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TOWARDS A MATHEMATICAL THEORY OF COMPLEX SOCIO-ECONOMICAL SYSTEMS BY FUNCTIONAL SUBSYSTEMS REPRESENTATION

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Abstract. This paper deals with the development of a mathematical theory for complex socio-economical systems. The approach is based on the methods of the mathematical kinetic theory for active particles, which describes the evolution of large systems of interacting entities which are carriers of specific functions, in our case economical activities. The method is implemented with the concept of functional subsystems constituted by aggregated entities which have the ability of expressing socio-economical purposes and functions.

1. **Introduction.** This paper deals with the development of a mathematical theory to model complex socio-economical systems, where individual behaviors and interactions may play a significant role on the evolution of the system.

The mathematical approach takes advantage of the kinetic theory for active particles [7] already applied in various fields of life sciences, e.g. modelling multicellular systems [9], [11], and social behaviors of interacting individuals [12], [13], [15]. This mathematical theory describes the evolution of the probability distribution over the microscopic state, called activity, of several interacting entities called active particles. The equation which models the evolution is derived by a conservation balance in the elementary volume of the space of the microscopic states, where the inlet and outlet flows are determined by interactions among active particles.

An additional essential tool for the theory is the concept of modules proposed by Hartwell [22] for the interpretation of complex biological systems. Briefly, a module is viewed as an aggregate of entities specialized to develop a well defined function referred to the system under consideration. A modification of the objectives of the investigation may possibly modify also the structure of the modules.

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This paper develops the above concepts introducing the idea of ***functional subsystem*** that refers to both the functions expressed by the socio-economical system, which is the system under consideration, and the observation and representation scale used in the mathematical modelling process. The whole system is composed by several interacting sub-systems, each related to a specific socio-economical function. The time-evolution of the system is described by differential equations delivered by a suitable development of the mathematical kinetic theory for active particles [7].

Before approaching the afore mentioned topics, it is worth analysing some conceptual aspects on the interaction between applied mathematics and social sciences. The main problem consists in investigating how the ***qualitative interpretation*** of social reality, that is delivered, including the interpretation of empirical data, by research activity in social sciences, can be transferred into a suitable ***quantitative description*** through mathematical equations. The above target needs the development of a dialogue between methods and traditions of two different disciplines with the additional difficulty of dealing with living systems, which have the ability to think and, as a consequence, to react to external actions, without following rules constant over time. This dialogue, however difficult, is necessary to a deeper understanding of the so called ***behavioral economics***, where deterministic rules may be stochastically perturbed by individuals behaviors. An additional difficulty to be taken into account is that individual behaviors not only show random fluctuations, but may be substantially modified by external environments, from depressive or panic situations to over-optimistic attitudes induced by misleading propaganda.

This paper aims at developing a first approach to the design of a mathematical theory focused on modelling the evolution in time of interacting socio-economical systems. The contents of the paper are organized in six sections, which follow the above introduction.

Section 2 first analyses some scaling issues related to the representation of economical systems and consequently proposes the concept of ***functional subsystem*** as an essential paradigm to develop the mathematical theory.

Section 3 shows how complex economical systems can be viewed as a network of several interacting functional subsystems, each of them having the ability to express economical functions and purposes. The proposed mathematical representation defines a probability distribution over their functional states. Models in which functional subsystems express deterministically their functions, should be considered as particular cases of the more general stochastic description.

Section 4 is devoted to the application of the above mentioned mathematical method to a variety of socio-economical systems, chosen among the ones described in Section 2. The relation of the approach proposed in this paper with respect to the existing literature is critically analysed in this section.

Section 5 deals with the derivation of a suitable mathematical structure, based on the kinetic theory for active particles, which leads to evolution equations over the probability distribution function introduced in Section 3. The contents refer to networks of interacting subsystems, both in the case of isolation from the outer world (***closed systems***), and in the case of stochastic interactions with the outer environment (***open systems***). Both mathematical structure, for closed and open systems, act as a paradigm for the derivation of models.

Section 6 shows how to implement this mathematical theory to derive a model by means of a methodological modelling approach to microscopic interaction. Subsequently, the method is applied, in Section 7, to the derivation of a model of competition for a secession.

Section 8 critically analyses some specific applications and provides an overlook on research perspectives, also focused on suitable developments of the mathematical structures proposed in Section 5, in order to enlarge the variety of socio-economical systems which can be described by mathematical models.

2. **From scaling to the concept of functional subsystems.** The first step towards the mathematical modelling of the behavior of real world systems consists in defining the proper observation and representation scale, for short the ***scaling problem***.

This issue is well documented in the physics of real systems [32], however the reasoning applies also to the case of socio-economical systems, with the additional difficulty of assessing the various and technically different components of the system. Moreover, dealing with socio-economical systems also needs to overcome the difficulties related to dealing with living systems, as individual behaviors can play a remarkable role in the evolution of economical phenomena.

A general rule to identify the appropriate observation scale when dealing with the inert matter in a real world system is the following: the microscopic scale identifies the smallest entities composing the system, the macroscopic scale, instead, refers to quantities which can be observed and measured. These observable quantities correspond to local averages over the states of interacting microscopic entities.

As an alternative, used in the sequel of this paper, it is possible to deal with the ***statistical (kinetic) representation***, where the state of the whole system is described by a suitable probability distribution over the microscopic state of the interacting elements. Macroscopic observable quantities are recovered by means of weighted moments of the distribution function.

In the case of living systems the assessment of what is small and what is large appears to be quite conventional and often related to the type of analysis that is developed. Moreover, living systems, differently from inert matter systems, are characterized by their ability of organizing specific strategies according to well defined objectives. This aspect is documented in the paper by Hartwell et al. [22] and developed in different fields of biology [10].

A preliminary step is the identification of microscopic and macroscopic scales. This aspect suggests to develop the definition of functional subsystems. The reference scaling in Economics is delivered by the concept of ***microeconomics*** and ***macroeconomics***. Although the above categorization shows some flexibility, it is well accepted that microeconomics refers to the interaction between economical agents, say individuals, households, enterprizes, and so on, that generate economical interactions and processes in the market. On the other hand, macroeconomics refers to aggregated economical processes, e.g. national income, unemployment, inflation, investment, international trade, which are the output both of the above interactions at the level of microscopic agents and of running of a country, e.g. macroeconomics is highly correlated with the development of economic policy and strategies.

The above reasonings are specifically referred to three examples of socio-economical systems which can be examined through the mathematical approach proposed in this paper. These applications are here analysed with both theoretical and applied tools.

The examples proposed in the following subsections refer to socio-economical phenomena, which have been recently analysed in Economics, even if from a different mathematical point of view. The innovative mathematical structure developed in this paper is proposed in order to give a unique and general mathematical approach to all these phenomena. A similar approach to socio-economical phenomena can be found in [19], [20], [25], [26], [33], where the influence of the social context on economical decisions is deeply considered and studied.

In particular, going through the economic literature of these last years, it is possible to find many attempts to explain socio-economical contemporary issues: as far as the example in Subsection 2.1 is concerned, it is possible to refer to [4], [5], [6], [16], [30], [31], where the mechanisms which rule country separations and even civil conflict secessions are properly dealt with.

Referring to the example of Subsection 2.2, concerning the democratization of a country, it is possible to refer to [2], [3], [23], [24], where a model has been proposed to explain how democracies or dictatorships can survive under a popular consensus or why the right of vote can be extended to all citizens. Additional valuable references are offered by Rubinstein [28].

Subsection 2.3 refers to *welfare politics*, focusing on the mechanisms according to which it is ruled by very general aspects of political economy. This issue is also dealt with in various papers [12], [13], [15], that apply methods of the discrete kinetic theory and stochastic games to social competition mathematical models. Methods derived from statistical mechanics are used to analyse political, economical, and social competitions phenomena [17], [18], [21], [29]. The use of multiple interaction games is proposed in papers[27], [8].

Finally, the last subsection proposes some additional reasonings on various aspects concerning further interpretations of the above socio-economical systems.

2.1. **Competition for a secession.** The overall system is a nation which, as visualized in Table 2.1, is decomposed into two or more subsystems identified by regional interest groups, which express, through specific actions taken by either their political parties or their peculiar interest groups, their attitude towards a process of secession by expressing a function u, which takes negative or positive values: when a certain region is expressing a positive value of u, then it is pro-secession, when it is expressing a negative value of u, then it is against it. Again, the absolute value of u measures the intensity of the expressed function.

Global nation			
Region1		Region2	
political parties	interest groups	political parties	interest group
$u > 0$ - pro	$u < 0$ - against	$u > 0$ - pro	$u < 0$ - against

2.1. Competition for a secession

2.2. **Democratization of a dictatorship.** The overall system is decomposed, as visualized in Table 2.2, into four subsystems: Dictator, Ministers, Parliament, and Citizens. All of them express, by technically different channels, a socio-economical function u, either to support the dictatorship, or to contrast it through dissidence. This function takes negative values when it is supporting, and positive values when it is contrasting. The absolute value of the function measures the expressed intensity of support or contrast.

Dictatorship							
Dictator		Ministers		Parliament		Citizens	
support	dissid.	support	dissid.	support	dissid.	support	dissid.
$u < 0$	$u > 0$	$u < 0$	$u > 0$	$u < 0$	$u > 0$	$u < 0$	$u > 0$

2.2.Democratization of a dictatorship

2.3. **Welfare politics.** The overall system is a nation which, as visualized in Table 2.3, is decomposed into subsystems identified by different interest groups, i.e. those groups which have some specific economical power within the political economical equilibria for a given country, which compete in order to determine "right" or "left" political economical decisions.

Country							
Firms		Banks		Political parties		Unions	
Left	Right	Left	Right	Left	Right	Left	Right
$u < 0$	$u > 0$	$u < 0$	$u > 0$	$u < 0$	$u > 0$	$u < 0$	$u > 0$

2.3. Welfare politics

2.4. **Additional reasonings.** The above examples want to show how real socio-economical phenomena can be interpreted and decomposed using the concept of functional subsystems. Each example shows a hierarchical decomposition from the global system to different subsystems, expressing the social activity u. This variable conceptually varies form one case to the other: this means clearly that the specific meaning of u is strictly related to the phenomenon under consideration.

Even the structure of the decomposition of the global system varies corresponding to the different examples shown above; this depends on which different aspects of the phenomenon it will be interesting to focus on.

Keeping this in mind, the afore-mentioned decomposition can be an applied starting point to understand the development of the mathematical framework which describes the subsystem itself. This framework analyses the evolution of the socio-economical variable u, as an effect of interactions among the different subsystems of the global structure. The different social actors of a dictatorship, (the dictator, the ministers, the members of the parliament, the citizens themselves), for example, usually interact to develop a certain kind of society. The consequence of this interaction can lead to the fall or the enhancement of the dictatorship itself: this process is described by the evolution of the variable u, introduced over the above schemes. Moreover, the same reasoning can be applied to the phenomena of secession or to political economical competition inside a certain country: again it is possible to describe whether a nation breaks up into many regions or whether a country is oriented toward liberal or toward socialist economy, as a consequence of the interactions between the subsystems identified in the corresponding examples.

Finally, it is important to remark that the above systems are viewed as closed with respect to the external world. If the system is open, interactions with the outer environment have to be specifically modelled.

3. **Socio-economical systems as interacting functional subsystems.** The overall socio-economical system is viewed, according to the analysis of Section 2, as a network of interacting functional sub-systems. Their representation is based on the assumption that each of them has the ability to express a specific function.

Bearing this in mind consider a network of n interacting functional subsystems whose function is identified by the variable $u \in \mathbb{R}$, where the value $u = 0$ separates the positive and negative valued ***functions*** expressed by each subsystem. The overall state of the system is described by the probability distributions:

$$f_i = f_i(t,u) : \quad [0,T] \times \mathbb{R} \to \mathbb{R}^+ \quad f_i \in L_1(\mathbb{R}) \,, \tag{1}$$

where the subscript refers to the ith subsystem, and where for each of them:

$$\lim_{|u|\to\infty} f_i = 0 \,, \qquad \int_{\mathbb{R}} f_i(t,u)\, du = 1 \,, \quad \forall\, t \geq 0 \,, \tag{2}$$

Each f_i has the structure of a probability density such that

$$\int_a^b f_i(t,u)\, du = P(t; u \in [a,b]) \tag{3}$$

is the probability that the ith subsystem expresses a function in the range $[a,b]$.

Remark 1. The variable u has to be regarded as a dimensionless quantity in units of a suitable reference quantity properly selected for each particular system. The fact that u ranges over an unbounded domain prevents the *a priori* identification of its upper and lower bounds, which in some cases may be technically difficult. On the other hand, if the upper or lower bounds are identified, the domain of u can be assumed as follows: $u \in D_u = [-1,1]$.

Remark 2. Let us consider the case where u is a scalar variable. In other words, the decomposition of the whole system into functional subsystems is organized in such a way that the expressed activity is a scalar. Therefore, i and u entirely describe the subsystem.

Macroscopic quantities can be computed, under the assumption $u^n f_i \in L_1(\mathbb{R})$, for $n = 1,2$, by weighted moments corresponding, for instance, to mean value, energy and variance, respectively:

$$L_i(t) = \int_{-\infty}^{\infty} u\, f_i(t,u)\, du \,, \tag{4}$$

$$E_i(t) = \int_{-\infty}^{\infty} u^2\, f_i(t,u)\, du \,, \tag{5}$$

$$\sigma_i(t) = \int_{-\infty}^{\infty} \big(u - L_i(t)\big)^2\, f_i(t,u)\, du \,. \tag{6}$$

Remark 3. The analysis of models, i.e. the solution of mathematical problems related to the application of models to real systems analysis, takes into account, at least in some cases, the initial value of the quantities L_i and E_i:

$$L_{i0} = L_i(t=0) \,, \qquad E_{i0} = E_i(t=0) \,.$$

In fact, interactions between functional subsystems can depend also on their initial properties.

The variable u refers to the specific function of each functional subsystem. In other words, the functional subsystem is referred to one function only; this means that large systems are decomposed, to reduce complexity, into several subsystems each expressing a scalar function. The possibility of dealing with subsystems expressing more than one function and the way to decompose them, is analysed in the last section referring to the examples proposed in Section 2.

The above assumption that functions are identified by continuous variables has the advantage that lower and upper bounds do not need to be fixed a priori, while it has to be acknowledged that discrete variables have the advantage that empirical data obtained by measurements can be related precisely to ranges of variability. This argument is posed in various papers dealing with modelling social dynamics [12], [13].

A brief indication on the use of discrete variables is given for the sake of completeness. In this case, the ***function*** is identified by a discrete variable

$$I_u = \{u_{-p} = -1, \ldots, u_0 = 0, \ldots, u_p = 1\} \,,$$

where the description of the state of each subsystem is delivered by the discrete probability density

$$f_i^j = f_i^j(t): \quad [0,T] \times \mathbb{R} \to \mathbb{R}_+^n, \qquad \sum_{j=1}^{p} f_i^j(t) = 1, \quad \forall\, t \geq 0, \tag{7}$$

for $i = 1, \dots, n$ and $j = -p, \dots, p$.

First and second order moments are given by sums as follows:

$$L_i(t) = \sum_{j=-p}^{p} u_j\, f_i^j(t), \qquad E_i(t) = \sum_{j=-p}^{p} u_j^2\, f_i^j(t). \tag{8}$$

Remark 4. The use of a discrete distribution is motivated by the technical difficulty to identify the state of functional subsystems by a continuous variable, while suitable ranges of the state, obtained by discretization, allow in a simpler manner the socio-political collocation.

Remark 5. Specific models may possibly include states $j = -p$ and/or $j = p$, where at the initial time one has $f_i^{-p} = 0$ and/or $f_i^p = 0$. Subsequently the dynamics may lead to an evolution where these states are characterized by some probability higher than zero.

Let us anticipate, with respect to the detailed analysis of the next section, that the mathematical structure candidate to derive specific models for closed systems can be formally written as follows:

$$\partial_t f_i = J_i[f](t,u) \quad i = 1, \dots, n, \tag{9}$$

where J_i is a suitable operator acting over the whole set of probability distributions $f = \{f_i\}$.

It is worth stressing that the formal structure (9), whose detailed expression will be given in the next section, acts as a paradigm for the derivation of specific models based, as we shall see, on a detailed modelling of the interactions in the network.

The above structure is meaningful for a close system, that is in absence of interactions with the outer environment. The modelling of open systems needs a representation of the outer functional subsystems by their actions on the inner system as follows:

$$A_{ji} = \varepsilon_{ji}\, g_j(t,v): \quad [0,T] \times \mathbb{R} \to \mathbb{R}^+ \quad g_j \in L_1(\mathbb{R}), \tag{10}$$

where ε_{ji} models the intensity over the ith subsystem due to the action of the jth agent represented by the probability density $g_j(t,v)$, where v is the expressed function and

$$\lim_{|u|\to\infty} g_j = 0, \qquad \int_{\mathbb{R}} g_j(t,u)\, du = 1, \quad \forall\; t \geq 0, \tag{11}$$

In this case the formal structure of the equation can be written as follows:

$$\partial_t f_i = J_i[f](t,u) + Q_i[f,g](t,u), \qquad i = 1, \dots, n, \tag{12}$$

where $g = \{g_j\}$.

Structures (9) and (12) will be derived for a system decomposed into a fixed number of subsystems. On the other hand, one also has to consider, as we shall see in the last section, the case of subsystems which have the ability to aggregate with other subsystems or to decompose each of them into two or more subsystems.

The corresponding structure for systems with discrete functions is written as follows:

$$\frac{d}{dt} f_i^j = J_{ij}[\mathbf{f}](t,u) + Q_{ij}[\mathbf{f},\mathbf{g}](t,u) \quad i = 1, \dots, n, \tag{13}$$

where $\mathbf{f} = \{f_{ij}\}$ and $\mathbf{g} = \{g_{ij}\}$.

4. **Representation of functional subsystems.** This section applies the representation rules proposed in Section 3 to the mathematical description of real socio-economical systems decomposed into a network of several interacting subsystems. Three specific systems are selected among those reported in Section 2 so that the application effectively refers to the real world.

4.1. **Representation of "Competition for a secession".** This phenomenon is modelled through the following decomposition: the global system is a nation, decomposed into two or more regions. Each region is decomposed into subsystems identified by different interest groups and political parties, which are to be assumed as those institutions having such an economical and political power as to enforce a secession process. Each group can express a variable u in favor or against secession; again let us suppose that positive values represent support to secession and negative ones opposition to it. The role of the distribution f_i is the same as before, indicating the probability that the interest group, labelled by i, is pro or against secession. $L_i(t)$ is the mean value of f_i, representing the mean desire of secession (***activation***) for the subsystem i. $E_i(t)$ is the mean energy with which the degree of secession desire is expressed (***activation energy***), and $\sigma_i(t)$ is the distribution variance.

4.2. **Representation of "Democratization of a dictatorship".** We model this phenomenon by decomposing the dictatorship-state into four functional subsystems, each of them representing a certain political institution: the dictator itself and its close "entourage", the ministers of the government, the parliament and the common citizens. Each subsystem expresses a variable $u \in \mathbb{R}$ which represents its attitude towards the dictatorship, namely whether the subsystem supports or opposes to it. Let us suppose that positive and negative values represent respectively dissidence and support. The overall state of the subsystem is described by the probability distribution f_i over u, which indicates the probability that one of the subsystems is in a state u at time t.

The macroscopic quantities defined in the previous section can assume a practical meaning. $L_i(t)$ is the mean value for the distribution f_i representing the mean degree of dissidence or support (***activation***) for the $i-th$ subsystem, $E_i(t)$ is the mean energy with which dissidence or support is expressed (***activation energy***), and $\sigma_i(t)$ is the variance of the distribution, from the mean $L_i(t)$.

4.3. **Representation of "Competition for a welfare politics".** This example refers to the decomposition of a state nation into different interest groups, which are supposed to be the political and economical powers able to compete for a certain political economical policy. Each group expresses a variable u which can be left (if negative valued) or right (if positive valued) oriented in terms of welfare state decisions. As before, $L_i(t)$ is the mean degree of political orientation (***activation***) for each subsystem, $E_i(t)$ the mean energy with which the political orientation is expressed (***activation energy***) and $\sigma_i(t)$ the variance from the defined mean, for each subsystem i.

5. **Mathematical structures for closed and open systems.** This section deals with the derivation, within the framework of the kinetic theory for active particles, of mathematical structures to model systems which do not (or do) interact with the outer world. These structures acts, as already mentioned, as a general paradigm for the derivation of models referred to specific socio-economical systems.

The first two next subsections deal with the derivation of the above mentioned frameworks, while the last section offers some additional reasoning on general issues to be critically analysed in the last section. The derivation technically refers to the recent papers by Bertotti and Delitala [13], [14] concerning models with discrete valued states of the variable u.

5.1. **Closed systems.** This section deals with the mathematical structure which can be used to describe a socio-economical *closed system* regarded as a set of interacting functional subsystems which do not interact with any system of the outer environment. In other words, we look for a mathematical structure of the type reported in Eq. (9).

Moreover, we focus on systems in which the number n of subsystems is constant: this means that we are not dealing with destructive or aggregative events. Interactions are supposed to be binary. Generalizations are considered in the last section.

The mathematical framework to describe microscopic interactions between two subsystems can be described by means of two different functions:

• the ***encounter rate*** $\eta_{ij}(u_*, u^*)$, that describes the rate of interactions between a subsystem i with state u_* and a subsystem j with state u^*.

• the ***transition probability density*** $B_{ij}(u_*, u*, u)$, that describes the probability density that a subsystem i with state u_* falls into the state u after the interaction with a subsystem j with state u^*.

The term $B_{ij}(u_*, u^*, u)$, according to its property of probability density, satisfies the following condition:

$$\forall u_*, u^* \in \mathbb{R}\,, \quad \forall i,j\;: \quad \int_{\mathbb{R}} B_{ij}(u_*, u^*, u)\, du = 1. \tag{14}$$

It is now possible to derive the equation which defines the evolution of the probability distribution over the microscopic state by a balance equation of the inlet and outlet flows in the elementary volume $[u, u + du]$ in the space of the microscopic states. Technical calculations yield:

$$\begin{aligned}
\partial_t f_i(t,u) &= G_i[f](t,u) - L_i[f](t,u) \\
&= \sum_{j=1}^{n} \int_{\mathbb{R}\times\mathbb{R}} \eta_{ij}(u_*, u^*) B_{ij}(u_*, u^*; u) f_i(t, u_*) f_j(t, u^*)\, du_*\, du^* \\
&- f_i(t,u) \sum_{j=1}^{n} \int_{\mathbb{R}} \eta_{ij}(u, u^*) f_j(t, u^*)\, du^*\,,
\end{aligned} \tag{15}$$

where G_i and L_i denote, respectively, the inflow and outflow, at time t, into (and out) the elementary volume $[u, u+du]$ of the space of the states for each population. This equation can describe a real economical systems after the identification of the key functions η and B.

Remark 6. As a consequence of (15), provided that f_i is integrable, the following equation holds:

$$\partial_t \int_{\mathbb{R}} f_i(t,u) du = 0\,, \quad \forall\; t \geq 0\,. \tag{16}$$

The above property implies that the zero-th order moment is preserved in time:

$$\int_{\mathbb{R}} f_i(t,u)\, du = \int_{\mathbb{R}} f_i(0,u)\, du\,, \tag{17}$$

while, due to interactions, the higher order moments evolve in time.

As far as discrete systems are concerned, taking into account a discretization of the socio-economical variable u such that $I_u = \{u_1, ..., u_h, ..., u_H.\}$, the microscopic interaction between two subsystems is described by the following discrete functions:

• the ***encounter rate*** η_{ij}^{pq}, which depends on the interacting subsystems i and j and on the socio-economical states, p and q respectively, of the interacting particles.

• the ***transition probability density*** $B_{ij}^{pq}(h) = B_{ij}(u_p, u_q; u_h)$, that describes the probability density that a candidate particle with state u_p of the subsystem i falls into the state u_h after the interaction with a field particle of the subsystem j with state u_q.

The transition probability density function is such that:

$$\forall i, j, \quad \forall p, q: \qquad \sum_{h=1}^{H} B_{ij}^{pq}(h) = 1. \tag{18}$$

In this case the balance equation describing the evolution of the probability distribution over the microscopic state leads to the following system of $n \times H$ ordinary differential equations:

$$\frac{df_i^h}{dt} = \sum_{j=1}^{n} \left(\sum_{p=1}^{H} \sum_{q=1}^{H} \eta_{ij}^{pq} B_{ij}^{pq}(h) f_i^p f_j^q - f_i^h \sum_{q=1}^{H} \eta_{ij}^{pq} f_j^q \right). \tag{19}$$

5.2. **Open systems.** Let us consider the derivation of a mathematical structure for socio-economical ***open systems***, that interact also with the outer environment. This means that, considering n interacting functional subsystems, the i-th subsystem interacts with external agents identified with the subscript $h = 1, ..., m$, where m is constant.

The external agent h has the ability to influence the socio-economical state u of the subsystem i, through a particular action identified by the variable v. The analysis developed in what follows is based on the assumption that the action of the outer agents can be modelled as follows:

$$\varepsilon_{hi}(t)\, g_h(v)\,, \tag{20}$$

where $\varepsilon_{ih} = \varepsilon_{ih}(t)$ is the intensity, that can depend on time, with which the agent h acts on the subsystem i; and $g_h(v)$ is the probability density associated to the variable v that characterizes the action of outer system.

Remark 7. Each external agent is regarded as a specific functional sub-system with the ability to interact with the functional sub-systems of the inner system. Both terms ε_{hi} and g_h are supposed to be given functions of their arguments. Specific examples are given in the next section.

The equation defining, for each functional sub-system, the evolution of the probability distribution over the microscopic state can be derived, as in the previous section, by a balance equation of the inlet and outlet flows in the elementary volume $[u, u + du]$ of the space of the microscopic states.

The mathematical framework to describe the microscopic interactions between a subsystem i and an external agent h needs the following interaction terms:

• The ***inner-outer encounter rate*** $\mu_{hi}(v_*, u^*)$ which describes the rate of interactions between an external agent h with state v_* and a subsystem i with state u^*.

• The ***inner-outer transition probability density*** $G_{hi}(v_*, u*, u)$, which describes the probability density that a subsystem i with state u^* falls into the state u after an interaction with an external agent h with state v_*.

The interaction term $G_{hi}(v_*, u^*, u)$ satisfies the following condition:

$$\forall v_*, u^* \in \mathbb{R}\,, \quad \forall h, i \qquad \int_{\mathbb{R}} G_{hi}(v_*, u^*, u)\, du = 1\,. \tag{21}$$

Technical calculations analogous to those we have seen in the preceding section yield:

$$\begin{aligned}\partial_t f_i(t,u) &= \sum_{j=1}^{n} \int_{\mathbb{R}\times\mathbb{R}} \eta_{ij}(u_*,u^*) B_{ij}(u_*,u^*;u) f_i(t,u_*) f_j(t,u^*)\, du_*\, du^* \\ &- f_i(t,u) \sum_{j=1}^{n} \int_{\mathbb{R}} \eta_{ij}(u,u^*) f_j(t,u^*)\, du^* \\ &+ \sum_{h=1}^{m} \int_{\mathbb{R}\times\mathbb{R}} \varepsilon_{hi}\, \mu_{hi}(v_*,u^*) G_{hi}(v_*,u^*;u) g_h(t,v_*) f_i(t,u^*)\, dv_*\, du^* \\ &- f_i(t,u) \sum_{h=1}^{m} \int_{\mathbb{R}} \varepsilon_{hi}\, \mu_{hi}(v_*,u) g_h(t,v_*)\, dv_*\,. \end{aligned} \tag{22}$$

In the case of discrete systems the subscript indicating the external agent h is replaced by the subscript l, since h indicates the discretization step. In this case we assume that the action of the outer agent can be modelled as $\varepsilon_{i\ell}\, g_l(v)$, where v is a discrete variable. The mathematical framework to describe the microscopic interactions between a subsystem i and an external agent ℓ needs the following discrete interaction terms:

- The ***inner-outer encounter rate*** $\mu_{i\ell}$ that describes the rate of interactions between an external agent l and a subsystem i and does not depend on the socio-economical state of the interacting particles.
- The ***inner-outer transition probability density*** $G_{i\ell}^{pq}(h) = G_{i\ell}^{pq}(v_q, u_p, u_h)$, that describes the probability density that a candidate particle with state u_p of the subsystem i falls into the state u_h after the interaction with a field particle of the external agent ℓ with state v_q.

The interaction term $G_{i\ell}^{pq}(v_q, u_p, u_h)$ satisfies the following condition:

$$\forall p, q, \forall i, \ell: \quad \sum_{h=1}^{H} G_{i\ell}^{pq}(h) = 1. \tag{23}$$

Technical calculations similar to the preceding one yield:

$$\begin{aligned}\frac{df_i^h}{dt} &= \sum_{j=1}^{n} \left(\sum_{p=1}^{H}\sum_{q=1}^{H} \eta_{ij}^{pq} B_{ij}^{pq}(h) f_i^p f_j^q - f_i^h \sum_{q=1}^{H} \eta_{ij}^{pq} f_j^q \right) \\ &+ \sum_{\ell=1}^{m} \left(\sum_{p=1}^{H}\sum_{q=1}^{H} \varepsilon_{i\ell}\mu_{i\ell}^{pq} G_{i\ell}^{pq}(h) g_\ell^q f_i^p - f_i^h \sum_{q=1}^{H} \varepsilon_{i\ell}\mu_{i\ell}^{pq} g_\ell^q \right). \end{aligned} \tag{24}$$

5.3. **Additional reasonings.** The mathematical structures (15),(22),(19)and (24) act as a general paradigm for the derivation of models if the terms η, B, ε and G, that are called here ***interaction parameters***, are properly identified. Their identification needs to be referred to a detailed analysis of the interactions between subsystems. Suitable examples will be given in Section 6.

It is worth stressing that two different classes of models can be considered:

- ***Predictive models*** which describe the evolution of the system, given an initial condition and interaction parameters obtained by a suitable interpretation of empirical data.
- ***Explorative models*** that offer a panorama of different types of evolution according to an explorative guess on interaction parameters. These models can be used to select a strategy among several conceivable ones, while the investigation can be further refined in the framework of optimal control problems.

Finally, it is worth providing a critical analysis of the approach of this paper compared with the the existing literature based on agents methods [29], that consist in using the concept of agents' preference, and trying to maximize some objective function, depending on the case study: the objective function can be the welfare of the population, or the benefit of the dictator, or the benefit of some party or group of interest in a country. This maximization process can lead to some trade-off equations, which determine the final outcome. This general approach is developed tanks to game theory, both deterministic and stochastic. However, defining a preference function for all individuals in the same group cannot capture, in some circumstances, all the varieties and intensities of opinions within a large group of people. This is the reason why we want to describe this kind of phenomena using the probabilistic concept of distribution of preferences: by this approach it is possible to describe and explain complex social, economical, and political behaviors and phenomena from a more comprehensive point of view.

Indeed, according to many works of the recent literature such as [28], it is extremely unrealistic to think that individuals are perfectly rational, in the sense that all of them want to maximize some fixed utility functions: in reality, opinions and preferences can dynamically vary across time. This means that if we observe a socio-economical phenomena during a time interval $[0, T]$, the preference of a certain individual about the final outcome of this phenomena will be very likely to change from $t = 0$ to $t = T$. As an example, if we are dealing with a democratization problem, all individuals and parties who may play a significant role in this phenomenon, can radically change their preference and as a consequence their efforts to support or hinder the democratization process. This is just a single example among those introduced in the preceding sections. For this reasons, these socio-economical phenomena can clearly be described by an evolution equation of the distribution on the so-called socio-economical activity u. Instead of fixed preferences, it is fairly more realistic to deal with dynamic preferences. Moreover, what in the real world determines the final outcome of an individual opinion or strength to reach the final socio-economical outcome is not deterministic: it depends on the kind and the intensity of the encounters (interactions) among the different groups of individuals; after the encounter with a specific individual my position can be strengthened or, on the opposite, can be dramatically weakened, through a stochastic process. All the afore-mentioned models can be analysed with this new perspective, trying to understand which mechanisms rule the reaching of a final outcome, depending on the history of every single individual between $[0, T]$. This means that starting from a state u depending on the subsystem, the socio economical variable may change after probabilistic encounters.

This is the reason why, with our approach, even the understanding of the way the phenomenon takes place can be more precise, considering both the microscopic and the macroscopic point of view. Indeed, our models allow to observe the modification of the opinion of specific individuals, and the convergence towards the macroscopic event corresponding to the final equilibrium.

6. **Modelling and mathematical problems.** This section is devoted to the development of a modelling approach based on the mathematical frameworks proposed in Section 5 corresponding to closed and open systems. These are formally given by Eqs. (9) and (12), that have been particularized by Eqs. (15),(22),(19) and (24). The method is essentially based on a detailed modelling of the interaction terms, so that the mathematical structures can be fully characterized.

Therefore, it is worth focusing on which specific mathematical functions can suitably describe the encounter rate and the transition probability density functions. Moreover, the evolution of the system can be computed if the initial conditions $f_{i0}(u) = f_i(t = 0) \quad i = 1, \dots, n$, are given. These conditions may influence the afore-mentioned identification.

Let us first consider the ***encounter rate***. The identification refers to assessment of the symmetric matrix $[\eta_{ij}]$, whose entries η_{ij} generally depend on the state of the interacting pairs.

Referring to the practical examples described in the previous sections, it is reasonable to think that, the more two subsystems are close in their specific state variable u, the more frequently they interact, namely the encounter rate increases with decreasing distance.

Referring to the role of the characteristics of the interacting pair, it seems realistic to believe that political institutions interact with higher frequency if they are close in their tasks and duties: i.e. the dictator *entourage* interacts more with his ministers than with the people.

As a consequence, the encounter rate matrix entries $\eta_{ij} = \eta_{ji}$, can be made depending on the macroscopic quantities L_i, L_j, which identify the mean socio-economical state of the system. In particular, if we define L_{i0} and L_{j0} as the mean socio-economical state of the subsystems i and j respectively, at the initial time $t = 0$, the *distance in state*

$$d_{ij}^0 =\mid L_{i0} - L_{j0} \mid,$$

can be defined.

Therefore, the encounter rate between the subsystem i and j can be described by $\eta_{ij}(d_{ij}^o)$, where in certain cases η_{ij} is an increasing function of d_{ij}^o, while in other cases is a decreasing function of d_{ij}^0. More in general, the encounter rate can be assumed to depend on the mean distance of the states at time t:

$$d_{ij}(t) =\mid L_i(t) - L_j(t) \mid \,. \tag{25}$$

Of course, if η_{ij} depends on time Eq.(15) needs to be properly rewritten to include this dependence. The simplest case is the one in which η_{ij} is simply a constant rate, not depending on any macroscopic quantity, in this case we indicate $\eta_{ij} = c_{ij}$, where c_{ij} is a suitable constant for the subsystems i and j.

The above reasoning focused on the interaction rate can be straightforwardly applied to open systems focused on the modelling of the rate μ_{ij} between the outer and inner system, treated as a constant or depending on the distance between the state of the interacting pairs.

Let us now consider the ***transition probability density functions*** B_{ij}. If the socio-variable u is defined over the whole real line $\mathbb{R}$, a reasonable assumption consists in defining B_{ij} as a *Gaussian density function*, $B_{ij}(m_{ij}(u_*, u^*), \sigma_{ij})$, where $[\sigma_{ij}]$ is a matrix, whose entries represent a dispersion factor assumed to be constant or depending on external factors, and $[m_{ij}(u_*, u^*)]$ is the matrix of the values around which the density is centered, namely the most probable value, depending on the two states of the interacting subsystems i and j.

The parameter m_{ij} can be modelled by taking into account the following law of change in state after the interaction between the subsystems i and j:

$$m_{ij} = u_* + \phi(u_*, u^*, \beta_{ij}), \tag{26}$$

where u_* is the state of the test subsystem and $\phi(u_*, u^*, \beta_{ij})$ its shift from the original value. $\phi(u_*, u^*, \beta_{ij})$ is a decreasing function in $\mid u_* \mid$, while β_{ij} is a parameter which determines the sign and the entity of this shift.

The function $\phi(u_*, u^*, \beta_{ij})$ can be assumed to depend on L_i and L_j since a change in state may depend on the mean value of the initial state of the subsystems. Moreover, also the case $\phi(u_*, u^*, \beta_{ij})$ depending on E_i and E_j can be studied, since a change in state can be due also to the energy values of the subsystems. In particular, if, as above, a distance in mean state is defined by $d_{ij} =\mid L_i - L_j \mid$, it can be assumed that $\phi = \phi(u_*, u^*, \beta_{ij}, d_{ij})$, meaning that it can be a decreasing or increasing function of such a distance.

Let us first introduce the mathematical entities which may characterize a competition for a secession. According to the definitions of the previous sections, the identification of the domain D_u for the socio-economical variable u is now necessary. If we deal with a country in which the Constitution and the law enforcement are well organized and respected, we define $D_u = [-1, 1]$: in nations where the democratic process is well established and it is not possible to overcome the limits settled by the laws, the variable u is not able to assume values external to the fixed interval $[-1, 1]$. In this case D_u corresponds to a bounded interval of the whole real line.

On the other hand, if we study a nation, where the juridical and political system is not powerful enough to guarantee the rule of law, we define $D_u = (-\infty, +\infty)$, since, in this case, the variable u can always reach extreme values. In this case D_u is no longer a bounded domain.

Remark 8. The structure reported in (6.2) does not hold in full generality. However, it can be justified considering that in every ordinary interaction, individuals generally agree, approaching their state u, or strongly disagree, getting more distant in their state u, after a common discussion.

Remark 9. Conservative interactions between i and j are such that $L_i + L_j = c_L$, where c_L is a constant. Interactions can also be conservative in energy if $E_i + E_j = c_E$, where again c_E is a constant. Conservation holds when the sum of mean states and energies do not vary before and after the interactions, although each mean state and energy can vary. This means that each subsystem can change moments, but the sum of the moments remains the same, during all the evolution. If the conservativeness of the interactions is assumed, the terms α_{ij} must be such to satisfy the constraint given by the conservative constraint equation for L_i or E_i.

7. **Modelling the competition for a secession.** This section develops the approach proposed in the preceding two sections to the modelling of competition for a secession in the case of a closed system. Specifically, the following topics are dealt with in the four following subsections:

i) Characterization of the model of competition for a secession;

ii) Sample simulations focused to visualize the role of the parameters of the model on the behavior of the solutions;

iii) A critical analysis to develop and improve the model.

7.1. **Characterization of the model.** Let us now consider the modelling of a competition for a secession, referred to a system corresponding, in this specific problem, to a nation isolated from the external environment.

Let us assume that the nation is divided into a richer and a poorer region, so that the richer part hopes to benefit from a secession in terms of national income and taxes. Many cases of nowadays reality in which this is happening, or has already happened, can be figured. Different oriented political parties and interest groups interact in the global nation in order to support the secession or not. The specific model proposed in what follows is characterized by a small number of variables and very simple interaction functions. The aim is to show that, even in a simple case, the model can describe some interesting phenomena. The last subsection critically analyses the model, proposing some guidelines to improve it.

Following the approach proposed in the preceding sections, the first step of the modelling consists in identifying the functional subsystems that compose the overall system. Accordingly, let us suppose to decompose the system into the following subsystems:

- Political parties,
- Unions,
- White collars.

Let us just focus on the above three subsystems to define a simple model suitable to describe the evolution of the system using a small number of parameters. Bearing in mind that each subsystems can be anyway represented in a much more complex way considering the different orientation (left, right and center) that are present in political parties, unions and white collars. We define the following probability densities:

- $f_1(t, u)$, describes the distribution over the state of the subsystem ***political parties***,
- $f_2(t, u)$, describes the distribution over the state of the subsystem ***unions***,

- $f_3(t,u)$, describes the distribution over the state of the subsystem ***white collars***.

The second step consists in modelling the interaction terms, namely the matrix of the encounter rates η_{ij} and the parameters of the transition probability density. Let us deal with the modelling of the encounter rate η_{ij} according to the following assumption:

Hp. 1. *$\eta_{ij} = \eta_{ji}$ is a constant*

$$\eta_{ij} = \eta_{0ij}. \tag{27}$$

In particular we suppose that η_{ij} attains the largest value when the interactions take place within the same subsystems:

$$\eta_{11} = \eta_{22} = \eta_{33} = \eta_0 = 1\,,$$

while

$$\eta_{12} = \eta_{21} = \alpha_1 < \eta_0, \qquad \eta_{13} = \eta_{31} = \alpha_2 < \alpha_1\eta_0, \qquad \eta_{23} = \eta_{32} = \alpha_3\eta_0\,,$$

where $\alpha_2 < \alpha_3 < \alpha_1$.

The values of η_{ij} are summarized in the following matrix:

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_2 \\ \alpha_1 & 1 & \alpha_3 \\ \alpha_2 & \alpha_3 & 1 \end{pmatrix}. \tag{28}$$

Remark 10. The entries of the above matrix are simply characterized by three parameters.

Let us now model the transition probability density function. The modelling is developed according to the following assumption consistent with the approach proposed in subsection (6.2).

Hp. 2. *$B_{ij}(m_{ij}(u_*,u^*),\sigma_{ij})$ is a Gaussian density function such that:*
i) σ_{ij} is a constant for each interacting pairs;
ii) the mean value is defined as follows as

$$m_{ij} = u_* + \beta_{ij}(u_* - u^*), \tag{29}$$

where β_{ij} is a small parameter which can assume either positive or negative value.

Remark 11. If the dispersion factor σ_{ij} tends to zero, the transition probability density function is as follows:

$$B_{ij}(u_*,u^*;u) = \delta(u - m_{ij}(u_*,u^*))\,, \tag{30}$$

where δ is the Dirac distribution function.

Moreover, we suppose that the interaction within the same subsystem does not change the state of the subsystem:

$$\beta_{11} = \beta_{22} = \beta_{33} = 0\,,$$

and that the probability distribution is symmetric:

$$\beta_{12} = \beta_{21} = \beta_1\,, \quad \beta_{13} = \beta_{31} = \beta_2\,, \quad \beta_{23} = \beta_{32} = \beta_3\,.$$

Instead β_1,β_2, and β_3 are positive if we suppose to deal with a *dialectic model* and negative if we are dealing with a *non dialectic model*. The values of β_{ij} are reported in the following matrix:

$$\begin{pmatrix} 0 & \beta_1 & \beta_2 \\ \beta_1 & 0 & \beta_3 \\ \beta_2 & \beta_3 & 0 \end{pmatrix}. \tag{31}$$

In the first case, namely dialectic model, we deal with a utopian reality in which parties always find a compromise solution, getting nearer in their opinions. In the second case, namely non dialectic model, we deal with a non realistic world in which every party strengthens and radicalizes its position after encounters.

This is the reason why, we propose a third model in which the parameters are the same as before, except for an extra parameter d_c, which represents a critical distance such that:

i) if $| u_* - u^* | < d_c \Rightarrow \beta_1, \beta_1, \beta_3 > 0$;

ii)if $| u_* - u^* | \geq d_c \Rightarrow \beta_1, \beta_1, \beta_3 < 0$.

This means that the model can become either dialectic or non dialectic depending on the distance in state of the interacting subsystems: if d_c is big enough the parties can reach an agreement while if d_c is not, there is a conflict solution. Notice that it is possible to define more than a single critical distance d_c, so that there are many intervals in the real line in which the model is dialectic and others in which it is not.

Taking into account Hp.1 and Hp.2, it is possible to write a detailed equation for the model for every density function f_i, $i = 1, 2, 3$ in the case $B_{ij}(u_*, u^*; u) = \delta(u - u_* + \beta_{ij}(u_* - u^*))$:

$$\begin{cases} \partial_t f_1(t,u) = \dfrac{\alpha_1}{| 1-\beta_1 |} \displaystyle\int_{\mathbb{R}} f_1(t, \frac{u-\beta_1 u^*}{1-\beta_1}) f_2(t,u^*) du^* \\ \qquad + \dfrac{\alpha_2}{| 1-\beta_2 |} \displaystyle\int_{\mathbb{R}} f_1(t, \frac{u-\beta_2 u^*}{1-\beta_2}) f_3(t,u^*) du^* - (\alpha_1+\alpha_2) f_1(t,u) \\ \partial_t f_2(t,u) = \dfrac{\alpha_1}{| 1-\beta_1 |} \displaystyle\int_{\mathbb{R}} f_2(t, \frac{u-\beta_1 u^*}{1-\beta_1}) f_1(t,u^*) du^* \\ \qquad + \dfrac{\alpha_3}{| 1-\beta_3 |} \displaystyle\int_{\mathbb{R}} f_2(t, \frac{u-\beta_3 u^*}{1-\beta_3}) f_3(t,u^*) du^* - (\alpha_1+\alpha_3) f_2(t,u) \\ \partial_t f_3(t,u) = \dfrac{\alpha_2}{| 1-\beta_2 |} \displaystyle\int_{\mathbb{R}} f_3(t, \frac{u-\beta_2 u^*}{1-\beta_2}) f_1(t,u^*) du^* \\ \qquad + \dfrac{\alpha_3}{| 1-\beta_3 |} \displaystyle\int_{\mathbb{R}} f_3(t, \frac{u-\beta_3 u^*}{1-\beta_3}) f_2(t,u^*) du^* - (\alpha_2+\alpha_3) f_3(t,u) \end{cases} \tag{32}$$

An analogous model can be derived in the case of discrete socio-variable $u \in [-1, 1]$. In this case, f_1, f_2, f_3 are discrete density functions for each subsystem Union, Political parties and White collars. The domain $[-1, 1]$ is defined, in this specific case, as follows

$$I_u = \{u_1 = -1\,,\ u_2 = -0.5\,,\ u_3 = 0\,,\ u_4 = 0.5\,,\ u_5 = 1\} \tag{33}$$

representing $H = 5$ different level of opinions. The encounter rate matrix is the same as before, while in a discrete model we need a different definition of the discrete probability density function $B_{ij}^{pq}(h) = B_{ij}(u_p, u_q; u_h)$, representing the probability density that a particle with state u_p of the population i falls in the state u_h, after the encounter with a particle with state u_q of the population j. We look for a discrete function such that

$$\forall i, j, \quad \forall p, q\ : \qquad \sum_{h=1}^{5} B_{ij}^{pq}(h) = 1. \tag{34}$$

Let us introduce, as before, a critical distance d_c. Following the continuous model, we can define the discrete transition probability density function as follows. If $p = q \Rightarrow B_{ij}^{pq}(h = p) = 1, \quad B_{ij}^{pq}(h \neq p) = 0$. If $p \neq q$, $| p - q | \leq d_c$, the model is dialectic, namely, if $p < q$

$$B_{ij}^{pq}(h = p+1) = \beta, \quad B_{ij}^{pq}(h = p) = 1 - \beta, \quad B_{ij}^{pq}(h \neq p, p+1) = 0;$$

if $p > q$

$$B_{ij}^{pq}(h = p-1) = \beta, \quad B_{ij}^{pq}(h = p) = 1-\beta, \quad B_{ij}^{pq}(h \neq p, p-1) = 0.$$

If $p \neq q$, $\mid p-q \mid \geq d_c$, the model is non dialectic, namely, if $p < q$

$$B_{ij}^{pq}(h = p-1) = \beta, \quad B_{ij}^{pq}(h = p) = 1-\beta, \quad B_{ij}^{pq}(h \neq p, p-1) = 0;$$

if $p > q$

$$B_{ij}^{pq}(h = p+1) = \beta, \quad B_{ij}^{pq}(h = p) = 1-\beta, \quad B_{ij}^{pq}(h \neq p, p+1) = 0,$$

where $\beta > 1/2$.

Taking into account the previous definitions for η_{ij} and B_{ij}^{pq}, it is possible to obtain the following 3×5 systems of ordinary differential equations:

$$\frac{df_i^h}{dt} = \sum_{j=1}^{3} \left(\sum_{p=1}^{5} \sum_{q=1}^{5} \eta_{ij} B_{ij}^{pq}(h) f_i^p f_j^q - f_i^h \sum_{q=1}^{5} \eta_{ij} f_j^q \right), \tag{35}$$

where $i = 1, 2, 3$ and $h = 1, 2, 3, 4, 5$.

7.2. **Simulations of a model of political conflicts.** Simulation are obtained by solving the initial value problems generated by model (32) or (35) linked to suitable initial conditions. The two problems can be formally written as follows:

$$\begin{cases} \partial_t f_i(t, u) = J_i[f](t, u), \\ \\ f_{i0}(u) = f_i(t = 0, u) \end{cases} \tag{36}$$

for $i = 1, \ldots, 3$ and

$$\begin{cases} \frac{df_1}{dt}(t, u) = J_i^h[f](t), \\ \\ f_{i0}^h = f_i^h(t = 0) \end{cases} \tag{37}$$

for $i = 1, \ldots, 3$; $h = 1, \ldots, 5$, and where the right-hand side terms correspond, respectively, to Eqs.(32) and (35).

A qualitative analysis of problem (36), for a class of equations with the same properties has been proved in paper [1] where it has been proven, by application of classical fixed point theorems, existence, uniqueness, and positivity of the solutions in the space of the functions integrable over the whole real line. Regularity of the solution can be proved under suitable assumptions on the smoothness of the initial data.

The qualitative analysis of Problem (37) needs a technically different approach. The existence proof is given in the paper by Bertotti and Delitala [13], who have been able also to show several properties on the asymptotic behavior of the solutions [14], both for closed and open systems, when the number of microscopic states is small. On the other hand, only partial results of the analysis are available, that makes the contribution of simulations as essential to depict the configurations reached asymptotically in time.

In particular, referring to the model described in the previous section, simulations are developed to obtain the time evolution of the three subsystems under consideration, respectively political parties, unions and white collars. The following parameters representing the encounter rate and the transition probability density: $\alpha_1 = 0.75$, $\alpha_3 = 0.45$, $\alpha_2 = 0.25$, and $\beta = 1$, are adopted, while the interval of opinions, spanning in the interval $[0, 1]$, is discretised into 5 different levels, representing the 5 different opinions which can be assumed within each subsystems.

Keeping all these parameters fixed, the goal of the simulations is to investigate the role of the critical distance d_c in the evolution of f_1, f_2 and f_3, given random uniformly distributed initial conditions. Simulations are developed in two distinct cases: the case in which the two extreme opinions (namely opinion 1 and opinion 5 representing the extreme attitude towards or against secession) cannot vary, although encountering people with other opinions; and the case in which the two extreme opinions are free to move between opinion 0 and opinion 1. All the figures reported hereby are such that continuous line corresponds to opinion 1, dashed line to opinion 2, thick line to opinion 3, dot-dashed line to opinion 4, and very thick line to opinion 5.

In the case with fixed extreme opinions, all subsystems have the same kind of behavior and the parameter d_c does not play a significant role in the evolution of f_1, f_2, f_3: for every value of $d_c = 0, 1, 2, 3, 4$ the behavior is the one shown in figures 1, 2 and 3. The two extreme opinions are the unique to survive for every d_c, no matter which subsystem examined: at the equilibrium all the individuals in each subsystems are split into two groups having different extreme ideas. The reason of this behavior is the following:

If extreme opinions are fixed, they play the role of attractor opinions, namely, once an individual reaches opinion 1 or 5, it cannot move further.

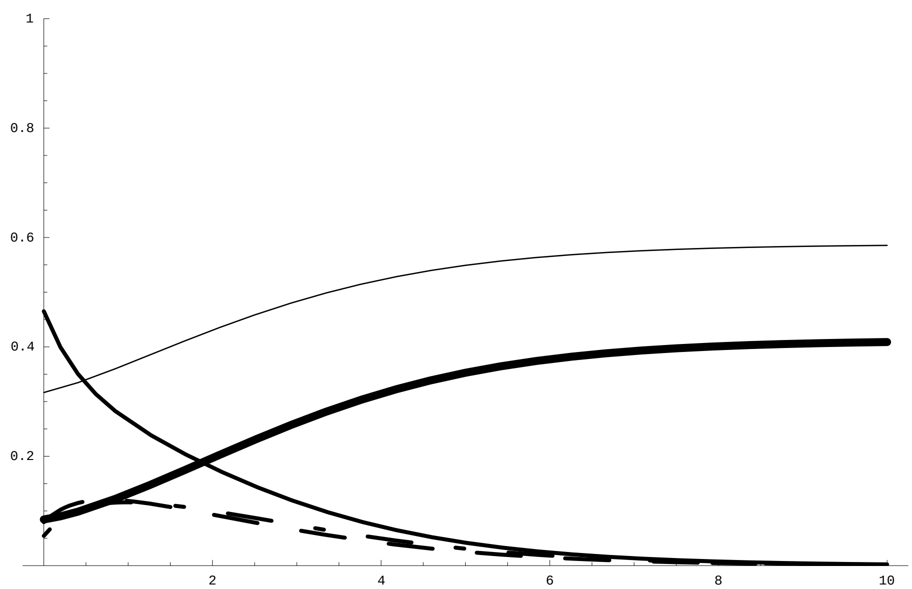

FIGURE 1. Evolution of the components of f_1 in the case of fixed extreme opinions

On the other hand, if extreme opinions are allowed to vary, the evolution scenario is completely different and the critical distance d_c does assume a significant role. When $d_c = 0$ or $d_c = 1$, namely the cases in which individuals tend to repulse different opinions, the evolution of f_1, f_2, f_3 is very similar to the one above, therefore figures are omitted.

When $d_c = 2$, the evolution begins to change: both a solution where opinion 1 and 5 survive as before is found, and a dialectic solutions appears, where the central opinion (opinion 3) is generally the dominant at equilibrium, sometimes with a second surviving opinion (opinion number 2 or 4). Notice that with $d_c = 2$ the convergence is much slower than that in the previous simulations. See figures 4, 5, 6, 7, 8, 9, 10, 11, and 12.

When $d_c = 3$, convergence is fast and the central opinion always survives, sometime jointly with opinions 2 or 4, while extreme opinions 1 and 5 never survive, as shown in figures 13, 14, 15, 16, 17, and 18.

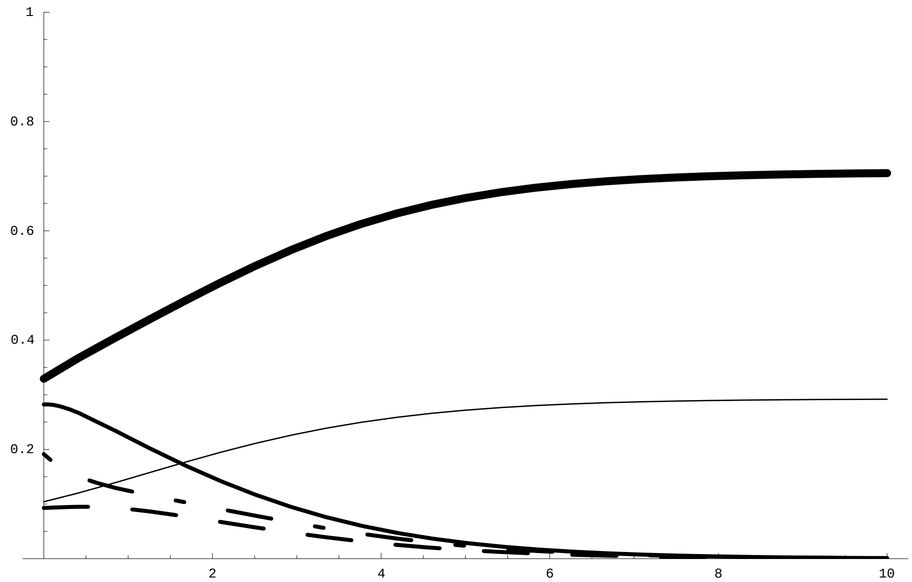

FIGURE 2. Evolution of the components of f_2 in the case of fixed extreme opinions

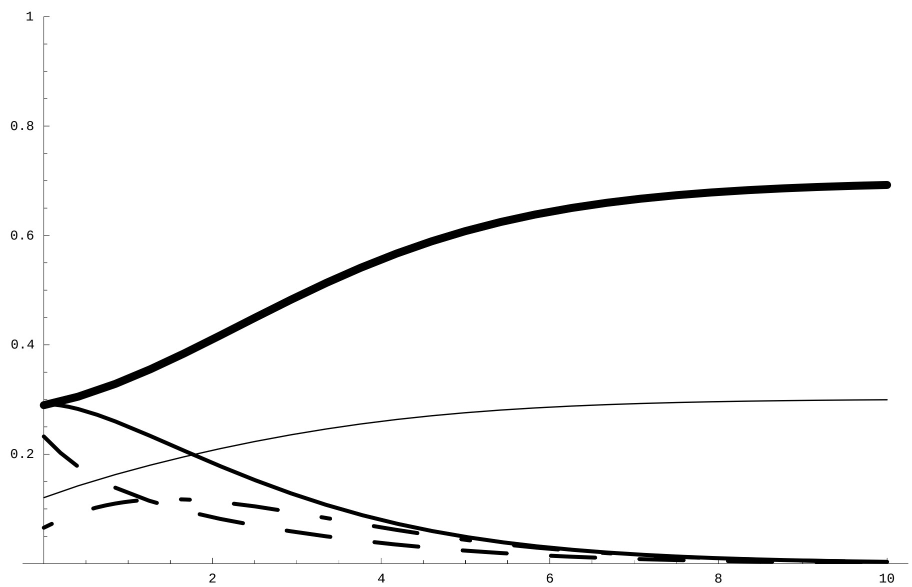

FIGURE 3. Evolution of the components of f_3 in the case of fixed extreme opinions

Finally, for $d_c = 4$, the opinion that survives is again the central one, generally alone, as shown in figures 13, 14,15.

In conclusion, as summarized in Table 1 (notice that i.c. means *initial condition*), when extreme opinions do not move, i.e. extremists remain extremists with no possibility to change, even if interacting with moderates, extreme opinions are the unique to survive.

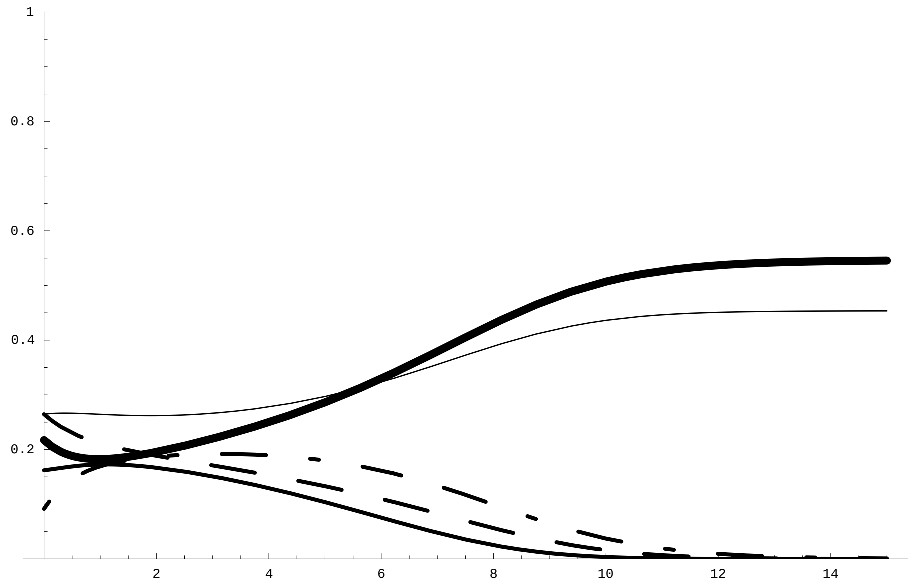

FIGURE 4. Evolution of the components of f_1, $d_c = 2$ extreme opinions survive

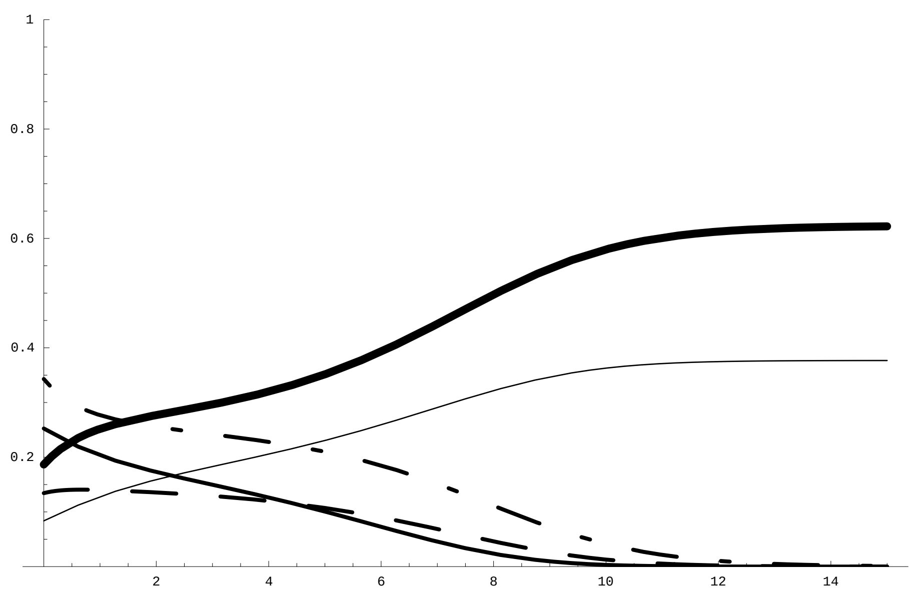

FIGURE 5. Evolution of the components of f_2, $d_c = 2$ extreme opinions survive

There are no significant differences with different d_cs: if radical people does not change their mind d_c has no role no matter how moderate other people are, sooner or later moderates will be attracted and affected by extremists. There is no possibility of a compromising solution.

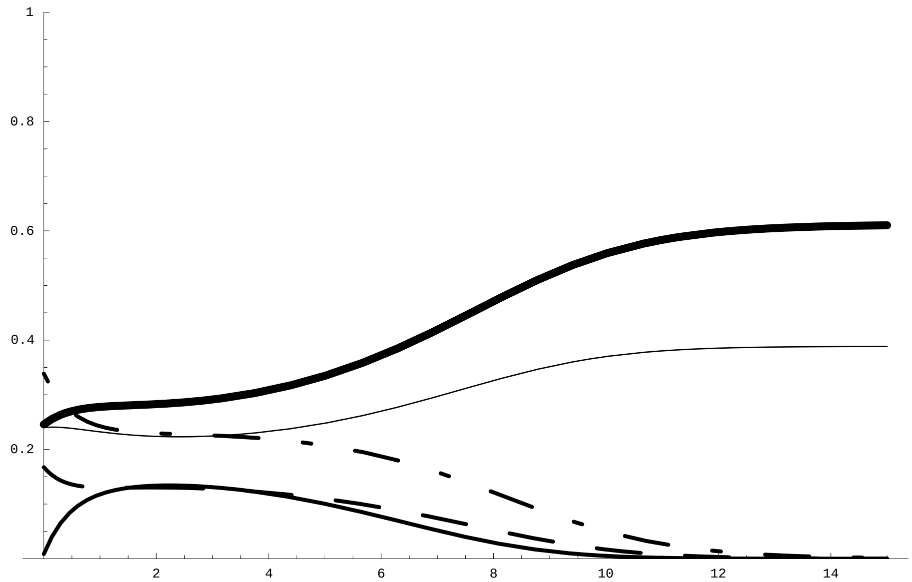

FIGURE 6. Evolution of the components of f_3, $d_c = 2$ extreme opinions survive

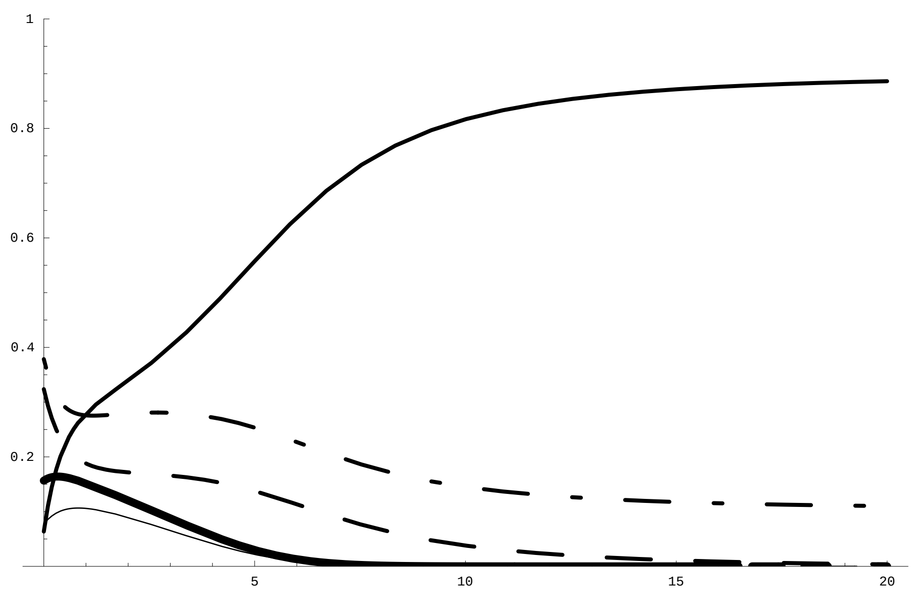

FIGURE 7. Evolution of the components of f_1, $d_c = 2$ central opinion survives

When extremists can move towards moderate opinions, as summarized in Table 2, d_c has a fundamental role. For low values of d_c results are very similar to the case of fixed extreme opinions, since people prefer to be repulsed than to be attracted by someone else's opinions. When $d_c = 2$, the transition takes place: compromise solutions begin to appear. It is not surprising that in this case convergence is slower: this is the minimum distance for which central opinion can survive and it takes longer to reach an equilibrium.

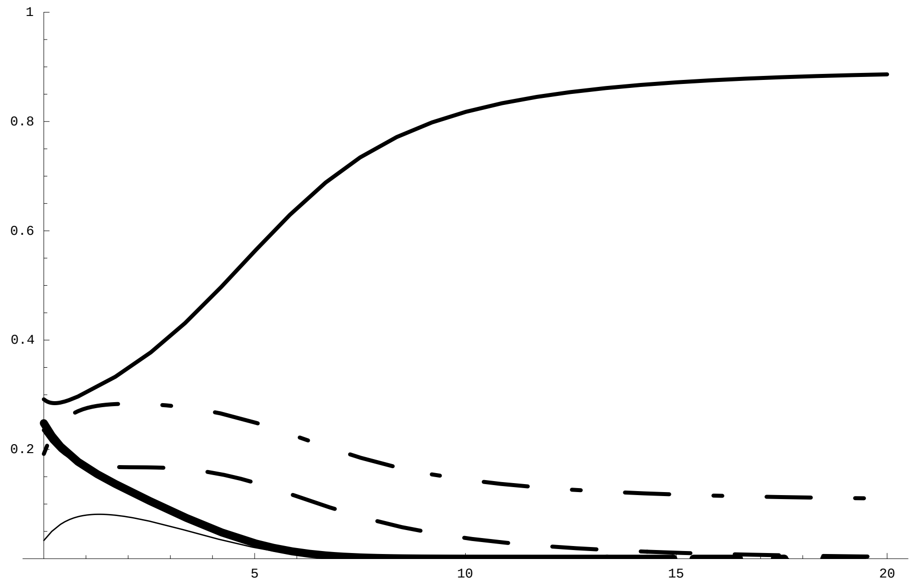

FIGURE 8. Evolution of the components of f_2, $d_c = 2$ central opinion survives

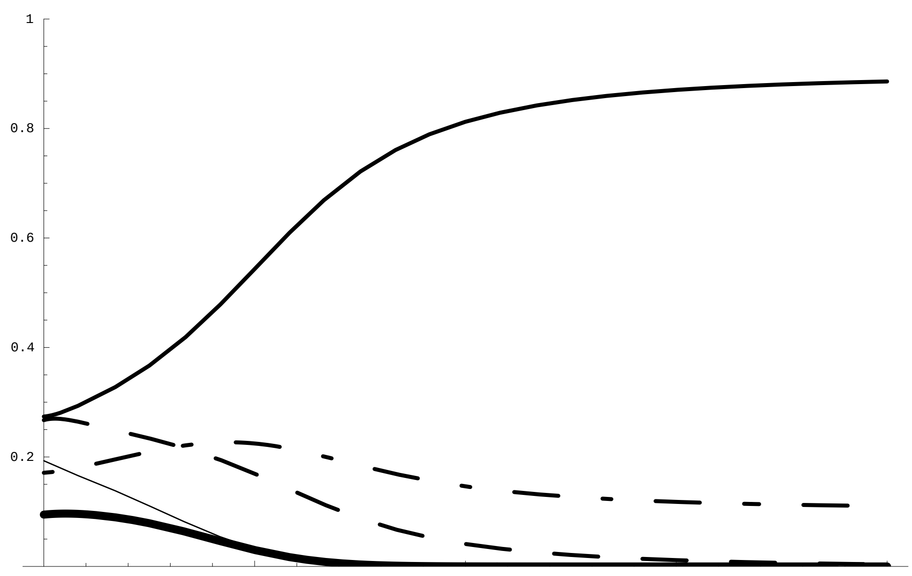

FIGURE 9. Evolution of the components of f_3, $d_c = 2$ central opinion survives

When $d_c = 3$, and especially $d_c = 4$, convergence is faster and central opinions always survive while extreme ones do not. If d_c is big, people are no longer repulsed but attracted by someone else's ideas and everyone goes towards the central moderate opinion, so that finally all the population agrees!

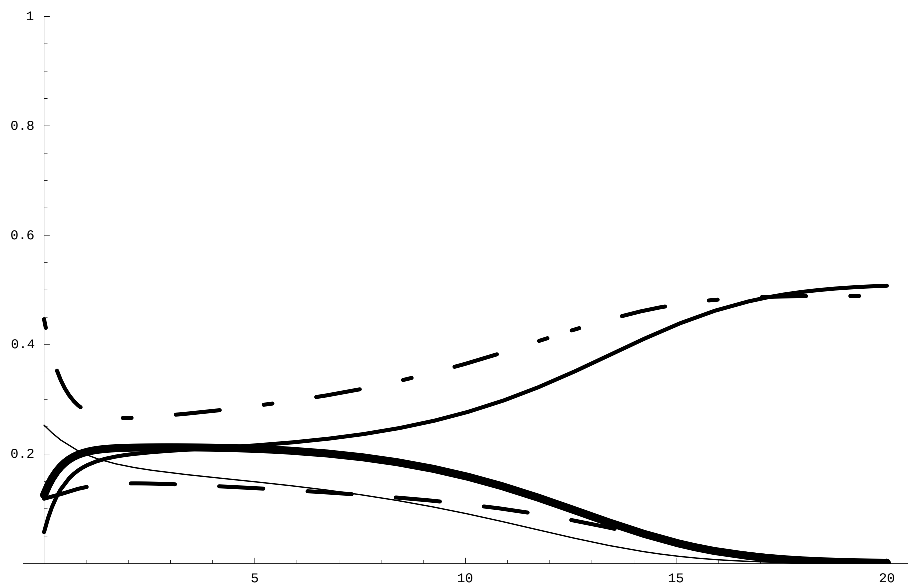

FIGURE 10. Evolution of the components of f_1, $d_c = 2$ two central opinions survive

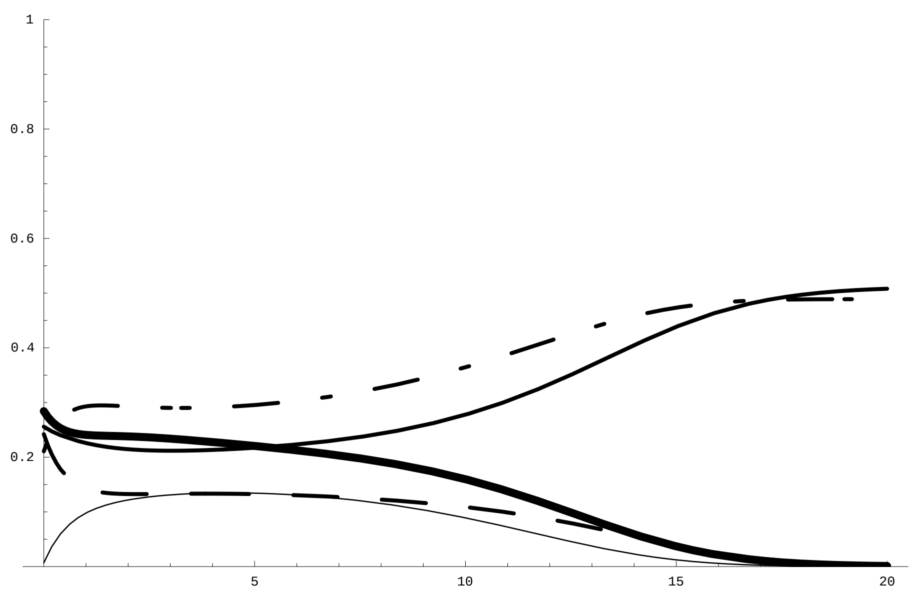

FIGURE 11. Evolution of the components of f_2, $d_c = 2$ two central opinions survive

7.3. **Reasonings on further modelling issues.** The mathematical model proposed and analysed in the preceding subsection has to be regarded as a simple application to show the theoretical approach of this paper applied to a specific case selected among those introduced in Section 2. Indeed, it is a toy model designed with explorative aims.

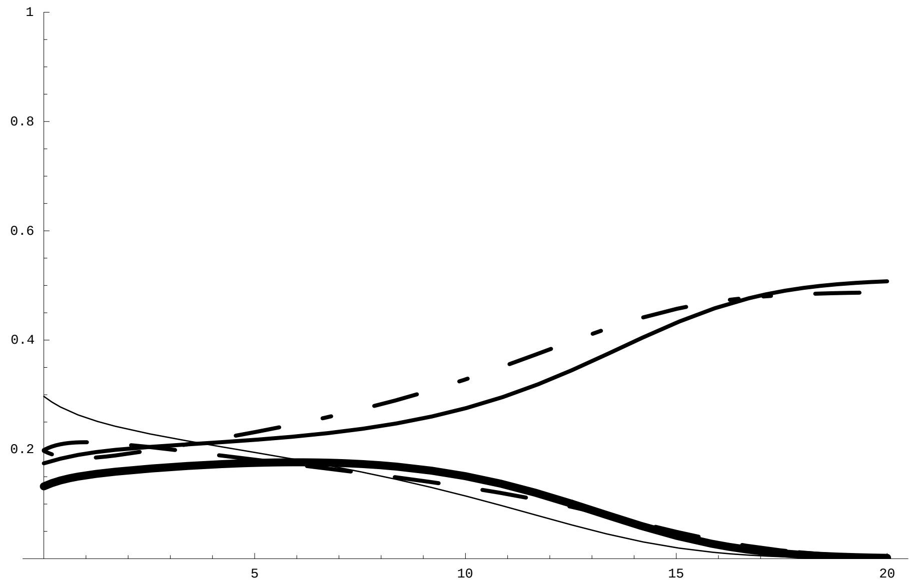

FIGURE 12. Evolution of the components of f_3, $d_c = 2$ two central opinions survive

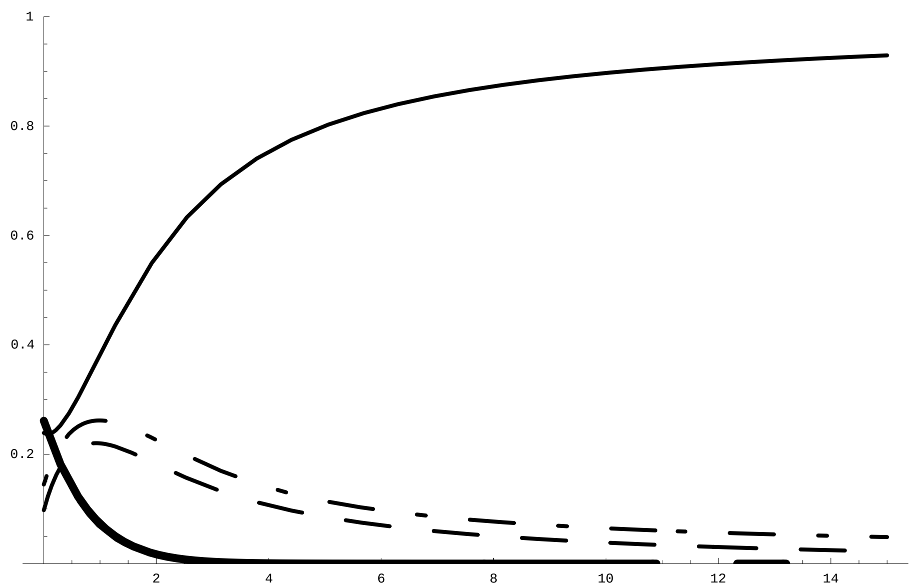

FIGURE 13. Evolution of the components of f_1, $d_c = 3, 4$ one central opinion survives

Some interesting aspects have been shown focused on the role of persuasion due to exchange of opinions related also to the presence of extremist positions. As we have seen, fixed extreme opinions play the role of attractor of opinions, that makes useless the exchange of opinions. This phenomenon is currently observed in politics where conflicts are often artificially radicalized to avoid a democratic dialogue, possibly based on ideas

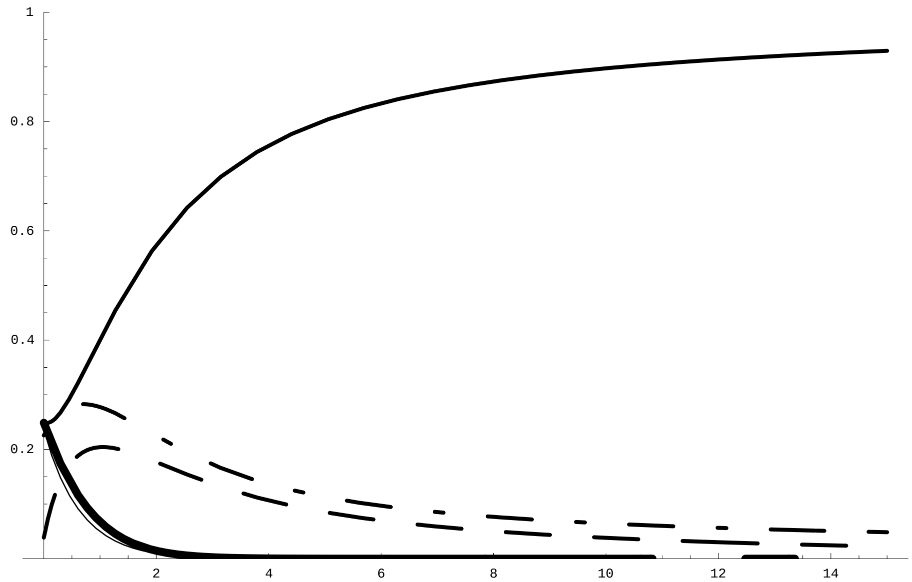

FIGURE 14. Evolution of the components of f_2, $d_c = 3, 4$ one central opinion survives

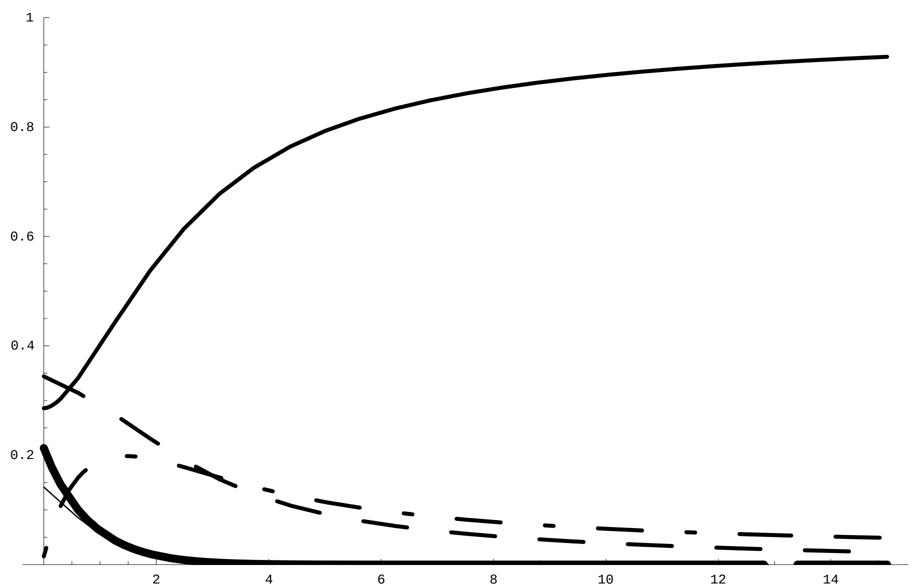

FIGURE 15. Evolution of the components of f_3, $d_c = 3, 4$ one central opinion survives

and social projects, that may eventually shift the majority of holders of a certain political idea towards an undesired direction.

The modelling has been developed in the case of absence of uniformly external actions. Specifically, an interesting modelling perspective refers to a detailed analysis of the role of external actions. The dynamics that has been shown in the simulations suggest to avoid

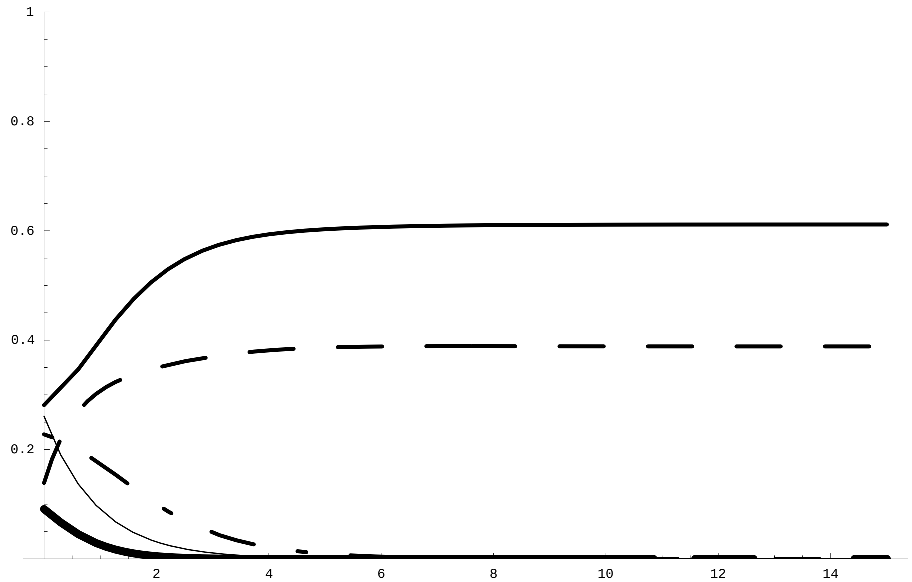

FIGURE 16. Evolution of f_1, $d_c = 3, 4$ two central opinions survive

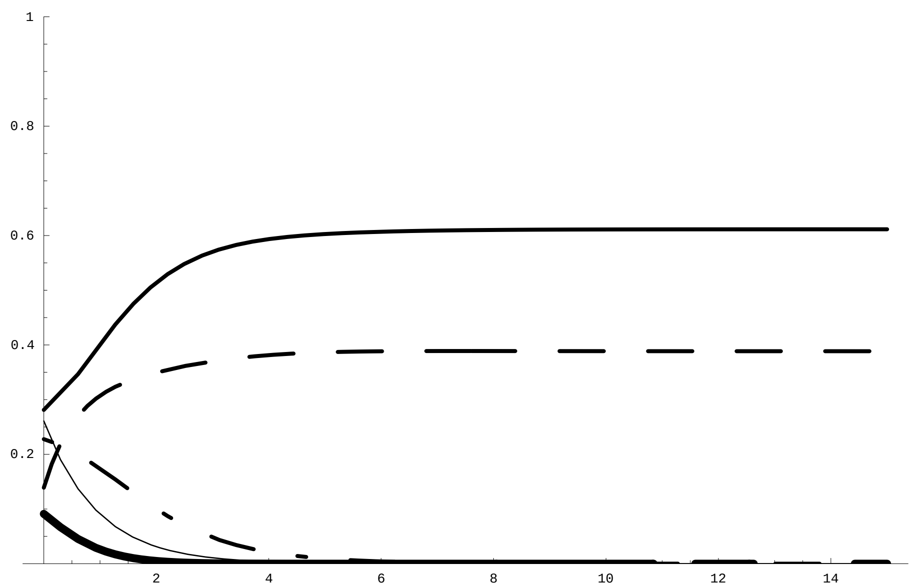

FIGURE 17. Evolution of f_2, $d_c = 3, 4$ two central opinions survive

external uniformly applied to the whole range of opinions. Rather, the actions should be concentrate on a few, may even be one, specific opinion. The investigation should also consider the cost of persuasive actions and the optimization of the whole persuasive action at fixed costs.

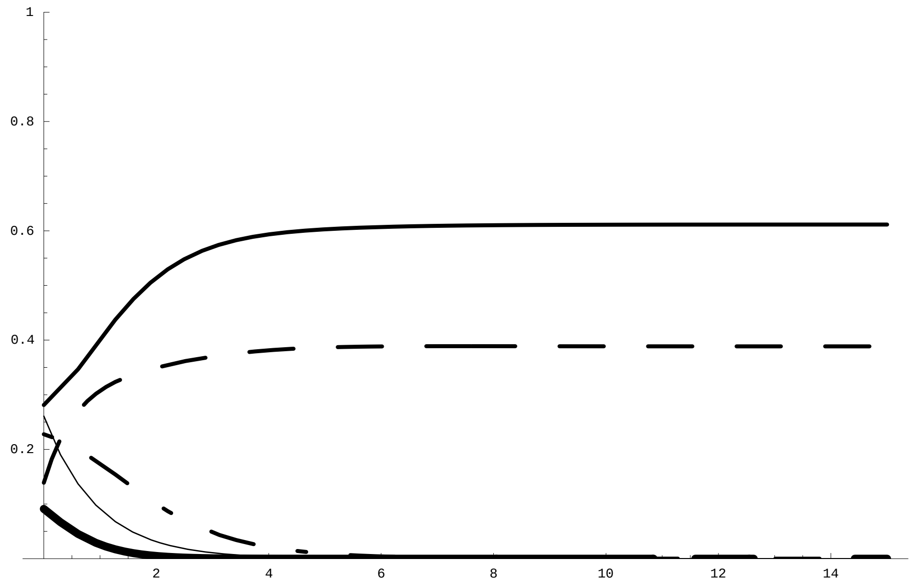

FIGURE 18. Evolution of f_3, $d_c = 3, 4$ two central opinions survive

Fixed opinions 1 and 5			
	opinions 1 and 5	opinions 2 and 4	opinion 3
$d_c = 0$	always	never	never
$d_c = 1$	always	never	never
$d_c = 2$	always	never	never
$d_c = 3$	always	never	never
$d_c = 4$	always	never	never

Summarizing Table 1 - Survivance of opinions

Mobile opinions			
	opinions 1 and 5	opinions 2 and 4	opinion 3
$d_c = 0$	always	never	never
$d_c = 1$	always	never	never
$d_c = 2$	depending on i.c.	depending on i.c.	depending on i.c.
$d_c = 3$	never	depending on i.c.	always
$d_c = 4$	never	depending on i.c.	always

Summarizing Table 2 - Survivance of opinions

8. **Critical analysis and perspectives.** A mathematical approach to social and behavioral economy has been developed in this paper based on the methods of the kinetic theory for active particles and on the characterization of functional subsystems to identify the socio-economical entities that interact in the environment under consideration. Interactions are modelled by stochastic games. An application to modelling the competition for a secession is also dealt with, although at a very preliminary stage with the aim of testing the potential ability of the mathematical approach for the applications and, specifically, to model phenomena of interest to the complex class of systems under consideration.

Considering that this paper aims at initiating a research program and is focused on methodological issues, leaving further applications to future activity, this final section proposes some further speculations related to a deeper understanding of the concept of functional subsystems and to a conceivable development of the methodological approach. Specifically, the three subsections, that conclude this paper deal with the following topics:

i) Decomposition of the whole system into functional subsystems with reference to scaling problems;

ii) Aggregation of functional subsystems into a new subsystem with greater size, or fragmentation of a functional subsystem into two subsystems with smaller size;

iii) Reasonings about the interaction of functional subsystems interconnected in networks.

The analysis is technically referred to the three specific classes of models proposed in the preceding sections. However, it remains at a preliminary level leaving a deeper insight to appropriate research programs. The contents of the following subsection refer to the already cited papers by Bertotti and Delitala [12], [13], [14], that offer a valuable source of ideas for the analysis of this paper. A closure on general issues is given in the last subsection.

8.1. **Additional reasoning on the concept of functional sub-systems.** The modelling approach proposed in this paper is based on the concept of functional subsystem considered as an aggregation of groups of interest expressing a common socio-economical function. It is worth stressing that this concept is flexible being related to specific events and issues that are object of modelling. Therefore, if the general context changes, the characterization and size of the functional subsystems also may have to be modified.

In principles, the function expressed by a subsystem can be a vector; however, the complexity of modelling interactions suggests to refine the identification of the subsystem by an additional decomposition to obtain that each of them expresses one function only. The examples reported in Section 2 clarify the above concept considering that, in each example, the identification of the groups of interest (identified as functional subsystems and reported in the first line of the three tables) refers to the general context (reported in the second line), while the function that is expressed is reported in the third line. When the general context and the socio-economical analysis changes, the identification of the groups of interest and their expression has to be modified.

8.2. **Aggregation and fragmentation of functional sub-systems.** The modelling approach is based on the assumption that the number of functional subsystems is constant in time. On the other hand, their aggregation or fragmentation can possibly occur in some circumstances. For instance the afore mentioned events may occur referred to the size of the subsystems or to the presence of radicalized opinions or interests.

The research perspective of including these events is definitely interesting. However, it cannot be pursued by a straightforward generalization of the approach proposed in this paper, while a further development of the mathematical structure developed in Section 5 cannot describe aggregation or fragmentation events unless properly modified. Some constructive suggestions are offered in Chapter 4 of Ref. [7], that may be possibly adapted to the class of systems under consideration.

8.3. **Interactions in networks.** The modelling dealt with in the preceding section considers interactions that do not depend on the geometry of the system and specifically on the localization of the interacting subsystems. This simplification is valid, when communications occur through delocalized devices: for instance media, internet, and so on. On the other hand, communications may be constrained by networks that organize and select the dialogue between pairs of functional subsystems.

The above research perspective needs, as in the previous case, an additional development of the mathematical framework that may possibly include multiple interactions [27], [8].

8.4. **Closure.** The contents of this paper has been focused on the modelling of complex socio-economical systems where individual behaviors play a crucial role in the interactions among functional subsystems. Interactions are modelled by stochastic games, that define the output of the interactions when the input states are given.

Interactions are stochastic as deterministic rules are not followed. Personal behaviors generate not only deviations from the most probable inputs, but may possibly change due to environmental conditions. For instance the onset of panic conditions.

The mathematical frameworks derived in Section 5 aim at offering the necessary background towards the derivation of specific models such as that of Section 6. Actually, this model can be regarded as a simple application proposed to show the application of the method. Although simulations have already shown the ability to describe interesting phenomena, further developments are necessary to enrich its descriptive ability. Indeed, some perspective ideas have already been considered, some of them induce further conceivable development of the mathematical approach.

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