrow F \qquad j = 1, 2, \dots, r$$

such that

(1) $G = \{g \in GL(n, F) \mid P_j(g) = 0 \quad j = 1, 2, \dots, r\}$
(2) $\overline{G} := \{g \in GL(n, \overline{F}) | P_j(g) = 0 \quad j = 1, 2, \dots, r\}$ is a subgroup of $GL(n, \overline{F})$.

An algebraic group $G \subset GL(n, F)$ is topologized by the topology it receives from F. In this topology G is a locally compact and totally disconnected topological group. $\overline{G}$, however, is topologized in a quite different way. $\overline{G}$ is an affine variety over the algebraically closed field $\overline{F}$. So $\overline{G}$ is topologized by the Zariski topology in which the closed sets are the algebraic sub-varieties of $\overline{G}$.

$g = [a_{ij}] \in \overline{G}$ is *unipotent* if all the eigenvalues of g are 1. A subgroup $H \subset \overline{G}$ is *unipotent* if every $g \in H$ is unipotent. An algebraic group $G \subset GL(n, F)$ is a *reductive* p-adic group if

- The only connected normal unipotent subgroup of $\overline{G}$ is the trivial one-element subgroup.

Terminology. G is *connected* if $\overline{G}$ is connected in the Zariski topology.

Examples. The groups $GL(n, F), SL(n, F), PGL(n, F), SO(n, F), Sp(2n, F)$ are connected reductive p-adic groups.

Example. With $n \geq 2$, let $UT(n, F)$ be the subgroup of $GL(n, F)$ consisting of all upper triangular matrices.

$$UT(n, F) := \{[a_{ij}] \in GL(n, F) \mid a_{ij} = 0 \text{ if } i > j\}$$

In $UT(n, \overline{F})$ consider the subgroup consisting of those $[a_{ij}] \in UT(n, \overline{F})$ such that $a_{ij} = 1$ if $i = j$. This is a non-trivial connected normal unipotent subgroup of $UT(n, \overline{F})$. So $UT(n, F)$ is not reductive.

5. The Smooth Dual

Let G be a reductive p-adic group.

Definition. A *representation* of G is a group homomorphism

$$\phi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

where V is a vector space over the complex numbers $\mathbb{C}$.

Definition. Two representations of G

$$\phi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

and

$$\psi : G \to \mathrm{Aut}_{\mathbb{C}}(W)$$

are *equivalent* if $\exists$ an isomorphism of $\mathbb{C}$ vector spaces $T : V \to W$ such that for all $g \in G$ there is commutativity in the diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi(g)} & V \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ W & \xrightarrow[\psi(g)]{} & W \end{array}$$

Definition. A representation

$$\phi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

of G is *irreducible* if $V \neq \{0\}$ and $\nexists$ a vector subspace W of V such that W is preserved by the action of G, $\{0\} \neq W$, and $W \neq V$.

Definition. A representation

$$\phi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

of G is *smooth* if for every $v \in V$,

$$G_v = \{g \in G \mid \phi(g)v = v\}$$

is an open subgroup of G.

The smooth dual of G, denoted $\widehat{G}$, is the set of equivalence classes of smooth irreducible representations of G.

$$\widehat{G} = \{\text{Smooth irreducible representations of } G\}/\sim$$

<u>Remark.</u> In any topological group an open subgroup is also closed. This is due to the fact that if H is an open subgroup of a topological group G, then any coset gH is an open subset of $G \Longrightarrow G - \{H\}$ is open $\Longrightarrow H$ is closed.

<u>Remark.</u> In a reductive p-adic group G, if U is any open set with the identity element e an element of U, then $\exists$ a compact open subgroup H with $H \subset U$.

6. Split Reductive p-adic Groups

Let $G \subset GL(n, F)$ be a reductive p-adic group. If r is a positive integer, then $(\overline{F}^\times)^r$ is the Cartesian product of r copies of $\overline{F}^\times$.

$$(\overline{F}^\times)^r := \overline{F}^\times \times \overline{F}^\times \times \cdots \times \overline{F}^\times$$

A *torus* in $\overline{G}$ is a closed connected abelian subgroup $\mathcal{T}$ of $\overline{G}$ such that for some positive integer r

$$\exists \text{ a bijection } \psi \colon \mathcal{T} \rightarrowtail (\overline{F}^\times)^r$$

with ψ both a group homomorphism and an isomorphism of $\overline{F}$ affine varieties.

G is *split* if $\exists$ an algebraic subgroup $\mathbb{T}$ of $GL(n, F)$ such that

(1) $\mathbb{T} \subset G$ and $\overline{\mathbb{T}} \subset \overline{G}$.

(2) $\overline{\mathbb{T}}$ is a maximal torus in $\overline{G}$.

(3) The bijection $\varphi\colon \overline{\mathbb{T}} \rightarrowtail (\overline{F}^{\times})^r$ can be chosen to be defined over F.

In (3) "defined over F" means that the polynomials which give the bijection $\psi\colon \overline{\mathbb{T}} \rightarrowtail (\overline{F}^{\times})^r$ have their coefficients in F. Note that in this case, when restricted to $\mathbb{T}$, ψ gives an isomorphism of locally compact totally disconected topological groups

$$\mathbb{T} \rightarrowtail (F^{\times})^r$$

If G is a split reductive p-adic group, then a subgroup $\mathbb{T}$ of G which satisfies conditions (1), (2), (3) is a *maximal p-adic torus* in G. Any two maximal p-adic tori in G are conjugate.

Examples. The groups $GL(n,F), SL(n,F), PGL(n,F), SO(n,F), Sp(2n,F)$ are connected split reductive p-adic groups.

Notation. Let $G \subset GL(n,F)$ be a reductive p-adic group. If E is a finite extension of F, then G_E is the intersection of $\overline{G}$ with $GL(n,E)$.

$$G_E := \overline{G} \cap GL(n,E)$$

For any finite extension E of F, G_E is a reductive p-adic group and G split implies G_E split.

Proposition. Let $G \subset GL(n,F)$ be a reductive p-adic group. Then $\exists$ a finite extension E of F such that G_E is split.

The proposition guarantees that the split reductive p-adic groups are plentiful within the reductive p-adic groups.

Let G be a reductive p-adic group. A connected algebraic subgroup B of $\overline{G}$ is a Borel subgroup of $\overline{G}$ if B is solvable and B is maximal among the connected solvable algebraic subgroups of $\overline{G}$. G is *quasi-split* if $\overline{G}$ has a Borel subgroup B defined over F — i.e. the polynomials determining B can be chosen to have their coefficients in F, where F is the p-adic field over which G is defined. Any split G is quasi-split.

7. The Local Langlands Conjecture

Let G be a connected reductive p-adic group. Associated to G is a connected reductive algebraic group over the complex numbers, denoted ${}^L G^{\,0}$.

Examples.

$$\begin{aligned}
{}^L GL(n,F)^{\,0} &= GL(n,\mathbb{C}) \\
{}^L SL(n,F)^{\,0} &= PGL(n,\mathbb{C}) \\
{}^L PGL(n,F)^{\,0} &= SL(n,\mathbb{C}) \\
{}^L SO(2n,F)^{\,0} &= SO(2n,\mathbb{C}) \\
{}^L SO(2n+1,F)^{\,0} &= Sp(2n,\mathbb{C}) \\
{}^L Sp(2n,F)^{\,0} &= SO(2n+1,\mathbb{C})
\end{aligned}$$

As above, F is the p-adic field over which G is defined. $Gal(\overline{F}\,|F)$ acts on ${}^LG^{\,0}$ as automorphisms of the complex algebraic group ${}^LG^{\,0}$. The Langlands dual group of G, denoted ${}^LG$, is the semidirect product group ${}^LG^{\,0} \rtimes Gal(\overline{F}\,|F)$.

$${}^LG := {}^LG^{\,0} \rtimes Gal(\overline{F}\,|F)$$

Via the usual topologies for ${}^LG^{\,0}$ and $Gal(\overline{F}\,|F)$, ${}^LG$ is a locally compact Hausdorff topological group. Since ${}^LG^{\,0}$ is connected and $Gal(\overline{F}\,|F)$ is totally disconnected, ${}^LG^{\,0}$ is the connected component of the identity in ${}^LG$.

A *torus* in ${}^LG^{\,0}$ is an algebraic subgroup T of ${}^LG^{\,0}$ such that for some positive integer r

$$\exists \text{ a bijection } \psi\colon T \rightarrowtail (\mathbb{C}^\times)^r = \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$$

with ψ both a group homomorphism and an isomorphism of complex affine varieties. Any two maximal tori in ${}^LG^{\,0}$ are conjugate. $g \in {}^LG^{\,0}$ is *semi-simple* if $\exists$ a torus T in ${}^LG^{\,0}$ with $g \in T$.

<u>Notation.</u>$({}^LG^{\,0})_{\mathrm{semi-simple}}$ denotes the set of all semi-simple elements in ${}^LG^{\,0}$.

As a set ${}^LG$ is ${}^LG^{\,0} \times Gal(\overline{F}\,|F)$. Thus $({}^LG^{\,0})_{\mathrm{semi-simple}} \times Gal(\overline{F}\,|F)$ is a subset of ${}^LG$.

$$({}^LG^{\,0})_{\mathrm{semi-simple}} \times Gal(\overline{F}\,|F) \subset {}^LG$$

$\beta \in {}^LG$ is said to be semi-simple if whenever $\pi\colon {}^LG \to \mathrm{Aut}_{\mathbb{C}}V$ is a finite dimensional representation of ${}^LG$, $\pi(\beta)$ is semi-simple. $({}^LG)_{\mathrm{semi-simple}}$ denotes the set of all semi-simple elements in ${}^LG$.
${}^LG_{\mathrm{semi-simple}}$ is a subset of $({}^LG^{\,0})_{\mathrm{semi-simple}} \times Gal(\overline{F}\,|F)$.

$${}^LG_{\mathrm{semi-simple}} \subset ({}^LG^{\,0})_{\mathrm{semi-simple}} \times Gal(\overline{F}\,|F)$$

A *Langlands parameter* for a connected reductive p-adic group G is a group homomorphism

$$\varphi\colon \mathcal{W}_F \times SL(2,\mathbb{C}) \longrightarrow {}^LG$$

such that

- φ is continuous.
- Restricted to $SL(2,\mathbb{C})$, φ is a morphism of complex algebraic groups $SL(2,\mathbb{C}) \to {}^LG^{\,0}$.
  (${}^LG^{\,0}$ is the connected component of the identity in ${}^LG$. Since φ is continuous and $SL(2,\mathbb{C})$is connected, φ must map $SL(2,\mathbb{C})$ to ${}^LG^{\,0}$.)
- There is commutativity in the diagram

$$\begin{array}{ccc} \mathcal{W}_F \times SL(2,\mathbb{C}) & \xrightarrow{\ \varphi\ } & {}^LG \\ \downarrow & & \downarrow \\ Gal(\overline{F}\,|F) & \xrightarrow[\ I\ ]{} & Gal(\overline{F}\,|F) \end{array}$$

  In this diagram $I\colon Gal(\overline{F}\,|F) \to Gal(\overline{F}\,|F)$ is the identity map $I(\beta) = \beta \ \ \forall \beta \in Gal(\overline{F}\,|F)$. The right vertical arrow is the quotient map obtained by dividing ${}^LG = {}^LG^{\,0} \rtimes Gal(\overline{F}\,|F)$ by the normal subgroup ${}^LG^{\,0}$.

The left vertical arrow is the composition
$\mathcal{W}_F \times SL(2,\mathbb{C}) \to \mathcal{W}_F \hookrightarrow Gal(\overline{F}\,|F)$ where the first map is the evident projection and the second map is the evident inclusion.

- $\varphi(\mathcal{W}_F) \subset ({}^LG)_{\text{semi-simple}}$

${}^LG^0$ acts on the set of Langlands parameters for G by conjugation, so the quotient set {Langlands parameters for $G\}/{}^LG^0$ can be formed.

The local Langlands conjecture (LL) asserts [14] that there is a (canonically defined) map of sets

$$\alpha_G \colon \widehat{G} \longrightarrow \{\text{Langlands parameters for } G\}/{}^LG^0$$

which is finite-to-one and has certain naturality properties. The fibers of α_G are referred to as L-packets.

If G is connected and split — or, more generally, if G is an inner form of a connected split reductive p-adic group — then the action of $Gal(\overline{F}\,|F)$ on ${}^LG^{\,0}$ is trivial. Hence in this case, ${}^LG$ as a group is the product group ${}^LG^{\,0} \times Gal(\overline{F}\,|F)$.

$$G \quad \text{connected and split} \implies \quad {}^LG = \; {}^LG^{\,0} \times Gal(\overline{F}\,|F)$$

Notation. For the rest of this section G will be a connected split reductive p-adic group or an inner form of a connected split reductive p-adic group.
"inner form" = "has same Langlands dual group".

It is then immediate that for such a G, the definition of Langlands parameter given above is equivalent to Langlands parameter defined as follows.

A *Langlands parameter* for a connected split reductive p-adic group G is a group homomorphism

$$\varphi \colon \mathcal{W}_F \times SL(2,\mathbb{C}) \longrightarrow \; {}^LG^{\,0}$$

such that

1. Restricted to $SL(2,\mathbb{C})$, φ is a morphism of complex algebraic groups.
2. Restricted to $\mathcal{W}_F$, φ is continuous where ${}^LG^{\,0}$ is topologized by viewing it as the underlying locally compact Hausdorff space of the complex analytic manifold ${}^LG^{\,0}$.
3. $\varphi(\mathcal{W}_F) \subset ({}^LG^{\,0})_{\text{semi-simple}}$.

(2) implies that $\varphi(\mathcal{I}_F)$ is a finite subgroup of ${}^LG^{\,0}$. $\varphi(\mathcal{W}_F)$ is the subgroup of ${}^LG^{\,0}$ generated by $\varphi(\mathcal{I}_F)$ and one additional semi-simple element of ${}^LG^{\,0}$. $\varphi(SL(2,\mathbb{C}))$ is either trivial (i.e. is the trivial one-element subgroup of ${}^LG^{\,0}$) or is an algebraic subgroup of ${}^LG^{\,0}$ isomorphic to $SL(2,\mathbb{C})$ or $PSL(2,\mathbb{C}) := SL(2,\mathbb{C})/\{I_2, -I_2\}$.

${}^LG^{\,0}$ acts on the set of Langlands parameters for G by conjugation, so the quotient set {Langlands parameters for $G\}/{}^LG^{\,0}$ can be formed.

The local Langlands conjecture (LL) asserts that there is a (canonically defined) map of sets

$$\alpha_G \colon \widehat{G} \longrightarrow \{\text{Langlands parameters for } G\}/{}^LG^{\,0}$$

which is surjective and finite-to-one and has certain naturality properties. The fibers of α_G are referred to as L-packets.

Example. Let $G = GL(1,F) = F^\times$. Then ${}^LG^{\,0} = GL(1,\mathbb{C}) = \mathbb{C}^\times$. Any morphism of complex algebraic groups $SL(2,\mathbb{C}) \to \mathbb{C}^\times$ is trivial, so in this example $\{$Langlands parameters for $G\}/{}^LG^{\,0} = \{$continuous homomorphisms $\mathcal{W}_F \to \mathbb{C}^\times\}$. The isomorphism of local class field theory

$$\mathcal{W}_F^{ab} \cong F^\times$$

gives a bijection

$$\{\text{continuous homomorphisms } \mathcal{W}_F \to \mathbb{C}^\times\} \longleftrightarrow \widehat{F^\times}$$

which verifies LL for this example and produces the point of view that the goal of LL is to extend local class field theory to non-abelian reductive p-adic groups.

Let D be an F-division algebra of dimension d^2 over its center F. With m a positive integer, $GL(m,D)$ denotes the connected reductive p-adic group consisting of all $m \times m$ invertible matrices with entries in D. Except for $GL(n,F)$, the groups $GL(m,D)$ are non-split. $GL(m,D)$ is an inner form of $GL(md,F)$ — i.e. $GL(m,D)$ and $GL(md,F)$ have the same Langlands dual group. Since $GL(n,F)$ is connected and split, each group $GL(m,D)$ is an inner form of a connected split reductive p-adic group.

The local Langlands correspondence was proved in [20, 28] for representations admitting non-zero Iwahori-fixed vectors. LL was proved for unipotent representations in [26]. LL was proved for $GL(n,F)$ in [21, 15, 16, 30, 33]. When combined with the Jacquet–Langlands correspondence, this proves LL for inner forms $GL(m,D)$ of $GL(md,F)$ [17, 7].

Each group $G = GL(m,D)$ has a reduced norm map $Nrd\colon GL(m,D) \to F^\times$.

$$Nrd\colon GL(m,D) \longrightarrow F^\times$$

Set $G^\sharp = ker(GL(m,D) \to F^\times)$. $G^\sharp$ is an inner form of $SL(md,F)$. LL for $G^\sharp$ is implied by LL for G — in the sense that each L-packet of $G^\sharp$ consists of the irreducible constituents of $\mathrm{Res}^G_{G^\sharp}(\Pi_\varphi(G))$ of an L-packet of G restricted to $G^\sharp$ — i.e. each L-packet of $G^\sharp$ consists of the irreducible constituents of the restriction to $G^\sharp$ of a smooth irreducible representation of G. LL was proved for the groups $G^\sharp$ in [17, 8].

For tempered representations of quasi-split classical groups, LL was proved in [2, 27], and extended to non-tempered representations in [9].

8. The Hecke Algebra - Bernstein Components

Let G be a reductive p-adic group.
G is locally compact so a (left-invariant) Haar measure dg can be chosen.
The Hecke algebra of G, denoted $\mathcal{H}G$, is then the convolution algebra of all locally-constant compactly-supported complex-valued functions $f : G \to \mathbb{C}$.

$$\begin{array}{rcl} (f+h)(g) & = & f(g)+h(g) \\ (f*h)(g_0) & = & \displaystyle\int_G f(g)h(g^{-1}g_0)dg \end{array} \quad \left\{ \begin{array}{l} g \in G \\ g_0 \in G \\ f \in \mathcal{H}G \\ h \in \mathcal{H}G \end{array} \right.$$

Definition. A *representation* of the Hecke algebra $\mathcal{H}G$ is a homomorphism of $\mathbb{C}$ algebras

$$\psi : \mathcal{H}G \to \mathrm{End}_{\mathbb{C}}(V)$$

where V is a vector space over the complex numbers $\mathbb{C}$.

Definition. A representation

$$\psi : \mathcal{H}G \to \mathrm{End}_{\mathbb{C}}(V)$$

of the Hecke algebra $\mathcal{H}G$ is *irreducible* if $\psi : \mathcal{H}G \to \mathrm{End}_{\mathbb{C}}(V)$ is not the zero map and $\nexists$ a vector subspace W of V with W preserved by the action of $\mathcal{H}G$ and $0 \neq W$ and $W \neq V$.

Definition. A *primitive ideal* in $\mathcal{H}G$ is an ideal I which is the null space of an irreducible representation. Thus an ideal $I \subset \mathcal{H}G$ is primitive if and only if $\exists$ an irreducible representation $\psi : \mathcal{H}G \to \mathrm{End}_{\mathbb{C}}(V)$ such that

$$0 \longrightarrow I \hookrightarrow \mathcal{H}G \xrightarrow{\psi} \mathrm{End}_{\mathbb{C}}(V)$$

is exact.

Remark. The Hecke algebra $\mathcal{H}G$ does not have a unit. $\mathcal{H}G$ does, however, have *local units* — i.e. if $a_1, a_2, a_3, \dots, a_l$ is any finite set of elements of $\mathcal{H}G$, then $\exists$ an idempotent $\omega \in \mathcal{H}G$ with

$$\omega a_j = a_j \omega = a_j \qquad j = 0, 1, 2, \dots, l$$

Definition. A representation

$$\psi : \mathcal{H}G \to \mathrm{End}_{\mathbb{C}}(V)$$

of the Hecke algebra $\mathcal{H}G$ is *non-degenerate* if $(\mathcal{H}G)V = V$ — i.e. for each $v \in V$, $\exists v_1, v_2, \dots, v_r \in V$ and $f_1, f_2, \dots, f_r \in \mathcal{H}G$ with $v = f_1 v_1 + f_2 v_2 + \dots + f_r v_r$.

Remark. Any irreducible representation of $\mathcal{H}G$ is non-degenerate.

Let

$$\phi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

be a smooth representation of G. Then ϕ integrates to give a non-degenerate representation of $\mathcal{H}G$.

$$f \mapsto \int_G f(g)\phi(g)dg$$
$$f \in \mathcal{H}G$$

This operation of integration gives an equivalence of categories

$$\begin{pmatrix} \text{Smooth} \\ \text{representations of } G \end{pmatrix} \cong \begin{pmatrix} \text{Non} - \text{degenerate} \\ \text{representations of } \mathcal{H}G \end{pmatrix}$$

In particular this gives a bijection (of sets)

$$\widehat{G} \longleftrightarrow \mathrm{Prim}(\mathcal{H}G)$$

where $\mathrm{Prim}(\mathcal{H}G)$ is the set of primitive ideals in $\mathcal{H}G$.
What has been gained from this bijection?
On $\mathrm{Prim}(\mathcal{H}G)$ have a topology : the Jacobson topology.
If S is a subset of $\mathrm{Prim}(\mathcal{H}G)$ then the closure $\overline{S}$ (in the Jacobson toplogy) of S is

$$\overline{S} = \{J \in \mathrm{Prim}(\mathcal{H}G) \mid J \supset \bigcap_{I \in S} I\}$$

Example. Let X be an affine variety over $\mathbb{C}$. $\mathcal{O}(X)$ denotes the co-ordinate algebra of X. $\mathrm{Prim}(\mathcal{O}(X))$ is the set of $\mathbb{C}$-rational points of X. The Jacobson topology is the Zariski topology.

Point set topology. In a topological space W two points w_1, w_2 are in the same *connected component* if and only if whenever U_1, U_2 are two open sets of W with $w_1 \in U_1, w_2 \in U_2$, and $U_1 \cup U_2 = W$, then $U_1 \cap U_2 \neq \emptyset$.
As a set, W is the disjoint union of its connected components. If each connected component is both open and closed, then as a topological space W is the disjoint union of its connected components.

$\widehat{G} = \mathrm{Prim}(\mathcal{H}G)$ (with the Jacobson topology) is the disjoint union of its connected components. Each connected component is both open and closed. The connected components of $\widehat{G} = \mathrm{Prim}(\mathcal{H}G)$ are known as the *Bernstein components.*

$\pi_o\mathrm{Prim}(\mathcal{H}G)$ denotes the set of connected components of $\widehat{G} = \mathrm{Prim}(\mathcal{H}G)$.
$\pi_o\mathrm{Prim}(\mathcal{H}G)$ is a countable set and has no further structure.
$\pi_o\mathrm{Prim}(\mathcal{H}G)$ is known as the *Bernstein spectrum* of G, and will be denoted $\mathfrak{B}(G)$.

$$\mathfrak{B}(G) := \pi_o\mathrm{Prim}(\mathcal{H}G)$$

For $\mathfrak{s} \in \mathfrak{B}(G)$, the Bernstein component of $\widehat{G} = \mathrm{Prim}(\mathcal{H}G)$ will be denoted $\widehat{G}_{\mathfrak{s}}$.

9. Infinitesimal Character-Tempered Dual

The main problem in the representation theory of reductive p-adic groups is:

Problem. Given a connected reductive p-adic group G, describe the smooth dual $\widehat{G} = \mathrm{Prim}(\mathcal{H}G)$.

A solution of this problem should include descriptions of

(1) The tempered dual
(2) The central characters
(3) The LL map $\alpha_G \colon \widehat{G} \longrightarrow \{\text{Langlands parameters for } G\}/{}^LG$

For (1), recall that a choice of (left-invariant) Haar measure for G determines a measure, the *Plancherel measure*, on $\widehat{G}$. The tempered dual of G is the support of the Plancherel measure. Equivalently, the tempered dual consists of those smooth irreducible representations of G whose Harish-Chandra character is tempered. Equivalently, let C^*_rG be the reduced C^* algebra of G. $\mathcal{H}G$ is a dense $*$-subalgebra of C^*_rG which is not holomorphically closed.

$$\mathcal{H}G \subset C^*_rG$$

Then the tempered dual of G consists of those irreducible representations of $\mathcal{H}G$ which can be extended to give an irreducible representation (in the sense of C^* algebras) of C_r^*G. The tempered dual of G will be denoted $\widehat{G}_{temp}$.

$$\widehat{G}_{temp} \subset \widehat{G}$$

For (2) Bernstein [12] [13] [29] assigns to each $\mathfrak{s} \in \mathfrak{B}(G) = \pi_o \mathrm{Prim}(\mathcal{H}G)$ a complex torus $T_\mathfrak{s}$ and a finite group $W_\mathfrak{s}$ which acts on $T_\mathfrak{s}$ as automorphisms of the affine variety $T_\mathfrak{s}$. Here "complex torus" means an algebraic group T, defined over the complex numbers $\mathbb{C}$, such that there exists an isomorphism of algebraic groups

$$T \cong \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times.$$

In general, $W_\mathfrak{s}$ acts on $T_\mathfrak{s}$ not as automorphisms of the algebraic group $T_\mathfrak{s}$ but only as automorphisms of the affine variety $T_\mathfrak{s}$. Bernstein then forms the quotient variety $T_\mathfrak{s}/W_\mathfrak{s}$ and proves that there is a surjective map (of sets) $\pi_\mathfrak{s}$ mapping the Bernstein component $\widehat{G}_\mathfrak{s}$ onto $T_\mathfrak{s}/W_\mathfrak{s}$. This map $\pi_\mathfrak{s}$ is known as the *infinitesimal character* or the *central character* or the *cuspidal support map.*

$$\begin{array}{c} \widehat{G}_\mathfrak{s} \\ \downarrow{\scriptstyle \pi_\mathfrak{s}} \\ T_\mathfrak{s}/W_\mathfrak{s} \end{array}$$

The central character encodes essential information about the representation theory of G. Thus any description of $\widehat{G}_\mathfrak{s}$ which did not include a calculation of the central character would be very incomplete.

Remark. $T_\mathfrak{s}$ is an algebraic group, defined over $\mathbb{C}$, which as an algebraic group is non-canonically isomorphic to $\mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$.

$$T_\mathfrak{s} \cong \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$$

Denote the maximal compact subgroup of $T_\mathfrak{s}$ by $T_\mathfrak{s}^{cpt}$. Although the action of $W_\mathfrak{s}$ on $T_\mathfrak{s}$ in general is not as automorphisms of the algebraic group $T_\mathfrak{s}$ (but only as automorphisms of the affine variety $T_\mathfrak{s}$) , this action always preserves $T_\mathfrak{s}^{cpt}$.

$$W_\mathfrak{s} \times T_\mathfrak{s}^{cpt} \longrightarrow T_\mathfrak{s}^{cpt}$$

$T_\mathfrak{s}$, $W_\mathfrak{s}$ and the action

$$W_\mathfrak{s} \times T_\mathfrak{s} \longrightarrow T_\mathfrak{s}$$

of $W_\mathfrak{s}$ on $T_\mathfrak{s}$ are usually quite easily calculated. $\widehat{G}_\mathfrak{s}$ and the central character $\pi_\mathfrak{s}$ can be (and very often are) extremely difficult to describe and calculate.

The ABPS conjecture states that $\widehat{G}_\mathfrak{s}$ has a very simple geometric structure given by the extended quotient.

10. Extended Quotient

Let Γ be a finite group acting on a complex affine variety X as automorphisms of the affine variety

$$\Gamma \times X \to X.$$

The quotient variety X/Γ is obtained by collapsing each orbit to a point. X/Γ is an affine variety.

For $x \in X$, Γ_x denotes the stabilizer group of x:

$$\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

$c(\Gamma_x)$ denotes the set of conjugacy classes of Γ_x. The extended quotient is obtained by replacing the orbit of x by $c(\Gamma_x)$. This is done as follows:

Set $\widetilde{X} = \{(\gamma, x) \in \Gamma \times X : \gamma x = x\}$. $\widetilde{X}$ is an affine variety and is a subvariety of $\Gamma \times X$. The group Γ acts on $\widetilde{X}$:

$$\begin{aligned} \Gamma \times \widetilde{X} &\to \widetilde{X} \\ \alpha(\gamma, x) &= (\alpha\gamma\alpha^{-1}, \alpha x), \qquad \alpha \in \Gamma, \quad (\gamma, x) \in \widetilde{X}. \end{aligned}$$

The extended quotient, denoted $X/\!/\Gamma$, is $\widetilde{X}/\Gamma$. Thus the extended quotient $X/\!/\Gamma$ is the usual quotient for the action of Γ on $\widetilde{X}$. The projection $\widetilde{X} \to X$, $(\gamma, x) \mapsto x$ is Γ-equivariant and so passes to quotient spaces to give a morphism of affine varieties

$$\rho\colon X/\!/\Gamma \to X/\Gamma.$$

This map will be referred to as the projection of the extended quotient onto the ordinary quotient.

The inclusion

$$\begin{aligned} X &\hookrightarrow \widetilde{X} \\ x &\mapsto (e, x) \qquad e = \text{identity element of } \Gamma \end{aligned}$$

is Γ-equivariant and so passes to quotient spaces to give an inclusion of affine varieties $X/\Gamma \hookrightarrow X/\!/\Gamma$. This will be referred to as the inclusion of the ordinary quotient in the extended quotient.
Notation. $X/\!/\Gamma$ with X/Γ removed will be denoted $X/\!/\Gamma - X/\Gamma$.

11. Approximate Statement of ABPS

Conjecture. Let G be a connected reductive p-adic group. Assume that G is quasi-split or that G is an inner form of $GL(n, F)$. Let $\widehat{G}_\mathfrak{s}$ be a Bernstein component of $\widehat{G}$. Let $T_\mathfrak{s}$ and $W_\mathfrak{s}$ be the complex torus and finite group [12] [13] [29] assigned by Bernstein to $\mathfrak{s} \in \mathfrak{B}(G)$. Denote by $\pi_\mathfrak{s}\colon \widehat{G}_\mathfrak{s} \to T_\mathfrak{s}/W_\mathfrak{s}$ and $\rho_\mathfrak{s}\colon T_\mathfrak{s}/\!/W_\mathfrak{s} \to T_\mathfrak{s}/W_\mathfrak{s}$ the central character and the projection of the extended quotient onto the ordinary quotient. Then :

$$\begin{array}{ccc} T_\mathfrak{s}/\!/W_\mathfrak{s} & & \widehat{G}_\mathfrak{s} \\ \Big\downarrow{\scriptstyle \rho_\mathfrak{s}} & \text{and} & \Big\downarrow{\scriptstyle \pi_\mathfrak{s}} \\ T_\mathfrak{s}/W_\mathfrak{s} & & T_\mathfrak{s}/W_\mathfrak{s} \end{array}$$

are almost the same.

As indicated above, $\widehat{G}_{\mathfrak{s}}$ and the central character $\pi_{\mathfrak{s}} \colon \widehat{G}_{\mathfrak{s}} \longrightarrow T_{\mathfrak{s}}/W_{\mathfrak{s}}$ are often very difficult to describe and calculate. The extended quotient $T_{\mathfrak{s}}//W_{\mathfrak{s}}$ and its projection $\rho_{\mathfrak{s}} : T_{\mathfrak{s}}//W_{\mathfrak{s}} \longrightarrow T_{\mathfrak{s}}/W_{\mathfrak{s}}$ onto the ordinary quotient are usually quite easily calculated.

What is the mathematical meaning of "are almost the same"? The precise statement of ABPS uses the extended quotient of the second kind.

12. Extended quotient of the second kind

Let Γ be a finite group acting as automorphisms of a complex affine variety X.

$$\Gamma \times X \to X.$$

For $x \in X$, Γ_x denotes the stabilizer group of x:

$$\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

Let $\mathrm{Irr}(\Gamma_x)$ be the set of (equivalence classes of) irreducible representations of Γ_x. The representations are on finite dimensional vector spaces over the complex numbers $\mathbb{C}$.

The *extended quotient of the second kind*, denoted $(X//\Gamma)_2$, is constructed by replacing the orbit of x (for the given action of Γ on X) by $\mathrm{Irr}(\Gamma_x)$. This is done as follows :

Set $\widetilde{X}_2 = \{(x,\tau) \,\big|\, x \in X \text{ and } \tau \in \mathrm{Irr}(\Gamma_x)\}$. Then Γ acts on $\widetilde{X}_2$.

$$\Gamma \times \widetilde{X}_2 \to \widetilde{X}_2,$$
$$\gamma(x,\tau) = (\gamma x, \gamma_* \tau),$$

where $\gamma_* \colon \mathrm{Irr}(\Gamma_x) \to \mathrm{Irr}(\Gamma_{\gamma x})$. $(X//\Gamma)_2$ is defined by :

$$(X//\Gamma)_2 := \widetilde{X}_2/\Gamma,$$

i.e. $(X//\Gamma)_2$ is the usual quotient for the action of Γ on $\widetilde{X}_2$. The projection $\widetilde{X}_2 \to X$ $(x,\tau) \mapsto x$ is Γ-equivariant and so passes to quotient spaces to give the projection of $(X//\Gamma)_2$ onto X/Γ.

$$\rho_2 \colon (X//\Gamma)_2 \longrightarrow X/\Gamma$$

Denote by triv_x the trivial one-dimensional representation of Γ_x. The inclusion

$$X \hookrightarrow \widetilde{X}_2$$
$$x \mapsto (x, \mathrm{triv}_x)$$

is Γ-equivariant and so passes to quotient spaces to give an inclusion

$$X/\Gamma \hookrightarrow (X//\Gamma)_2$$

This will be referred to as the inclusion of the ordinary quotient in the extended quotient of the second kind.

Let $\mathcal{O}(X)$ be the coordinate algebra of the complex affine variety X and let $\mathcal{O}(X) \rtimes \Gamma$ be the crossed-product algebra for the action of Γ on $\mathcal{O}(X)$. There are canonical bijections

$$\mathrm{Irr}(\mathcal{O}(X) \rtimes \Gamma) \longleftrightarrow \mathrm{Prim}(\mathcal{O}(X) \rtimes \Gamma) \longleftrightarrow (X//\Gamma)_2,$$

where $\mathrm{Prim}(\mathcal{O}(X) \rtimes \Gamma)$ is the set of primitive ideals in $\mathcal{O}(X) \rtimes \Gamma$ and $\mathrm{Irr}(\mathcal{O}(X) \rtimes \Gamma)$ is the set of (equivalence classes of) irreducible representations of $\mathcal{O}(X) \rtimes \Gamma$. The irreducible representation of $\mathcal{O}(X) \rtimes \Gamma$ associated to $(x, \tau) \in (X/\!/\Gamma)_2$ is

$$\mathrm{Ind}_{\mathcal{O}(X)\rtimes\Gamma_x}^{\mathcal{O}(X)\rtimes\Gamma}(\mathbb{C}_x \otimes \tau).$$

Here $\mathbb{C}_x \colon \mathcal{O}(X) \to \mathbb{C}$ is the irreducible representation of $\mathcal{O}(X)$ given by evaluation at $x \in X$. $\mathrm{Ind}_{\mathcal{O}(X)\rtimes\Gamma_x}^{\mathcal{O}(X)\rtimes\Gamma}$ is induction from $\mathcal{O}(X) \rtimes \Gamma_x$ to $\mathcal{O}(X) \rtimes \Gamma$.

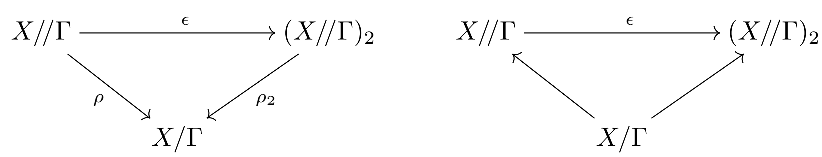
$\mathrm{Prim}(\mathcal{O}(X) \rtimes \Gamma)$ is endowed with the Jacobson topology, which makes it a nonseparated algebraic variety. This structure can be transferred via the canonical bijection $\mathrm{Prim}(\mathcal{O}(X) \rtimes \Gamma) \longleftrightarrow (X/\!/\Gamma)_2$ to $(X/\!/\Gamma)_2$. Hence $(X/\!/\Gamma)_2$ is a non-separated complex algebraic variety. In many examples $(X/\!/\Gamma)_2$ is not an affine variety.

From the non-commutative geometry point of view, $\mathcal{O}(X) \rtimes \Gamma$ is the coordinate algebra of the non-commutative affine algebraic variety $(X/\!/\Gamma)_2$.

13. Comparison of the two extended quotients

With X, Γ as above, there is a non-canonical bijection $\epsilon\colon X/\!/\Gamma \to (X/\!/\Gamma)_2$ with commutativity in the diagrams

(1)



To construct the non-canonical bijection ϵ, some choices must be made. Let ψ be a family of bijections (one bijection ψ_x for each $x \in X$)

$$\psi_x : c(\Gamma_x) \to \mathrm{Irr}(\Gamma_x)$$

such that for all $x \in X$:

(1) $\psi_x(e) = \mathrm{triv}_x \qquad e = \text{identity element of } \Gamma_x$
(2) $\psi_{\gamma x}([\gamma g \gamma^{-1}]) = \phi_x([g]) \circ \mathrm{Ad}_\gamma^{-1}$ for all $g \in \Gamma_x, \gamma \in \Gamma$
(3) $\psi_x = \psi_y$ if $\Gamma_x = \Gamma_y$ and x, y belong to the same connected component of the variety $X^{\Gamma_x} := \{z \in X \mid \gamma z = z \;\; \forall \gamma \in \Gamma_x\}$

Such a family of bijections will be referred to as a c-Irr system. ψ induces a map $\widetilde{X} \to \widetilde{X}_2$ which preserves the X-coordinates. By property (2) this map is Γ-equivariant, so it descends to give a bijection

$$\epsilon = \epsilon_\psi : X/\!/\Gamma \to (X/\!/\Gamma)_2.$$

such that the diagrams (1) commute. Property (3) is not really needed, but serves to exclude some rather awkward and unpleasant choices of ψ.

As remarked above, the crossed-product algebra $\mathcal{O}(X) \rtimes \Gamma$ can be viewed as the coordinate algebra of the non-commutative affine algebraic variety $(X/\!/\Gamma)_2$.

There are some intriguing similarities and differences between the two finite-type $\mathcal{O}(X/\Gamma)$-algebras $\mathcal{O}(X/\!/\Gamma)$ and $\mathcal{O}(X) \rtimes \Gamma$. In many examples (e.g. if X is connected and the action of Γ on X is neither trivial nor free) these two algebras are not

Morita equivalent. However, these two algebras always have the same periodic cyclic homology.

$$HP_*(\mathcal{O}(X) \rtimes \Gamma) \cong HP_*(\mathcal{O}(X/\!/\Gamma)) \cong H^*(X/\!/\Gamma; \mathbb{C})$$

$H^*(X/\!/\Gamma; \mathbb{C})$ is the cohomology (in the usual sense of algebraic topology), with coefficients $\mathbb{C}$, of the underlying locally compact Hausdorff topological space of the complex affine variety $X/\!/\Gamma$.

In the example relevant to the representation theory of reductive p-adic groups (i.e. $\Gamma = W_{\mathfrak{s}}$, $X = T_{\mathfrak{s}}$) these two finite-type $\mathcal{O}(X/\Gamma)$-algebras are very often — conjecturally always — equivalent via a weakening of Morita equivalence referred to as "geometric equivalence". See the appendix for the definition of "geometric equivalence".

The finite group $W_{\mathfrak{s}}$ is often an extended finite Coxeter group i.e. $W_{\mathfrak{s}}$ is often a semi-direct product for the action of a finite abelian group A on a finite Weyl group W:

$$W_{\mathfrak{s}} = W \rtimes A.$$

Due to this, in many examples there is a clear preferred choice of c-Irr system for the action of $W_{\mathfrak{s}}$ on $T_{\mathfrak{s}}$.

For the groups $GL(m, D)$ every Bernstein component has the two extended quotients $T_{\mathfrak{s}}/\!/W_{\mathfrak{s}}$, $(T_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_2$ canonically in bijection. The reason for this is that all the isotropy groups for the action of $W_{\mathfrak{s}}$ on $T_{\mathfrak{s}}$ are finite Cartesian products of symmetric groups. The classical theory of Young tableaux (or the Springer correspondence) then applies to give a canonical bijection between Irr and the set of conjugacy classes — i.e. a canonical c-Irr system for the action of $W_{\mathfrak{s}}$ on $T_{\mathfrak{s}}$.

<u>Remark.</u> If S is a set and Γ is a group acting on S, then (in an evident way) the two extended quotients $S/\!/\Gamma$ and $(S/\!/\Gamma)_2$ can be formed.

14. Statement of the ABPS conjecture

Let G be a connected reductive p-adic group. Assume that G is quasi-split or that G is an inner form of $GL(n, F)$. Let $\mathfrak{s}$ be a point in the Bernstein spectrum of G.

$$\mathfrak{s} \in \mathfrak{B}(G) = \pi_o \mathrm{Prim}(\mathcal{H}G)$$

The ABPS conjecture [1] - [6] consists of the following five statements.

(1) The central character

$$\pi_{\mathfrak{s}} : \widehat{G}_{\mathfrak{s}} \to T_{\mathfrak{s}}/W_{\mathfrak{s}}$$

is one-to-one if and only if the action of $W_{\mathfrak{s}}$ on $T_{\mathfrak{s}}$ is free.

(2) There is a canonically defined commutative triangle

$$\begin{array}{ccc} & (T_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_2 & \\ \swarrow & & \searrow \\ \widehat{G}_{\mathfrak{s}} & \longrightarrow & \{\text{Langlands parameters}\}^{\mathfrak{s}}/{}^L G^{\,0} \end{array}$$

in which the left slanted arrow is bijective and the horizontal arrow is the map of the local Langlands correspondence.
{Langlands parameters}$^{\mathfrak{s}}$ is those Langlands parameters whose L-packets have non-empty intersection with $\widehat{G}_{\mathfrak{s}}$.
The maps in this commutative triangle are canonical.

(3) The canonical bijection

$$(T_{\mathfrak{s}}//W_{\mathfrak{s}})_2 \longrightarrow \widehat{G}_{\mathfrak{s}}$$

comes from a canonical geometric equivalence of the two unital finite-type $\mathcal{O}(T_{\mathfrak{s}}/W_{\mathfrak{s}})$-algebras $\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}$ and $\mathcal{H}_{\mathfrak{s}}$. See the appendix for details on "geometric equivalence".

(4) The canonical bijection

$$(T_{\mathfrak{s}}//W_{\mathfrak{s}})_2 \longrightarrow \widehat{G}_{\mathfrak{s}}$$

maps $(T_{\mathfrak{s}}^{cpt}//W_{\mathfrak{s}})_2$ onto $\widehat{G}_{\mathfrak{s}} \cap \widehat{G}_{temp}$.

(5) A c-Irr system can be chosen for the action of $W_{\mathfrak{s}}$ on $T_{\mathfrak{s}}$ such that the resulting (non-canonical) bijection

$$\epsilon\colon T_{\mathfrak{s}}//W_{\mathfrak{s}} \longrightarrow (T_{\mathfrak{s}}//W_{\mathfrak{s}})_2$$

when composed with the canonical bijection $(T_{\mathfrak{s}}//W_{\mathfrak{s}})_2 \to \widehat{G}_{\mathfrak{s}}$ gives a bijection

$$\mu_{\mathfrak{s}} : T_{\mathfrak{s}}//W_{\mathfrak{s}} \longrightarrow \widehat{G}_{\mathfrak{s}}$$

which has the following six properties:

Notation for Property 1:
Within the smooth dual $\widehat{G}$, there is the tempered dual
$\widehat{G}_{temp}$ = {smooth tempered irreducible representations of G} $/\sim$
$T_{\mathfrak{s}}^{cpt}$ = maximal compact subgroup of $T_{\mathfrak{s}}$.
$T_{\mathfrak{s}}^{cpt}$ is a compact real torus. The action of $W_{\mathfrak{s}}$ on $T_{\mathfrak{s}}$ preserves $T_{\mathfrak{s}}^{cpt}$, so the compact orbifold $T_{\mathfrak{s}}^{cpt}//W_{\mathfrak{s}}$ can be formed.

Property 1 of the bijection $\mu_{\mathfrak{s}}$:
The bijection $\mu_{\mathfrak{s}} : T_{\mathfrak{s}}//W_{\mathfrak{s}} \longrightarrow \widehat{G}_{\mathfrak{s}}$ maps $T_{\mathfrak{s}}^{cpt}//W_{\mathfrak{s}}$ onto $\widehat{G}_{\mathfrak{s}} \cap \widehat{G}_{temp}$, and hence restricts to a bijection

$$\mu_{\mathfrak{s}}\colon T_{\mathfrak{s}}^{cpt}//W_{\mathfrak{s}} \longleftrightarrow \widehat{G}_{\mathfrak{s}} \cap \widehat{G}_{temp}$$

Property 2 of the bijection $\mu_{\mathfrak{s}}$:
For many $\mathfrak{s} \in \mathfrak{B}(G)$ the diagram

$$\begin{array}{ccc} T_{\mathfrak{s}}//W_{\mathfrak{s}} & \xrightarrow{\ \mu_{\mathfrak{s}}\ } & \widehat{G}_{\mathfrak{s}} \\ & {\scriptstyle\rho_{\mathfrak{s}}}\searrow \quad \swarrow{\scriptstyle\pi_{\mathfrak{s}}} & \\ & T_{\mathfrak{s}}/W_{\mathfrak{s}} & \end{array}$$

does not commute.

Property 3 of the bijection $\mu_{\mathfrak{s}}$:

In the possibly non-commutative diagram

$$\begin{array}{ccc} T_{\mathfrak{s}}/\!/W_{\mathfrak{s}} & \xrightarrow{\mu_{\mathfrak{s}}} & \widehat{G}_{\mathfrak{s}} \\ & {\scriptstyle\rho_{\mathfrak{s}}}\searrow \quad \swarrow{\scriptstyle\pi_{\mathfrak{s}}} & \\ & T_{\mathfrak{s}}/W_{\mathfrak{s}} & \end{array}$$

the bijection $\mu_{\mathfrak{s}} : T_{\mathfrak{s}}/\!/W_{\mathfrak{s}} \longrightarrow \widehat{G}_{\mathfrak{s}}$ is continuous where the affine variety $T_{\mathfrak{s}}/\!/W_{\mathfrak{s}}$ has the Zariski topology and $\widehat{G}_{\mathfrak{s}} \subset \mathrm{Prim}(\mathcal{H}G)$ has the Jacobson topology — and the composition

$$\pi_{\mathfrak{s}} \circ \mu_{\mathfrak{s}} : T_{\mathfrak{s}}/\!/W_{\mathfrak{s}} \longrightarrow T_{\mathfrak{s}}/W_{\mathfrak{s}}$$

is a morphism of complex affine algebraic varieties.

Property 4 of the bijection $\mu_{\mathfrak{s}}$:
There is an algebraic family

$$\theta_z : T_{\mathfrak{s}}/\!/W_{\mathfrak{s}} \longrightarrow T_{\mathfrak{s}}/W_{\mathfrak{s}}$$

of finite morphisms of algebraic varieties, with $z \in \mathbb{C}^\times$, such that

$$\theta_1 = \rho_{\mathfrak{s}}, \quad \theta_{\sqrt{q}} = \pi_{\mathfrak{s}} \circ \mu_{\mathfrak{s}}, \quad \text{and} \quad \theta_{\sqrt{q}}(T_{\mathfrak{s}}/\!/W_{\mathfrak{s}} - T_{\mathfrak{s}}/W_{\mathfrak{s}}) = R(\pi_{\mathfrak{s}}).$$

Here q is the order of the residue field of the p-adic field F over which G is defined and $R(\pi_{\mathfrak{s}}) \subset T_{\mathfrak{s}}/W_{\mathfrak{s}}$ is the sub-variety of reducibility [4]. Setting

$$Y_z = \theta_z(T_{\mathfrak{s}}/\!/W_{\mathfrak{s}} - T_{\mathfrak{s}}/W_{\mathfrak{s}})$$

a flat family of sub-schemes of $T_{\mathfrak{s}}/W_{\mathfrak{s}}$ is obtained with

$$Y_1 = R(\rho_{\mathfrak{s}}), \qquad Y_{\sqrt{q}} = R(\pi_{\mathfrak{s}}).$$

Remark. Both $\rho_{\mathfrak{s}}$ and $\pi_{\mathfrak{s}}$ are surjective finite-to-one maps. For $x \in T_{\mathfrak{s}}/W_{\mathfrak{s}}$, denote by $\#(x, \rho_{\mathfrak{s}})$, $\#(x, \pi_{\mathfrak{s}})$ the number of points in the pre-image of x using $\rho_{\mathfrak{s}}$, $\pi_{\mathfrak{s}}$. The numbers $\#(x, \pi_{\mathfrak{s}})$ are of interest in representation theory. Within $T_{\mathfrak{s}}/W_{\mathfrak{s}}$ the algebraic sub-varieties $R(\rho_{\mathfrak{s}})$, $R(\pi_{\mathfrak{s}})$ are defined by

$$R(\rho_{\mathfrak{s}}) := \{x \in T_{\mathfrak{s}}/W_{\mathfrak{s}} \mid \#(x, \rho_{\mathfrak{s}}) > 1\}$$

$$R(\pi_{\mathfrak{s}}) := \{x \in T_{\mathfrak{s}}/W_{\mathfrak{s}} \mid \#(x, \pi_{\mathfrak{s}}) > 1\}$$

Property 5 of the bijection $\mu_{\mathfrak{s}}$ (Correcting cocharacters):
For each irreducible component $\mathbf{c}$ of the affine variety $T_{\mathfrak{s}}/\!/W_{\mathfrak{s}}$ there is a cocharacter (i.e. a homomorphism of algebraic groups)

$$h_c : \mathbb{C}^\times \longrightarrow T_{\mathfrak{s}}$$

such that

$$\theta_z[w, t] = b(h_c(z) \cdot t)$$

for all $[w, t] \in \mathbf{c}$.
Let $b\colon T_{\mathfrak{s}} \longrightarrow T_{\mathfrak{s}}/W_{\mathfrak{s}}$ be the quotient map. Here, as above, points of $\widetilde{T}_{\mathfrak{s}}$ are pairs (w, t) with $w \in W_{\mathfrak{s}}$, $t \in T_{\mathfrak{s}}$ and $wt = t$. $[w, t]$ is the point in $T_{\mathfrak{s}}/\!/W_{\mathfrak{s}}$ obtained by applying the quotient map $\widetilde{T}_{\mathfrak{s}} \to T_{\mathfrak{s}}/\!/W_{\mathfrak{s}}$ to (w, t).
Remark. The equality

$$\theta_z[w, t] = b(h_c(z) \cdot t)$$

is to be interpreted thus:
Let Z_1, Z_2, ..., Z_r be the irreducible components of the affine variety $T_\mathfrak{s}//W_\mathfrak{s}$ and let h_1, h_2, ..., h_r be the cocharacters as in the statement of Property 5. Let

$$\nu_\mathfrak{s} \colon \widetilde{T_\mathfrak{s}} \longrightarrow T_\mathfrak{s}//W_\mathfrak{s}$$

be the quotient map.
Then irreducible components X_1, X_2, ..., X_r of the affine variety $\widetilde{T_\mathfrak{s}}$ can be chosen with

- $\nu_\mathfrak{s}(X_j) = Z_j$ for $j = 1, 2, \ldots, r$
- For each $z \in \mathbb{C}^\times$ the map $m_z \colon X_j \to T_\mathfrak{s}/W_\mathfrak{s}$, which is the composition

$$\begin{array}{ccccc} X_j & \longrightarrow & T_\mathfrak{s} & \longrightarrow & T_\mathfrak{s}/W_\mathfrak{s} \\ (w,t) & \longmapsto & h_j(z)t & \longmapsto & b(h_j(z)t), \end{array}$$

makes the diagram

$$\begin{array}{ccc} X_j & \xrightarrow{\nu_\mathfrak{s}} & Z_j \\ & {\scriptstyle m_z} \searrow \quad \swarrow {\scriptstyle \theta_z} & \\ & T_\mathfrak{s}/W_\mathfrak{s} & \end{array}$$

commutative. Note that $h_j(z)t$ is the product of $h_j(z)$ and t in the algebraic group $T_\mathfrak{s}$.

Remark. The conjecture asserts that to calculate the central character

$$\pi_\mathfrak{s} \colon \widehat{G}_\mathfrak{s} \longrightarrow T_\mathfrak{s}/W_\mathfrak{s}$$

two steps suffice:
Step 1: Calculate the projection of the extended quotient onto the ordinary quotient

$$\rho_\mathfrak{s} \colon T_\mathfrak{s}//W_\mathfrak{s} \longrightarrow T_\mathfrak{s}/W_\mathfrak{s}$$

Step 2: Determine the correcting cocharacters.

The cocharacter assigned to $T_\mathfrak{s}/W_\mathfrak{s} \hookrightarrow T_\mathfrak{s}//W_\mathfrak{s}$ is always the trivial cocharacter mapping $\mathbb{C}^\times$ to the unit element of $T_\mathfrak{s}$. So all the non-trivial correcting is taking place on $T_\mathfrak{s}//W_\mathfrak{s} - T_\mathfrak{s}/W_\mathfrak{s}$.

Notation for Property 6.
If S and V are sets, a *labelling* of S by V is a map of sets $\lambda \colon S \to V$.
Property 6 of the bijection $\mu_\mathfrak{s}$ (L-packets):
As in Property 5, let $\{Z_1, \ldots, Z_r\}$ be the irreducible components of the affine variety $T_\mathfrak{s}//W_\mathfrak{s}$, and let $\{h_1, h_2, \ldots, h_r\}$ be the correcting cocharacters.
Then a finite set V and a labelling $\lambda \colon \{Z_1, Z_2, \ldots, Z_r\} \to V$ exist such that $\lambda(Z_i) = \lambda(Z_j) \Longrightarrow h_i = h_j$ and:
For every two points $[w,t]$ and $[w',t]$ of $T_\mathfrak{s}//W_\mathfrak{s}$:

$$\mu_\mathfrak{s}[w,t] \text{ and } \mu_\mathfrak{s}[w',t'] \text{ are in the same } L\text{-packet}$$

if and only

(i) $\theta_z[w,t] = \theta_z[w',t']$ for all $z \in \mathbb{C}^\times$;
(ii) $\lambda[w,t] = \lambda[w',t']$, where λ has been lifted to a labelling of $T_\mathfrak{s}//W_\mathfrak{s}$ in the evident way.

Remark. An L-packet can have non-empty intersection with more than one Bernstein component. The conjecture does not address this issue. The conjecture only describes the intersections of L-packets with any one given Bernstein component.

In brief, the conjecture asserts that — once a Bernstein component has been fixed — intersections of L-packets with that Bernstein component consisting of more than one point are "caused" by repetitions among the correcting cocharacters. If, for any one given Bernstein component, the correcting cocharacters $h_1, h_2, \ldots, h_r$ are all distinct, then (according to the conjecture) the intersections of L-packets with that Bernstein component are singletons.

A Langlands parameter

$$\mathcal{W}_F \times SL(2,\mathbb{C}) \longrightarrow {}^LG$$

determines a homomorphism of complex algebraic groups

$$SL(2,\mathbb{C}) \longrightarrow {}^LG^{\,0}$$

Let $T({}^LG^{\,0})$ be the maximal torus of ${}^LG^{\,0}$. By restricting the homomorphism of complex algebraic groups $SL(2,\mathbb{C}) \to {}^LG^{\,0}$ to the maximal torus of $SL(2,\mathbb{C})$ a cocharacter

$$\mathbb{C}^\times \longrightarrow T({}^LG^{\,0})$$

is obtained. In examples, these are the correcting cocharacters.

15. Two Theorems

As above D is an F-division algebra of dimension d^2 over its center F. With m a positive integer, $GL(m,D)$ denotes the connected reductive p-adic group consisting of all $m \times m$ invertible matrices with entries in D. Except for $GL(n,F)$, these groups are non-split. $GL(m,D)$ is an inner form of $GL(md,F)$. Hence all the groups $GL(m,D)$ are inner forms of connected split reductive p-adic groups.

Theorem 1. *The ABPS conjecture is valid for $G = GL(m,D)$. In this case, for each Bernstein component $\widehat{G}_{\mathfrak{s}} \subset \widehat{G}$, all three maps in the commutative triangle*

$$\begin{array}{ccc} & (T_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_2 & \\ \swarrow & & \searrow \\ \widehat{G}_{\mathfrak{s}} & \longrightarrow & \{\text{Langlands parameters}\}^{\mathfrak{s}}/{}^LG^{\,0} \end{array}$$

are bijective.

Assume now that G is connected and split.
As in section 6 above, let $\mathbb{T}$ be a maximal p-adic torus in G. An irreducible smooth representation ϕ of G

$$\phi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

is in the *principal series* iff $\exists$ a smooth irreducible representation χ of $\mathbb{T}$, i.e. a smooth character

$$\chi \colon \mathbb{T} \longrightarrow \mathbb{C}^\times$$

with ϕ a sub-quotient of $\mathrm{Ind}_{\mathbb{T}}^G(\chi)$. Here $\mathrm{Ind}_{\mathbb{T}}^G$ is smooth parabolic induction (of smooth representations) from $\mathbb{T}$ to G. Any Bernstein component $\widehat{G}_{\mathfrak{s}}$ in $\widehat{G}$ either has empty intersection with the principal series or is completely contained within

the principal series.
The Langlands parameters

$$\varphi\colon \mathcal{W}_F \times SL(2,\mathbb{C}) \longrightarrow {}^L G^{\,0}$$

for the principal series have a very simple form. They are those Langlands parameters φ such that when restricted to $\mathcal{W}_F$, φ is trivial on $\mathcal{W}_F^{der}$ — i.e. the restriction of φ to $\mathcal{W}_F$ factors through $\mathcal{W}_F^{ab}$.

$$\mathcal{W}_F^{ab} := \mathcal{W}_{\mathcal{F}}/\mathcal{W}_F^{der}$$

Due to the isomorphism of local class field theory

$$\mathcal{W}_F^{ab} \cong F^\times$$

such a Langlands parameter φ can be viewed as a map

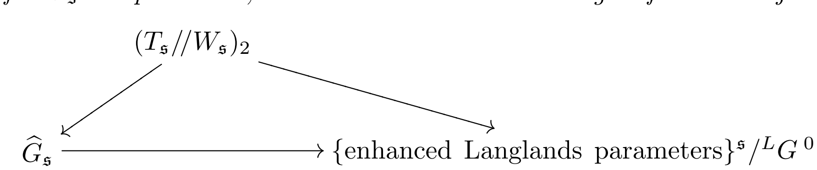
$$\varphi\colon F^\times \times SL(2,\mathbb{C}) \longrightarrow {}^L G^{\,0}$$

An *enhancement* [22, 20, 1, 32] of such a φ is a pair (φ,σ) where σ is an irreducible representation of the finite group $\pi_0(Z_{^L G^{\,0}}(\text{Image}\varphi))$ which occurs in $H^*(\mathcal{BV}(\varphi);\mathbb{C})$. The notation is:
Notation.

- $\text{Image}(\varphi) = \varphi(F^\times \times SL(2,\mathbb{C}))$ is the image of φ.
- $Z_{^L G^{\,0}}(\text{Image}\,\varphi)$ is the centralizer (in ${}^L G^{\,0}$) of $\text{Image}(\varphi)$.
- $\pi_0(Z_{^L G^{\,0}}(\text{Image}\,\varphi))$ is the finite group whose elements are the connected components of $Z_{^L G^{\,0}}(\text{Image}\,\varphi)$
- $\mathcal{BV}(\varphi)$ is the algebraic variety of all Borel subgroups of ${}^L G^{\,0}$ which contain $\varphi(B_2)$ — where B_2 is the standard Borel subgroup in $SL(2,\mathbb{C})$ i.e. B_2 is the subgroup of $SL(2,\mathbb{C})$ consisting of all upper triangular matrices in $SL(2,\mathbb{C})$. A*Borel subgroup* of ${}^L G^{\,0}$ is a connected solvable algebraic subgroup B of ${}^L G^{\,0}$ which is maximal among the connected solvable algebraic subgroups of ${}^L G^{\,0}$.
- $H^*(\mathcal{BV}(\varphi);\mathbb{C})$ is the cohomology (in the usual sense of algebraic topology), with coefficients $\mathbb{C}$, of the underlying locally compact Hausdorff space of the complex algebraic variety $\mathcal{BV}(\varphi)$.
- $Z_{^L G^{\,o}}(\text{Image}\,\varphi)$ acts on $\mathcal{BV}(\varphi)$ by conjugation, thus determining a representation (whose vector space is $H^*(\mathcal{BV}(\varphi)\,;\,\mathbb{C})$) of the finite group $\pi_0(Z_{^L G^{\,0}}(\text{Image}\,\varphi))$.

A hypothesis in Theorem 2 is that the p-adic field F satisfies a mild restriction on its residual characteristic, depending on G. Granted this restriction, Theorem 2 states that ABPS is valid for any Bernstein component $\widehat{G}_{\mathfrak{s}}$ in the principal series.

Theorem 2. *Let G be a connected split reductive p-adic group. Assume that the residual characteristic of the local field F is not a torsion prime for G. Let $\widehat{G}_{\mathfrak{s}}$ be a Bernstein component in the principal series of G. Then the ABPS conjecture is valid for $\widehat{G}_{\mathfrak{s}}$. In particular, there is a commutative triangle of natural bijections*



In this triangle {enhanced Langlands parameters}$^{\mathfrak{s}}/{}^LG^{\,0}$ *is the set of enhanced Langlands parameters for the Bernstein component* $\widehat{G}_{\mathfrak{s}}$ *(modulo conjugation by* ${}^LG^{\,0}$*).*

Theorem 2 extends and generalizes [20] and [28]. Note that [28] assumes that for ramified inducing characters (i.e. all Bernstein components except the Iwahori component) the center of G is connected. In Theorem 2 there is no hypothesis that the center of G is connected.

Brief indication of proof. In the commutative triangle of Theorem 2, the left slanted arrow is defined and is proved bijective by using the representation theory of affine Hecke algebras [31]. The right slanted arrow is defined and is proved bijective by applying the Springer correspondence for affine Weyl groups [19].

Proofs of Theorems 1 and 2 are given in [7].

What happens if G is a connected reductive p-adic group which is not quasi-split? Many Bernstein components $\widehat{G}_{\mathfrak{s}}$ have the geometric structure as in the statement of ABPS above. However, in some examples there are Bernstein components $\widehat{G}_{\mathfrak{s}}$ which are canonically in bijection not with $(T_{\mathfrak{s}}//W_{\mathfrak{s}})_2$ but with $(T_{\mathfrak{s}}//W_{\mathfrak{s}})_2$-twisted by a 2-cocyle. See [7] for the definition of $(T_{\mathfrak{s}}//W_{\mathfrak{s}})_2$-twisted by a 2-cocycle. The authors of this note are currently formulating a precise statement of ABPS for connected reductive p-adic groups which are not quasi-split. Our precise statement will be given elsewhere.

16. Appendix : Geometric Equivalence

Let X be a complex affine variety and let $k = \mathcal{O}(X)$ be its coordinate algebra. Equivalently, k is a unital algebra over the complex numbers which is commutative, finitely generated, and nilpotent-free. A k-algebra is an algebra A over the complex numbers which is a k-module (with an evident compatibility between the algebra structure of A and the k-module structure of A). A is of *finite type* if as a k-module A is finitely generated. This appendix will introduce — for finite type k-algebras — a weakening of Morita equivalence called *geometric equivalence.*

The new equivalence relation preserves the primitive ideal space (i.e. the set of isomorphism classes of simple A-modules) and the periodic cyclic homology. However, the new equivalence relation permits a tearing apart of strata in the primitive ideal space which is not allowed by Morita equivalence. The ABPS conjecture asserts that the finite type algebra which Bernstein constructs [12][13][29] for any given Bernstein component of a reductive p-adic group is geometrically equivalent to the coordinate algebra of the associated extended quotient — and that the geometric equivalence can be chosen so that the resulting bijection between the Bernstein component and the extended quotient has properties as in the statement of ABPS.

16.1. k-algebras.

Let X be a complex affine variety and $k = \mathcal{O}(X)$ its coordinate algebra.

A *k-algebra* is a $\mathbb{C}$-algebra A such that A is a unital (left) k-module with:

$$\lambda(\omega a) = \omega(\lambda a) = (\lambda\omega)a \quad \forall(\lambda, \omega, a) \in \mathbb{C} \times k \times A$$

and

$$\omega(a_1a_2) = (\omega a_1)a_2 = a_1(\omega a_2) \quad \forall(\omega, a_1, a_2) \in k \times A \times A.$$

Denote the center of A by $Z(A)$

$$Z(A) := \{c \in A \mid ca = ac \; \forall a \in A\}$$

k-algebras are not required to be unital.
Remark.Let A be a unital k-algebra. Denote the unit of A by 1_A. $\omega \mapsto \omega 1_A$ is then a unital morphism of $\mathbb{C}$-algebras $k \to Z(A)$. Thus a unital k-algebra is a unital $\mathbb{C}$-algebra A with a given unital morphism of $\mathbb{C}$-algebras

$$k \longrightarrow Z(A).$$

Let A, B be two k-algebras. A morphism of k-algebras is a morphism of $\mathbb{C}$-algebras

$$f\colon A \to B$$

which is also a morphism of (left) k-modules,

$$f(\omega a) = \omega f(a) \quad \forall(\omega, a) \in k \times A.$$

Let A be a k-algebra. A representation of A [or a (left) A-module] is a $\mathbb{C}$-vector space V with given morphisms of $\mathbb{C}$-algebras

$$A \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V) \qquad\qquad k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$$

such that

(1) $k \to \mathrm{Hom}_{\mathbb{C}}(V, V)$ is unital

and

(2) $(\omega a)v = \omega(av) = a(\omega v) \; \forall(\omega, a, v) \in k \times A \times V$.

Two representations

$$A \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V_1, V_1) \qquad\qquad k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V_1, V_1)$$

and

$$A \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V_2, V_2) \qquad\qquad k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V_2, V_2)$$

are equivalent (or isomorphic) if $\exists$ an isomorphism of $\mathbb{C}$ vector spaces $T\colon V_1 \to V_2$ which intertwines the two A-actions and the two k-actions.

A representation is irreducible if $A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$ is not the zero map and $\nexists$ a sub-$\mathbb{C}$-vector space W of V with:

$$\{0\} \neq W \quad , \quad W \neq V$$

and

$$\omega w \in W \quad \forall\, (\omega, w) \in k \times W$$

and

$$aw \in W \quad \forall\, (a, w) \in A \times W$$

$\mathrm{Irr}(A)$ denotes the set of equivalence classes of irreducible representations of A.

16.2. **Spectrum preserving morphisms of k-algebras.** Definition. An ideal I in a k-algebra A is a *k-ideal* if $\omega a \in I \; \forall\, (\omega, a) \in k \times I$.
An ideal $I \subset A$ is *primitive* if $\exists$ an irreducible representation

$$\varphi\colon A \to \mathrm{Hom}_{\mathbb{C}}(V,V) \qquad\qquad k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V,V)$$

with

$$I = \mathrm{Kernel}(\varphi)$$

That is,

$$0 \to I \hookrightarrow A \xrightarrow{\varphi} \mathrm{Hom}_{\mathbb{C}}(V,V)$$

is exact.
Remark. Any primitive ideal is a k-ideal. $\mathrm{Prim}(A)$ denotes the set of primitive ideals in A. The map $\mathrm{Irr}(A)\to$ $\mathrm{Prim}(A)$ which sends an irreducible representation to its primitive ideal is a bijection if A is a finite type k-algebra. Since k is Noetherian, any k-ideal in a finite type k-algebra A is itself a finite type k-algebra.

Definition. Let A, B be two finite type k-algebras, and let $f\colon A \to B$ be a morphism of k-algebras. f is *spectrum preserving* if

(1) Given any primitive ideal $I \subset B$, $\exists$ a unique primitive ideal $L \subset A$ with $L \supset f^{-1}(I)$

and

(2) The resulting map
$$\mathrm{Prim}(B) \to \mathrm{Prim}(A)$$
is a bijection.

Definition. Let A, B be two finite type k-algebras, and let $f\colon A \to B$ be a morphism of k-algebras. f is *spectrum preserving with respect to filtrations* if $\exists$ k-ideals

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = A \qquad \text{in } A$$

and k ideals

$$0 = L_0 \subset L_1 \subset \cdots \subset L_{r-1} \subset L_r = B \qquad \text{in } B$$

with $f(I_j) \subset L_j$, $(j = 1, 2, \ldots, r)$ and $I_j/I_{j-1} \to L_j/L_{j-1}$, $(j = 1, 2, \ldots, r)$ is spectrum preserving.

16.3. **Algebraic variation of k-structure.** Notation. If A is a $\mathbb{C}$-algebra, $A[t,t^{-1}]$ is the $\mathbb{C}$-algebra of Laurent polynomials in the indeterminate t with coefficients in A.

Definition. Let A be a unital $\mathbb{C}$-algebra, and let

$$\Psi\colon k \to Z\left(A[t,t^{-1}]\right)$$

be a unital morphism of $\mathbb{C}$-algebras. Note that $Z\left(A[t,t^{-1}]\right) = Z(A)[t,t^{-1}]$. For $\zeta \in \mathbb{C}^\times = \mathbb{C} - \{0\}$, $\mathrm{ev}(\zeta)$ denotes the "evaluation at ζ" map:

$$\mathrm{ev}(\zeta)\colon A[t,t^{-1}] \to A$$
$$\sum a_j t^j \mapsto \sum a_j \zeta^j$$

Consider the composition

$$k \xrightarrow{\Psi} Z\left(A[t,t^{-1}]\right) \xrightarrow{\mathrm{ev}(\zeta)} Z(A).$$

Denote the unital k-algebra so obtained by A_ζ. The underlying $\mathbb{C}$-algebra of A_ζ is A. Assume that for all $\zeta \in \mathbb{C}^\times$, A_ζ is a finite type k-algebra. Then for $\zeta, \zeta' \in \mathbb{C}^\times$, $A_{\zeta'}$ is obtained from A_ζ by an *algebraic variation of k-structure.*

16.4. **Definition and examples.** With k fixed, geometric equivalence (for finite type k-algebras) is the equivalence relation generated by the two elementary moves:

- Spectrum preserving morphism with respect to filtrations
- Algebraic variation of k-structure

Thus two finite type k-algebras A, B are *geometrically equivalent* if $\exists$ a finite sequence $A = A_0, A_1, \dots, A_r = B$ with each A_j a finite type k-algebra such that for $j = 0, 1, \dots, r-1$ one of the following three possibilities is valid:

(1) A_{j+1} is obtained from A_j by an algebraic variation of k-structure.
(2) There is a spectrum preserving morphism with respect to filtrations $A_j \to A_{j+1}$.
(3) There is a spectrum preserving morphism with respect to filtrations $A_{j+1} \to A_j$.

To give a geometric equivalence relating A and B, the finite sequence of elementary steps (including the filtrations) must be given. Once this has been done, a bijection of the primitive ideal spaces and an isomorphism of periodic cyclic homology [10][11]are determined.

$$\mathrm{Prim}(A) \longleftrightarrow \mathrm{Prim}(B) \qquad HP_*(A) \cong HP_*(B)$$

Remark. If two unital finite type k-algebras A, B are Morita equivalent (as k-algebras) then they are geometrically equivalent [7].

Example 1. Let G be a connected reductive complex Lie group with maximal torus T. W denotes the Weyl group

$$W = N_G(T)/T$$

and $X^*(T)$ is the character group of T. $N_G(T)$ is the normalizer (in G) of T. The semi-direct product $X^*(T) \rtimes W$ is the affine Weyl group of G. For each non-zero complex number q, there is the affine Hecke algebra $\mathcal{H}_q(G)$. This is an affine Hecke algebra with equal parameters and $\mathcal{H}_1(G)$ is the group algebra of the affine Weyl group:

$$\mathcal{H}_1(G) = \mathbb{C}[X^*(T) \rtimes W] = \mathcal{O}(T) \rtimes W$$

Using the action of W on T, form the quotient variety T/W and let k be its coordinate algebra,

$$k = \mathcal{O}(T/W)$$

For all $q \in \mathbb{C}^\times$, $\mathcal{H}_q(G)$ is a unital finite type k-algebra. Let J be Lusztig's asymptotic algebra. [23, 24, 25].
Except for q in a finite set of roots of unity (none of which is 1) Lusztig constructs a morphism of k-algebras

$$\phi_q \colon \mathcal{H}_q(G) \longrightarrow J$$

which is spectrum preserving with respect to filtrations. The algebra $\mathcal{H}_q(G)$ is viewed as a k-algebra via the canonical isomorphism

$$\mathcal{O}(T/W) \cong Z(\mathcal{H}_q(G))$$

Lusztig's map ϕ_q maps $Z(\mathcal{H}_q(G))$ to $Z(J)$ and thus determines a unique k-structure for J such that the map ϕ_q is a morphism of k-algebras. J with this k-structure will be denoted J_q. $\mathcal{H}_q(G)$ is then geometrically equivalent to $\mathcal{H}_1(G)$ by the three elementary steps

$$\mathcal{H}_q(G) \rightsquigarrow J_q \rightsquigarrow J_1 \rightsquigarrow \mathcal{H}_1(G).$$

The second elementary step (i.e. passing from J_q to J_1) is an algebraic variation of k-structure. The first elementary step uses Lusztig's map ϕ_q, and the third elementary step uses Lusztig's map ϕ_1. Hence (provided q is not in the exceptional set of roots of unity—none of which is 1) $\mathcal{H}_q(G)$ is geometrically equivalent to $\mathcal{H}_1(G) = \mathbb{C}[X^*(T) \rtimes W] = \mathcal{O}(T) \rtimes W$.
As observed in section 12 above, $\mathrm{Irr}(\mathcal{H}_1(G)) = (T/\!/W)_2$. Thus the geometric equivalence of $\mathcal{H}_q(G)$ to $\mathcal{H}_1(G)$ determines a bijection

$$(T/\!/W)_2 \longleftrightarrow \mathrm{Irr}(\mathcal{H}_q(G))$$

Example 2. With notation as in example 1, let $\mathcal{H}_{\mathbf{q}}(X^*(T) \rtimes W)$ be the affine Hecke algebra of $X^*(T) \rtimes W$ with unequal parameters $\mathbf{q} = \{q_1, \ldots, q_k\}$. We assume that $q_i \in \mathbb{R}_{>0}$. $\mathcal{S}_{\mathbf{q}}(X^*(T) \rtimes W)$ denotes the Schwartz completion of $\mathcal{H}_{\mathbf{q}}(X^*(T) \rtimes W)$. In [31] a morphism of Fréchet algebras

$$\mathcal{S}_1(X^*(T) \rtimes W) \to \mathcal{S}_{\mathbf{q}}(X^*(T) \rtimes W)$$

is constructed which is spectrum preserving with respect to filtrations. However, the existence of a geometric equivalence between the finite type $\mathcal{O}(T/W)$-algebras $\mathcal{O}(T) \rtimes W$ and $\mathcal{H}_{\mathbf{q}}(X^*(T) \rtimes W)$ is still an open question in the case when $\mathbf{q}$ contains unequal parameters q_i.

Example 3. Let G be a connected reductive p-adic group, and let $\mathfrak{s} \in \mathfrak{B}(G)$ be any point in the Bernstein spectrum of G. $\mathcal{H}_{\mathfrak{s}}$ denotes the unital finite type $\mathcal{O}(T_{\mathfrak{s}}/W_{\mathfrak{s}})$ algebra which Bernstein assigns to $\mathfrak{s}$. $\mathcal{H}_{\mathfrak{s}}$ has the property

$$\mathrm{Irr}(\mathcal{H}_{\mathfrak{s}}) = \widehat{G}_{\mathfrak{s}}$$

The ABPS conjecture asserts that the three unital finite type $\mathcal{O}(T_{\mathfrak{s}}/W_{\mathfrak{s}})$ algebras

$$\mathcal{H}_{\mathfrak{s}} \qquad \mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}} \qquad \mathcal{O}(T_{\mathfrak{s}}/\!/W_{\mathfrak{s}})$$

are geometrically equivalent. Results and conjectures of G. Lusztig [23, 24, 25] are relevant to this assertion. According to the ABPS conjecture, the geometric equivalence between $\mathcal{H}_{\mathfrak{s}}$ and $\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}$ is canonical and gives the left slanted arrow in the statement of ABPS.

Remark Let A be a finite type k-algebra. $A_{\mathbb{C}}$ denotes the underlying $\mathbb{C}$-algebra of A — i.e. $A_{\mathbb{C}}$ is obtained from A by forgetting the action of k on A. Then an irreducible $A_{\mathbb{C}}$ module in a canonical way becomes an irreducible A module. This gives bijections

$$\mathrm{Irr}(A) \longleftrightarrow \mathrm{Irr}(A_{\mathbb{C}}) \qquad \mathrm{Prim}(A) \longleftrightarrow \mathrm{Prim}(A_{\mathbb{C}})$$

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