A Singularly Perturbed Convection Diffusion Turning Point Problem with an Interior Layer

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Abstract

A linear singularly perturbed interior turning point problem with a continuous convection coefficient is examined in this paper. Parameter uniform numerical methods composed of monotone finite difference operators and piecewise-uniform Shishkin meshes, are constructed and analysed for this class of problems. A refined Shishkin mesh is placed around the location of the interior layer and we consider disrupting the centre point of this fine mesh away from the point where the convection coefficient is zero. Numerical results are presented to illustrate the theoretical parameter-uniform error bounds established.

Keywords: Singularly Perturbed; Shishkin mesh; Interior Turning Point

1 Introduction

Interior layers exhibiting a hyperbolic tangent profile can arise in solutions of singularly perturbed quasilinear problems of the form

\[ \varepsilon u'' - uu' - b(x)u = f(x), \quad x \in (0,1), b \geq 0, u(0) > 0, u(1) < 0; \quad (1.1) \]

when the singular perturbation parameter \( \varepsilon > 0 \) can be arbitrarily small. Asymptotic expansions can be used to locate the interior point \( p \), where \( u(p) = 0 \), to within an \( O(\varepsilon) \) neighbourhood of some known point \( d^* \). That is, \( p \in (d^* - C\varepsilon, d^* + C\varepsilon) \) (see Howes [5]).

In this paper, we examine numerical methods for a class of linear problems associated with the above nonlinear problem. For example, problems of the form

\[ \varepsilon y'' - \tanh\left(\frac{d - x}{2\varepsilon}\right)y' - y = f, \quad x \in (0,1), \quad y(0) > 0 > y(1), \quad d \in (0,1) \quad (1.2) \]

will be studied. For such a problem, an interior layer forms about the interior point \( d \). A parameter-uniform numerical method [1], based on an upwind finite difference operator and a piecewise-uniform Shishkin mesh ([1]), will be constructed and analysed in this paper for these type of problems. We will also consider the effect of centring the piecewise-uniform mesh at some point \( d^N \in (d - C\varepsilon \ln N, d + C\varepsilon \ln N) \). The resulting numerical analysis may prove useful in any future examination of the nonlinear problem (1.1), where the point \( p \) is only known to lie within some interval \( (d^* - C\varepsilon, d^* + C\varepsilon) \).

Singularly perturbed turning point problems of the form

\[ -\varepsilon y'' - a(x)y' + b(x)y = f(x), \quad x \in (0,1), \quad a(d) = 0, \quad d \in (0,1), \quad b > 0 \]

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have been studied by several authors (Farrell [3], Berger et al.[2]). In this case, where the
convective coefficient $a$ is independent of $\varepsilon$, then the nature of any interior layer is different to
the problem being considered in the current paper. Depending on the quantity
$|a'(x)| \geq \frac{1}{2}|a'(d)|$, $x \in (0,1)$ is placed on this problem, whereas in (1.2), the convective coefficient
$\tilde{a}:= 2 \tanh(\frac{d-x}{2\varepsilon})$ satisfies $|\tilde{a}'(x)| \leq |\tilde{a}'(d)|$, $x \in (0,1)$.

Exponential interior layers can be generated by considering linear problems with discontinuous
coefficients of the form: Find $y \in C^1(0,1)$ such that

$$
\varepsilon y'' - a(x)y' - b(x)y = f(x), \quad x \in (0, d) \cup (d, 1) \\
a(x) \geq \alpha > 0, \quad x < d, \quad a(x) \leq -\alpha < 0, \quad a(d^+) \neq a(d^-).
$$

Interior layers of exponential type form in the vicinity of the point of discontinuity in the
coefficient $a$ (Farrell et al. [4]). The numerical analysis associated with such problems relies
heavily on the fact that the coefficient $a$ is strictly bounded away from zero in the interval
$(0, d) \cup (d, 1)$.

Finally, in [6], we examined a boundary turning point problem of the form: Find $y \in C^1(0,1)$
such that

$$
\varepsilon y'' + a(x)y' = f(x), \quad x \in (0,1), \quad y(0), \ y(1) \text{ given}.
$$

The presence of the coefficient

$$
a_\varepsilon(x) \geq C(1 - e^{-\alpha x/\varepsilon}), \quad a_\varepsilon(0) = 0,
$$
generated a boundary layer of exponential type in the vicinity of the boundary point $x = 0$. For
this problem, the turning point at $x = 0$. In the current paper, the point where the convection
coefficient vanishes is in the interior of the domain. As a consequence, the analysis for the
current problem is significantly different to the analysis presented in [6]. In particular, the
discrete error analysis is more intricate. In §2 we state the class of problems examined in this
paper and derive a priori bounds on the derivatives of the solutions. In §3, we construct and
analyse a set of numerical methods for this class of problems. The numerical methods consist of
an upwind finite difference operator on piecewise uniform meshes, which are fine in the vicinity
of the interior layer. In the final section, some numerical results are presented to illustrate the
theoretical error bounds. Note that throughout the paper, the notation $f^{(k)}$ denotes the $k$-th
derivative of $f$, $C$ denotes a generic constant that is independent of $\varepsilon$ and $N$, and $\|\cdot\|$ denotes
the maximum pointwise norm.

\section{Continuous Problem}

Consider the following problem class on the unit interval $\Omega = (0,1)$. Find $y_\varepsilon$ such that

$$
L_\varepsilon y_\varepsilon(x) := (\varepsilon y_\varepsilon'' - a_\varepsilon(x)y_\varepsilon' - b_\varepsilon(x)) = f(x), \quad x \in \Omega, \quad y_\varepsilon(0) = y_0 > 0, \quad y_\varepsilon(1) = y_1 < 0, \\
av_\varepsilon, b, f \in C^2((0,1) \setminus \{d\}) \cap C^0[0,1], \quad b(x) \geq 0, \quad x \in \Omega,
$$

$$
av_\varepsilon(x) > 0 \text{ for } x \in [0,d), \quad a_\varepsilon(d) = 0, \quad a_\varepsilon(x) < 0 \text{ for } x \in (d,1].
$$

We will show that the solution to $(P_\varepsilon)$ exhibits an interior layer in a neighbourhood of the point
d. Additional restrictions on the function $a_\varepsilon$ are listed in (2.1) below.

**Assumptions on the coefficient $a_\varepsilon$ in $(P_\varepsilon)$**
Denote $\Omega^- := (0, d)$ and $\Omega^+ := (d, 1)$. Define the limiting functions $a^-_0$ and $a^+_0$ as $a^-_0(x) := \lim_{x \to 0} a_\varepsilon(x)$, $x \in [0, d)$ and $a^+_0(x) := \lim_{x \to 0^+} a_\varepsilon(x)$, $x \in (d, 1]$ and $a^+_0(x) := \lim_{x \to d^+} a_\varepsilon(x)$. Assume the following conditions on $a_\varepsilon$:

$$|a_\varepsilon(x)| > |\alpha_\varepsilon(x)|, \quad x \neq d, \quad \alpha_\varepsilon(x) := \theta \tanh(r(d-x)/\varepsilon), \quad \theta > 2r > 0, \quad x \in \bar{\Omega}, \quad (2.1a)$$

$$\int_{t=0}^x |a'_\varepsilon(t)| \, dt \leq C, \quad x \in \bar{\Omega}, \quad (2.1b)$$

$$\varphi_\varepsilon^\pm(x) := (a_\varepsilon^\pm - a_\varepsilon(x)) \text{ satisfies } |\varphi_\varepsilon^\pm(x)| \leq |\varphi_\varepsilon^\pm(d)|e^{\frac{q}{\varepsilon^2}(d-x)}, \quad x \in \Omega^\pm. \quad (2.1c)$$

Note that (2.1a) implies $a_\varepsilon^+(x) \geq \theta, x \in \Omega^-$ and $a_\varepsilon^+(x) \leq -\theta, x \in \Omega^+$. The differential operator $L_\varepsilon$ defined in problem $(P_\varepsilon)$ satisfies the following minimum principle.

**Theorem 2.1.** Let $L_\varepsilon$ be the differential operator defined in $(P_\varepsilon)$ and $z \in C^2(\Omega) \cap C^0(\bar{\Omega})$. If $\min \{z(0), z(1)\} \geq 0$ and $L_\varepsilon z(x) \leq 0$ for $x \in \Omega$, then $z(x) \geq 0$ for all $x \in \bar{\Omega}$.

**Proof.** The proof is by contradiction. Assume that there exist a point $p \in \bar{\Omega}$ such that $z(p) < 0$. It follows from the hypotheses that $p \notin \{0, 1\}$. Define the auxiliary function $u = ze^{\frac{1}{2r}\int_{t=0}^x \alpha_\varepsilon(t) \, dt}$ and note that $u(p) < 0$. Choose $q \in \Omega$ such that $u(q) = \min u(x) < 0$. Therefore, from the definition of $q$, we have $u'(q) = 0$ and $u''(q) > 0$. But then, using $a_x^2 > 2a_x'$, we have

$$L_\varepsilon z(q) = [\varepsilon u'' + \frac{1}{4\varepsilon}(\alpha_\varepsilon(2a_x - \alpha_\varepsilon) - 2\varepsilon a_x' + 4\varepsilon b)(-u)](q)e^{-\frac{1}{2r}\int_{t=0}^x \alpha_\varepsilon(t) \, dt} > 0$$

which is a contradiction. \qed

**Lemma 2.1.** Assuming $(2.1)$, there exists a unique solution, $y_\varepsilon$, of $(P_\varepsilon)$ such that

$$|y_\varepsilon^{(k)}(x)| \leq C\varepsilon^{-k}, \quad k = 0, 1, 2, \quad x \in \bar{\Omega}.$$

**Proof.** Define the two barrier functions $\psi^\pm(x) := \|f\|_{2(\Omega)}[(x-d)\alpha_\varepsilon(x) + \theta] + \max \{|y_0|, |y_1|\} \pm y_\varepsilon(x)$, which are nonnegative at $x = 0, 1$. From (2.1) we have

$$\varepsilon a_x'' - a_x a_x' \begin{cases} \geq \varepsilon a_x'' - a_x a_x' = (\frac{2r}{\theta} - 1)\alpha_x a_x' \begin{cases} \geq 0, \quad x \leq d, \\ \leq 0, \quad x \geq d. \end{cases} \end{cases} \quad (2.2a)$$

and

$$2\varepsilon a_x' - a_x a_x \leq 2\varepsilon a_x' - a_x^2 = (\frac{2r}{\theta} - 1)\alpha_x^2 - 2\theta r \leq -2\theta r, \quad x \in \bar{\Omega}. \quad (2.2b)$$

Using the definition of the problem $(P_\varepsilon)$ and the inequalities $(2.2)$, we can deduce that

$$L_\varepsilon \psi^\pm \leq \frac{\|f\|_{2(\Omega)}}{2\theta r}[(x-d)(\varepsilon a_x'' - a_x a_x') + (2\varepsilon a_x' - a_x a_x)] + \|f\| \leq 0.$$

Hence, using Theorem 2.1, we have $\|y_\varepsilon\| \leq C$ on $\bar{\Omega}$.

We now bound the derivatives of $y_\varepsilon$. Using $(P_\varepsilon)$ and the assumptions in (2.1) we have for any $x \in \bar{\Omega}$:

$$\int_{t=0}^x \varepsilon y_x''(t) \, dt = \varepsilon y_x'(x) - \varepsilon y_x'(0) = \int_{t=0}^x (f + by_x + a_x y_x')(t) \, dt, \quad (2.3a)$$

$$\left| \int_{t=0}^x (a_x y_x')(t) \, dt \right| \leq \|a_x y_x\|_0 + \|y_x\| \int_{t=0}^x |a_x'(t)| \, dt \leq C, \quad (2.3b)$$

$$\left| \int_{t=0}^x (by_x + f)(t) \, dt \right| \leq \|b\| \|y_x\| + \|f\|. \quad (2.3c)$$
By the Mean Value Theorem \( \exists z \in (0, \varepsilon) \) s.t.

\[
\varepsilon |y'_\varepsilon(z)| \leq 2\|y_\varepsilon\|.
\]  

(2.3d)

Using (2.3) with \( x = z \) we obtain

\[
\varepsilon |y'_\varepsilon(0)| \leq \|f\| + \left(2 + \|b\| + 2\|a_\varepsilon\| + \int_0^\varepsilon |a'_\varepsilon(t)| \, dt\right) \|y_\varepsilon\| \leq C.
\]

Then for any \( x \in [0, 1] \) we have

\[
\varepsilon |y'_\varepsilon(x)| \leq 2\|f\| + \left(2 + 2 \left(\|b\| + 2\|a_\varepsilon\| + \int_0^x |a'_\varepsilon(t)| \, dt\right)\right) \|y_\varepsilon\|.
\]

Hence \( \|y'_\varepsilon\| \leq C\varepsilon^{-1} \) and use the differential equation in \((P_\varepsilon)\) to bound \( y''_\varepsilon \).

We split the problem \((P_\varepsilon)\) into left and right problems around \( x = d \) as follows. Define

\[
y_\varepsilon(x) = \begin{cases} y_L(x), & x \leq d, \\ y_R(x), & x \geq d, \end{cases}
\]

where the left and right problems are defined by

\[
L_\varepsilon y_L(x) = f(x), \quad x \in \Omega^-, \\
y_L(0) = y_0, \quad y_L(d) = y_\varepsilon(d), \quad (P_L) \\
L_\varepsilon y_R(x) = f(x), \quad x \in \Omega^+ \\
y_R(0) = y_\varepsilon(d), \quad y_R(1) = y_1, \quad (P_R).
\]

From [6], we have the existence of a unique \( y_L \) and \( y_R \) s.t. \( \|y^{(k)}_{L,R}\| \leq C\varepsilon^{-k}, \ k = 0, 1, 2 \). We decompose each \( y_{L,R} \) into the sum of a regular component, \( v_{L,R} \), and a layer component, \( w_{L,R} \). If \( a_\varepsilon \) satisfied the bound \( a_\varepsilon \geq C > 0 \) for all \( x \in (0, d) \), then, as in [4], we would simply define the left regular component as the solution of \( L_\varepsilon v_L = f \) with suitable boundary conditions and the left layer component as the solution of \( L_\varepsilon w_L = 0, \ w_L(0) = 0, \ w_L(1) = (y_\varepsilon - v_L)(0) \). Since \( a_\varepsilon \) does not satisfy such a lower bound, we study the problem

\[
L^- v_L(x) := \varepsilon v''_L + a_0^- v'_L + b v_L = f(x), \quad x \in \Omega^-, \quad v_L(0) = y_0, \quad v_L(d) = (v_0 + \varepsilon v_1)(d),
\]  

(2.4a)

where \( v_L = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 \) and the subcomponents \( v_0, v_1, v_2 \) satisfy

\[
(a_0^- v'_0 + b v_0)(x) = -f(x), \quad x \in (0, d), \quad v_0(0) = y_0, \quad (2.4b)
\]

\[
(a_0^- v'_1 + b v_1)(x) = \begin{cases} v''_0(x), & x \in (0, d) \\ \lim_{t \downarrow d} v''_0(t), & x = d \end{cases}, \quad v_1(0) = 0, \quad (2.4c)
\]

\[
L^- v_2(x) = -v''_1(x), \quad x \in \Omega^-, \quad v_2(0) = v_2(d) = 0. \quad (2.4d)
\]

Note that in problem (2.4a), the coefficient \( a_\varepsilon \), of the first derivative term, has been replaced by the strictly positive \( a_0^- \) defined in (2.1). We incorporate the error \((L_\varepsilon - L^-) v_L \) into the layer component \( w_\varepsilon \), which is, noting (2.1c), defined as the solution of

\[
L_\varepsilon w_L(x) = \varphi^- v'_L(x), \quad x \in \Omega^-, \quad w_L(0) = 0, \quad w_L(d) = (y_\varepsilon - v_L)(d). \quad (2.4e)
\]

Similarly \( v_R \) and \( w_R \) satisfy

\[
L^+ v_R(x) := \varepsilon v''_R + a_0^+ v'_R - b v_R = f(x), \quad x \in \Omega^+, \quad v_R(d) = v^*, \quad v_R(1) = y_1, \quad (2.5a)
\]
\[ L_\varepsilon w_R(x) = \varphi^+_\varepsilon v'_R(x), \quad x \in \Omega^+, \quad w_R(d) = (y_\varepsilon - v_R)(d), \quad w_R(1) = 0, \quad (2.5b) \]

where \( v^* \) is chosen in an analogous fashion to \( v_L(d) \) so that we may bound the derivatives of \( v_R \) appropriately.

**Lemma 2.2.** If \( v_L \) and \( w_L \) are the solutions of (2.4) and \( v_R \) and \( w_R \) are the solutions of (2.5) then for \( k = 0, 1, 2, 3 \), we have the following bounds on the derivatives of \( v_L \) and \( w_L \): 

\[
\begin{align*}
|v_L^{(k)}(x)| &\leq C(1 + \varepsilon^{2-k}) \quad \text{and} \quad |w_L^{(k)}(x)| \leq C\varepsilon^{-k}e^{-\frac{\theta}{\varepsilon}(d-x)}, \quad x \in \Omega^-, \\
|v_R^{(k)}(x)| &\leq C(1 + \varepsilon^{2-k}) \quad \text{and} \quad |w_R^{(k)}(x)| \leq C\varepsilon^{-k}e^{-\frac{\theta}{\varepsilon}(d-x)}, \quad x \in \Omega^+,
\end{align*}
\]

where \( \theta \) is given in (2.1a).

**Proof.** Analysis of problems (2.4a) can be carried out effectively in the same manner as \([1, \S 3.3]\) to establish the bounds

\[
\|v_0^{(k)}\|, \|v_1^{(k)}\| \leq C \quad \text{and} \quad \|v_2^{(k)}\| \leq C\varepsilon^{-k}, \quad k = 0, 1, 2, 3. \quad (2.6)
\]

Consider the barrier functions

\[
\psi^\pm(x) = me^{-\frac{\theta}{\varepsilon} \int_{t=x}^d \alpha_\varepsilon(t) dt \pm w_L(x)}, \quad x \in \Omega^-, \quad m := \max \{ |w_L(d), \frac{2}{\varepsilon} \|v'_L\| \varepsilon \},
\]

which are nonnegative at \( x = 0, 1 \). Using (2.1), (2.6) we have

\[
L_\varepsilon \psi^\pm(x) \leq -\frac{m\theta r}{2\varepsilon} e^{-\frac{\theta}{\varepsilon}(d-x)} + |\delta_\varepsilon(0)|\|v'_L\| e^{-\frac{\theta}{\varepsilon}(d-x)} \leq 0.
\]

Note that \( \int_{t=x}^d \alpha_\varepsilon(t) dt = \frac{\theta}{\varepsilon} \ln \cosh (\frac{\varepsilon}{\theta}(d-x)) \). Thus we can use (2.1) and \( \cosh t \geq \frac{1}{2} e^t, \ t > 0 \), to show that

\[
e^{-\frac{\theta}{\varepsilon}(d-x)} \leq e^{-\frac{1}{2} \int_{t=x}^d \alpha_\varepsilon(t) dt} \leq 2\varepsilon e^{-\frac{\theta}{\varepsilon}(d-x)}.
\]

Hence using the minimum principle in Theorem 2.1 we obtain \( |w_L(x)| \leq C e^{-\frac{\theta}{\varepsilon}(d-x)} \). The bounds on the derivatives of \( w_L \) are established as in [6]. Analysis of \( v_R \) and \( w_R \) is performed in the same manner. \( \square \)

### 3 The Discrete Problem and Error Analysis

Given the bounds in Lemma 2.2 on the solution, it is natural to refine the mesh in the vicinity of the point \( d \). We examine such a mesh below. In addition, we consider the effect of centring the mesh at some other point \( d^N \) near the point \( d \).

Consider the following finite difference method. Find \( Y_\varepsilon \) such that

\[
L_\varepsilon^N Y_\varepsilon(x_i) := (\varepsilon \delta^2 - a_\varepsilon D - b) Y_\varepsilon(x_i) = f(x_i), \quad x_i \in \Omega^N, \quad Y_\varepsilon(0) = y_0, \quad Y_\varepsilon(1) = y_1,
\]

\[
D := \begin{cases} 
D^- & \text{if } a_\varepsilon(x_i) \geq 0, \\
D^+ & \text{if } a_\varepsilon(x_i) < 0, \end{cases} \quad \delta^2 Z(x_i) := \frac{D^+(Z(x_i) - Z(x_{i-1}))}{(h_{i+1} + h_i)/2}, \quad h_i := x_i - x_{i-1},
\]

where \( D^\pm \) are the standard forward and backward finite difference operators. We define the
piecewise uniform mesh, $\Omega^N_\varepsilon$, with a refined mesh centred at $d^N$ by
\[
|d^N - d| < p\sigma, \quad p \leq \frac{1}{2}, \quad \mu = \frac{1}{2} \min \{d^N, 1 - d^N\}, \quad \sigma := \min \{\mu, \varepsilon \ln N\},
\]
(3.1a)
\[
H_0 := \frac{1}{N}(d^N - \sigma), \quad h := \frac{1}{N}\sigma, \quad H_1 := \frac{1}{N}(1 - d^N - \sigma),
\]
(3.1b)
\[
\bar{\Omega}^N_\varepsilon := \begin{cases} 
  x_i = H_0 i, & 0 \leq i \leq \frac{N}{4}, \\
  x_i = \frac{x}{N} + h(i - \frac{N}{4}), & \frac{N}{4} < i \leq \frac{3N}{4}, \\
  x_i = x + H_1(i - \frac{3N}{4}), & \frac{3N}{4} < i \leq N.
\end{cases}
\]
(3.1c)
\[
\Omega^N_\varepsilon := \bar{\Omega}^N_\varepsilon \setminus \{x_0, x_N\}.
\]

Note, if $d^N = d$ then set $p = 0$. If $d^N \neq d$, then the mesh is not aligned to the point $d$.

To prove the existence of a discrete solution to $(P^N_\varepsilon)$, we construct discrete analogues of the barrier functions used in Lemma 2.1. However, in place of $\alpha_\varepsilon$ in these barrier functions, we will construct and use a discrete analogue $A_\varepsilon$.

We identify the nearest mesh point to the left of $d$ in this non-aligned mesh as $x_0$. That is
\[
x_Q := \max \{x_i | x_i \leq d\}.
\]
(3.2)

Define the following mesh functions
\[
S(x_i) := (1 + \rho)^{Q-i} - (1 - \rho)^{Q-i}, \quad C(x_i) := (1 + \rho)^{Q-i} + (1 - \rho)^{Q-i},
\]
(3.3a)
\[
T(x_i) := \frac{S(x_i)}{C(x_i)}, \quad \rho = rh\varepsilon^{-1}, \quad 0 \leq i \leq N.
\]
(3.3b)
The mesh functions $S$, $C$ and $T$ can be thought of as discrete analogues of the $2\sinh$, $2\cosh$ and $\tanh$ functions centred at $x_Q$ respectfully. Note that from (3.1), we have $\rho \leq CN^{-1}\ln N \rightarrow 0$ as $N \rightarrow \infty \forall \varepsilon$. Some properties of these mesh functions are given in Lemma A.1 in Appendix A.

Define the operators
\[
D_h^+ Z(x_i) := \frac{1}{h}(Z(x_i) - Z(x_{i-1})), \quad D_h^- := D_h^+ Z(x_{i+1}), \quad \delta_h^2 := \frac{1}{h}((D_h^+ - D_h^-)Z(x_i)).
\]
(3.4)

Observe that in the fine mesh; $D_h^- = D^-$ and in the coarse mesh; $D_h^- \neq D^-$. The purpose of defining such operators is for convenience. We present identities and inequalities that result when these operators are applied to the mesh function $T$ in Lemma A.2 in Appendix A.

We now present the discrete function $A_\varepsilon$ which will replace $\alpha_\varepsilon$ in discrete analogues of the barrier functions in Lemma 2.1. We distinguish between the cases $\sigma < \mu$ ($\sigma \equiv C\varepsilon \ln N$) and $\sigma = \mu$ ($\sigma \equiv C$). When $\sigma < \mu$, we define $A_\varepsilon$ as follows
\[
A_\varepsilon(x_i) := \begin{cases} 
  A_\varepsilon(x_{\frac{N}{4}}) + (x_{\frac{N}{4}} - x_i)\frac{5\rho}{2\mu}N^{-(1-2p)}, & 0 \leq i < \frac{N}{4}, \\
  \theta T(x_i), & \frac{N}{4} \leq i \leq \frac{3N}{4}, \\
  A_\varepsilon(x_{\frac{3N}{4}}) - (x_{\frac{3N}{4}} - x_i)\frac{5\rho}{2\mu}N^{-(1-2p)}, & \frac{3N}{4} < i \leq N.
\end{cases}
\]
(3.5a)
When $\sigma = \mu$, we consider the case $\mu = \frac{1}{2}(1 - d^N)$ (where $H_0 \geq h = H_1$) and define $A_\varepsilon$ as follows
\[
A_\varepsilon(x_i) := \begin{cases} 
  A_\varepsilon(x_{\frac{N}{4}}) + \frac{H_2}{h} \theta(T(x_i) - T(x_{\frac{N}{4}})), & 0 \leq i < \frac{N}{4}, \\
  \theta T(x_i), & \frac{N}{4} \leq i \leq N.
\end{cases}
\]
(3.5b)
When $\mu = \frac{dN}{r}$ (where $H_0 = h \leq H_1$) we can define $A_\varepsilon$ in the same manner. We prove convergence of $A_\varepsilon$ to $\alpha_\varepsilon$ in the following lemma, the proof of which is given in Appendix B.

**Lemma 3.1.** For sufficiently large $N$, the function $\alpha_\varepsilon$ defined in (2.1) and the mesh function $A_\varepsilon$ defined in (3.5) satisfies:

$$|(\alpha_\varepsilon - A_\varepsilon)(x_i)| \leq CN^{-1}(\ln N)^2 + CN^{-(1-2p)}.$$

By the assumption (2.1a), and assuming $N$ is sufficiently large, we have

$$|a_\varepsilon(x_i)| \geq |A_\varepsilon(x_i)|, \quad x_i \in \bar{\Omega}_N.$$

(3.6)

The finite difference operator $L_\varepsilon^N$ satisfies the following discrete minimum principle.

**Theorem 3.1.** Let $L_\varepsilon^N$ be the difference operator defined in $(P_\varepsilon^N)$ and $Z$ be a mesh function on $\bar{\Omega}_N$. If $\min \{Z(x_0), Z(x_N)\} \geq 0$ and $L_\varepsilon^N Z(x_i) \leq 0$ for $x_i \in \Omega_N$, then $Z(x_i) \geq 0$ for all $x_i \in \bar{\Omega}_N$.

**Proof.** Suppose that $Z(x_q) = \min, Z(x_i) < 0$. If follows from the hypotheses that $q \notin \{0, N\}$ and $x_q \in \bar{\Omega}_N$. Since $Z(x_q)$ is the minimum value we have $D^- Z(x_q) \leq 0$, $D^+ Z(x_q) \geq 0$ and $\delta^2 Z(x_q) \geq 0$. If $x_q < d$, to avoid a contradiction we must have $L_\varepsilon^N Z(x_q) \leq 0$ but if $b(x_q) > 0$ then $L_\varepsilon^N Z(x_q) > 0$ or if $b(x_q) = 0$ then $Z(x_{q-1}) = Z(x_q) = Z(x_{q+1}) < 0$. Repeat this argument will eventually lead us to conclude that either $L_\varepsilon^N Z(x_q) > 0$ or $Z(x_0) < 0$ for $x_q < d$ which is a contradiction. If $x_q > d$ the same argument leads to a contradiction and finally a contradiction can also be established for $x_q = d$. \hfill \Box

**Lemma 3.2.** There exists a unique solution, $Y_\varepsilon$, of $(P_\varepsilon^N)$ such that

$$|Y_\varepsilon(x_i)| \leq C, \quad x_i \in \bar{\Omega}_N.$$

**Proof.** We first establish the following list of inequalities;

$$\varepsilon \delta^2 A_\varepsilon(x_i) - A_\varepsilon(x_i) D^- A_\varepsilon(x_i) \geq 0, \quad 0 < i < Q, \quad (3.7a)$$

$$\varepsilon \delta^2 A_\varepsilon(x_i) - A_\varepsilon(x_i) D^+ A_\varepsilon(x_i) \leq 0, \quad Q < i < N, \quad (3.7b)$$

$$A_\varepsilon(x_i) A_\varepsilon(x_{i-1}) - \frac{2\varepsilon}{h_{i+1} + h_i} (h_i D^- + h_{i+1} D^+) A_\varepsilon(x_i) \geq \theta r, \quad 0 < i \leq Q, \quad (3.7c)$$

$$A_\varepsilon(x_{i+1}) A_\varepsilon(x_i) - \frac{2\varepsilon}{h_{i+1} + h_i} (h_i D^- + h_{i+1} D^+) A_\varepsilon(x_i) \geq \theta r, \quad Q < i < N, \quad (3.7d)$$

which are discrete counterparts to the inequalities (2.2). For convenience in this proof, we write $A_\varepsilon(x_i) = A_i$ and $h_{i+1} + h_i = \Sigma h$. Recall from the assumption (2.1) that $\theta > 2r$. We first consider the case where $\sigma < \mu$. Using (3.5a) we can show that

$$D^- A_\varepsilon(x_i) = \{- \frac{5r}{2dN} N^{-(1-2p)}, \quad 0 < i \leq \frac{N}{4}, \quad \frac{3N}{4} < i \leq N, \quad \theta D^- T_i, \quad \frac{N}{4} < i \leq \frac{3N}{4}\} \quad (3.8a)$$

and

$$\delta^2 A_\varepsilon(x_i) = \begin{cases} 0, & \max \{i, N-i\} < \frac{N}{4} \\ \frac{2(D^+ - D^-)(A_i)}{h + H_0}, & i = \frac{N}{4}, \frac{3N}{4} \\ \theta \delta h T_i, & \frac{3N}{4} < i < N. \end{cases} \quad (3.8b)$$

For (3.7a), the result is trivial for $0 < i < \frac{N}{4}$. For $i = \frac{N}{4}$, using Lemma A.2 and (3.8), for sufficiently large $N$, we have

$$\varepsilon \delta^2 A_i - A_i D^- A_i = (\frac{N}{2dN} + A_i)(-D^- A_i) + \frac{N}{2dN} \varepsilon D^+ A_i \geq \frac{5r\theta}{4\mu} N^{-(1-2p)} - \frac{N}{2dN}(5r\theta N^{-2(1-p)}) \geq 0.$$
For $\frac{N}{4} < i < Q$, use (A.6g). Bound (3.7b) in the same manner. For (3.7c), for $0 < i \leq \frac{N}{4}$, using (A.6), for sufficiently large $N$, we have

$$A_i A_{i-1} - \frac{2\varepsilon}{\Sigma_h} (h_i D^- + h_{i+1} D^+)A_i \geq A_i A_{i-1} \geq A_i^2 \geq (1-C\rho)^2 \theta^2 \geq \frac{1}{2} \theta^2 > \theta r.$$ 

For $\frac{N}{4} < i \leq Q$, use (A.6h). The bound (3.7d) is established in the same manner.

Next, consider the case where $\sigma = \mu = \frac{1}{2}(1-d^N)$. Using (3.5b) we can show that

$$D^- A_\varepsilon(x_i) = \theta D^- T_i, \quad \delta^2 A_\varepsilon(x_i) = \{ \frac{\theta}{\Sigma_h} \theta_d^2 T_i, \quad i < \frac{N}{4}, \quad \frac{2\theta}{\Sigma_h} \theta_d^2 T_i, \quad i = \frac{N}{4}, \quad \theta_d^2 T_i, \quad i > \frac{N}{4} \}.$$ 

Using Lemma A.2 and $H_0 \geq h = H_1$, we can prove (3.7a)-(3.7d), in the same manner as the case $\sigma < \mu$. The case $\sigma = \mu = \frac{d^N}{2}$ follows suit.

Define the discrete barrier functions

$$\psi^\pm(x_i) := \|f\|_{\theta r} \max_{x \in \Omega^N} (x_i - x_Q) A_\varepsilon(x_i) + \max_{x \in \Omega^N} |A_\varepsilon(x_i) + Y_\varepsilon(x_i)|,$$ 

which are nonnegative at $x = 0, 1$. Using (3.5a) and Lemma A.2 we can find

$$D^\pm[(x_i - x_Q) A_\varepsilon(x_i)] = (x_i - x_Q) D^\pm A_\varepsilon(x_i) + A_\varepsilon(x_i \pm 1) \begin{cases} \geq 0, & i \leq Q, \\ \leq 0, & i \geq Q. \end{cases}$$

Using this, we can show that

$$L_\varepsilon^N \psi^\pm(x_i) \leq \begin{cases} \|f\|_{\theta r} [(x_i - x_Q) \varepsilon \delta^2 A_\varepsilon - A_\varepsilon D^- A_i] - (A_\varepsilon A_{i-1} - \frac{2\varepsilon}{\Sigma_h} (h_i D^- + h_{i+1} D^+)A_i)] + \|f\|, & 0 < i \leq Q, \\ \|f\|_{\theta r} [(x_i - x_Q) \varepsilon \delta^2 A_\varepsilon - A_\varepsilon D^- A_i] - (A_{i+1} - \frac{2\varepsilon}{\Sigma_h} (h_i D^- + h_{i+1} D^+)A_i)] + \|f\|, & Q < i < N. \end{cases}$$

Using the bounds in (3.7), we have $L_\varepsilon^N \psi^\pm(x_i) \leq 0$ for $x_i \in \Omega^N_\varepsilon$. Thus using Theorem 3.1, we have $|Y_\varepsilon| \leq C$ on $\Omega^N_\varepsilon$.

As with the continuous problem $(P_c)$, we split the discrete problem $(P_\varepsilon^N)$ into left and right discrete problems centred around $x = x_Q$ as follows. Define

$$Y_\varepsilon(x_i) = \begin{cases} Y_L(x_i), & x_i \leq x_Q, \\ Y_R(x_i), & x_i \geq x_Q, \end{cases}$$

where the discrete left and right problems are defined by

$$\begin{align*}
L_\varepsilon^N Y_L(x_i) &= f(x_i), & x_i \in \Omega^N_\varepsilon \cap (0, x_Q), \\
Y_L(0) &= y_0, & Y_L(x_Q) = Y_\varepsilon(x_Q),
\end{align*}$$

$$\begin{align*}
L_\varepsilon^N Y_R(x_i) &= f(x_i), & x_i \in \Omega^N_\varepsilon \cap (0, x_Q), \\
Y_R(x_Q) &= Y_\varepsilon(x_Q), & Y_R(1) = y_1.
\end{align*}$$

We decompose each $Y_{L\setminus R}$ into the sum of a regular component, $V_{L\setminus R}$, and a layer component, $W_{L\setminus R}$. We define $V_L$ as the solution of

$$L^N V_L(x_i) := (\varepsilon \delta^2 + a_0^- D^- - b)V_L(x_i) = f(x_i), \quad x_i \in \Omega^N_\varepsilon \cap (0, x_Q)$$

$$V_L(0) = y_0, \quad V_L(x_Q) = (V_0 + \varepsilon V_1)(x_Q),$$

(3.12a)
where $V_L = V_0 + \varepsilon V_1 + \varepsilon^2 V_2$ and $V_0, V_1, V_2$ satisfy
\begin{equation}
(a_0^- D^+ + b)V_0(x_i) = -f(x_i), \quad x_i \in \Omega_e^N \cap (0, x_Q), \quad V_0(0) = y_0, \quad (3.12b)
\end{equation}
\begin{equation}
(a_0^- D^+ + b)V_i(x_i) = \begin{cases} \delta^2 V_0(x_i), & x_i \in \Omega_e^N \cap (0, x_Q) \\ \delta^2 V_0(x_{i-1}), & x_i = x_Q \end{cases}, \quad V_1(0) = 0, \quad (3.12c)
\end{equation}
\begin{equation}
L^N V_2(x_i) = -\delta^2 V_1(x_i), \quad x_i \in \Omega_e^N \cap (0, x_Q), \quad V_2(0) = V_2(x_Q) = 0. \quad (3.12d)
\end{equation}

We incorporate the error $(L_e - L^N) V_L$ into the discrete layer component, $W_L$, which is, noting (2.1c), defined as the solution of the following discrete problem
\begin{equation}
L^N W_L(x_i) = \varphi^- D^- V_L(x_i), \quad x_i \in \Omega_e^N \cap \Omega^-, \quad (3.12e)
\end{equation}
\begin{equation}
W_L(x_i) = 0, \quad W_L(x_Q) = (Y_L - V_L)(x_Q). \quad (3.12f)
\end{equation}

Similarly $V_R$ and $W_R$ satisfy
\begin{equation}
L^N V_R(x_i) := (a_0^+ D^+ - b)V_R(x_i) = f(x_i), \quad x_i \in \Omega_e^N \cap (x_Q, 1), \quad (3.13a)
\end{equation}
\begin{equation}
V_R(x_Q) = v^*, \quad V_R(1) = y_1, \quad (3.13b)
\end{equation}
\begin{equation}
L^N W_R(x_i) = \varphi^+ D^+ V_R(x_i), \quad x_i \in \Omega_e^N \cap (x_Q, 1), \quad (3.13c)
\end{equation}
\begin{equation}
W_L(x_i) = (Y_R - V_R)(x_Q), \quad W_R(1) = 0. \quad (3.13d)
\end{equation}

where $v^*$ is defined analogously to $V_L(x_Q)$ in (3.12). We now determine bounds on $V_L, W_L, V_R, D^\pm V_L, D^\pm V_R$ and on the error $V_L - v_L, W_L - v_L$. But first we present a mesh function that we will use in the barrier functions for the layer component, along with some of its properties. The proof of this Lemma is given in Appendix C.

**Lemma 3.3.** For sufficiently large $N$, the mesh function defined as
\begin{equation}
\hat{W}(x_i) := \prod_{j=i+1}^Q \left( 1 + \frac{\alpha e_j h_j}{2 \varepsilon} \right) -1, \quad 0 \leq i < Q, \quad \hat{W}(x_Q) := 1
\end{equation}
satisfies
\begin{equation}
\hat{W}(x_i) \geq (1 - C \rho) e^{-\frac{\rho}{\pi} (d-x_i)}, \quad 0 \leq i \leq Q, \quad (3.14a)
\end{equation}
\begin{equation}
\hat{W}(x_i) \leq C e^{-\frac{\rho}{\pi} (d-x_i)}, \quad \frac{N}{4} \leq i \leq Q (\sigma < \mu), \quad 0 \leq i \leq Q (\sigma = \mu). \quad (3.14b)
\end{equation}
\begin{equation}
|\hat{W}(x_i)| \leq CN^{-(1-p)}, \quad i \leq \frac{N}{4} (\sigma < \mu). \quad (3.14c)
\end{equation}

**Lemma 3.4.** If $v_L, w_L$ are the solutions of (2.4), $v_R, w_R$ are the solutions of (2.5), $V_L, W_L$ are the solutions of (3.12) and $V_R, W_R$ are the solutions of (3.13) then we have the following bounds
\begin{equation}
|D^- V_L(x_i)| \leq C, \quad x_i \in \Omega_e^N \cap (0, x_Q); \quad |(V_L - v_L)(x_i)| \leq CN^{-1} x_i, \quad x_i \in \Omega_e^N \cap [0, x_Q], \quad (3.15a)
\end{equation}
\begin{equation}
|D^+ V_R(x_i)| \leq C, \quad x_i \in \Omega_e^N \cap [x_Q, 1); \quad |(V_R - v_R)(x_i)| \leq CN^{-1} (1 - x_i), \quad x_i \in \Omega_e^N \cap [x_Q, 1], \quad (3.15b)
\end{equation}
\begin{equation}
|W_L(x_i)| \leq \begin{cases} Ce^{-\frac{\rho}{\pi} (d-x_i)} + CN^{-(1-p)}, & N \leq i \leq Q (\sigma < \mu), \quad 0 \leq i \leq Q (\sigma = \mu), \\ CN^{-(1-p)}, & 0 \leq i \leq \frac{N}{4} (\sigma < \mu). \end{cases} \quad (3.15c)
\end{equation}
\begin{equation}
|W_R(x_i)| \leq \begin{cases} Ce^{-\frac{\rho}{\pi} (x_i-d)} + CN^{-(1-p)}, & Q \leq i \leq \frac{3N}{4} (\sigma < \mu), \quad Q \leq i \leq N (\sigma = \mu), \\ CN^{-(1-p)}, & \frac{3N}{4} \leq i \leq N (\sigma < \mu). \end{cases} \quad (3.15d)
\end{equation}
Proof. If $Z$ is a mesh function on $\Omega^N_x \cap [0,x_Q]$ then we can easily prove that:

\[
\text{if } Z|_{x_0} \geq 0 \text{ and } (a_0D - b)Z|_{\Omega^N_x \cap [0,x_Q]}|_{x_0} \geq 0 \text{ then } Z|_{\Omega^N_x \cap [0,x_Q]} \geq 0. \tag{3.15}
\]

Consider the functions $\psi^\pm(x_i) = y_0 + \frac{h_i}{4} x_i \pm 0(x_i)$. Using (2.1), (3.12b) and (3.15) we can show that $\|V_0\|, \|\delta^2 V_0\|, \|\delta^2 V_0\| \leq C$. Similarly we can show that $\|V_1\|, \|D^{-1}v\|, \|\delta^2 V_1\| \leq C$. For (3.12d), using the results in [1, §3.3] we have $|V_2(x_i)| \leq C x_i, x_i \in \Omega^N_x \cap [0,x_Q]$. Repeat [1, §3.5, Lemma 3.14] and obtain $\|D^{-1}V_2\| \leq C$. Thus it follows that $\|D^{-1}V_L\| \leq C$.

Using (2.4) with (3.13) we can define the errors $E_0 := V_0 - v_0, E_1 := V_1 - v_1$ and $E := V_0 - v_0$ as the solutions of

\[
(a_0D - b)E_0(x_i) = a_0(v'_0 - D^{-1}v_0)(x_i), \quad x \in \Omega^N_x \cap (0,x_Q), \quad E_0(0) = 0, \tag{3.16a}
\]

\[
(a_0D - b)E_1(x_i) = (a_0(v'_1 - D^{-1}v_1) - (v''_0 - \delta^2 V_0))(x_i), \quad x_i \in \Omega^N_x \cap (0,x_Q), \quad E_1(0) = 0, \tag{3.16b}
\]

\[
L^N_x E(x_i) = (L^N - L^N_v) v_L(x_i), \quad x_i \in \Omega^N_x \cap (0,x_Q), \quad E(0) = 0, \quad E(x_Q) = (E_0 + \varepsilon E_1)(x_Q). \tag{3.16c}
\]

Using Lemma 2.2, we can use standard truncation error estimates to show that

\[
\|v_0' - D^{-1}v_0\| \leq C\|v_0''\| \max h_i \leq CN^{-1}, \tag{3.17a}
\]

\[
\|v_0'' - \delta^2 v_0\| \leq C\|v_0''\| \max h_i \leq CN^{-1}, \tag{3.17b}
\]

\[
\|v_1' - D^{-1}v_1\| \leq C\|v_0''\| \max h_i \leq CN^{-1}, \tag{3.17c}
\]

\[
\|L^N - L^N_x v_L\| \leq C(\|v_0''\| + \|v_L''\|) \max h_i \leq CN^{-1}. \tag{3.17d}
\]

Using suitable barrier functions, we can easily show that $|E_0(x_i)| \leq CN^{-1} x_i$. Using $(v_0' - D^{-1}v_0)(x_i) = \frac{1}{h_i} \int_{x_{i-1}}^{x_i} v_0'(x_i) - v_0'(t) \, dt$ we can show that

\[
|\{v_0'' - \delta^2 V_0\}(x_i)| \leq |\{v_0'' - \delta^2 v_0\}(x_i)| + |\delta^2 E_0(x_i)| \leq CN^{-1} + \frac{CN^{-1}}{C, \quad i \neq \frac{N}{4}, \quad i = \frac{N}{4}}.
\]

Rewrite (3.16b) as

\[
E_1(x_i) = [E_1(x_{i-1}) + \frac{h_i}{a_0}(a_0(v'_1 - D^{-1}v_1) - (v''_0 - \delta^2 V_0))(x_i)](1 + \frac{bh_i}{a_0}(x_i))^{-1}
\]

\[= \sum_{j=1}^{i} \prod_{k=j}^{i} (1 + \frac{bh_i}{a_0}(x_k))^{-1} h_j(a_0(v'_1 - D^{-1}v_1) - (v''_0 - \delta^2 V_0))(x_j). \tag{3.17e}
\]

Using $1 + h_i(a_0(x_i))^{-1} < 1$ and using (3.17) we have

\[
|E_1(x_i)| \leq i|h_i||CN^{-1} + C \sum_{j=1}^{i} h_j(v_0'' - \delta^2 V_0)(x_j)|. \tag{3.17f}
\]

For $i < \frac{N}{4}$, we clearly have $|E_1(x_i)| \leq CN^{-1} + C|h_i||CN^{-1} \leq CN^{-1}$ and for $i \geq \frac{N}{4}$ we have $|E_1(x_i)| \leq CN^{-1} + C h_{\frac{N}{4}} + C|h_i|(i - \frac{N}{4})CN^{-1} \leq CN^{-1}$. Use suitable barrier functions with (3.16c) to find that $|E(x_i)| \leq CN^{-1} x_i$ for $x_i \in \Omega^N_x \cap [0,x_Q]$. The proof for the bounds on $V_R$ can be established in an analogous manner.

We now bound the layer component $W_L$. We first establish a few inequalities. Using $\frac{1}{2}e^t \leq
cosh $t \leq e^t$, $t > 0$ and using (2.1), (A.5) and Lemma A.2, bound $\varphi^-$ and $a_\varepsilon$ for sufficiently large $N$ as follows
\begin{equation}
\dot{W}(x_{i-1}) \geq e^{-\frac{\theta}{\pi}(d-x_i + h)} \geq \frac{1}{2} e^{-\frac{\theta}{\pi}(d-x_i)}, \quad \frac{N}{4} < i < Q (\sigma < \mu), \quad 0 < i < Q (\sigma = \mu)
\end{equation}
\begin{equation}
\alpha_{\varepsilon}(x_i) \geq \frac{\theta}{2}, \quad 0 \leq i \leq \frac{N}{4}, (\sigma < \mu)
\end{equation}
\begin{equation}
|\varphi^-(x_i)| \leq N^{-(1-p)}, \quad 0 \leq i \leq \frac{N}{4}, (\sigma < \mu).
\end{equation}

Consider the functions $\psi^\pm(x_i) = m_1 \dot{W}(x_i) + m_2 N^{-(1-p)} x_i \pm W_L(x_i)$, $m_2 := \frac{\theta}{2} ||\varphi^-||D^- V_L||$, $m_1 := \max \{|W_L||, \frac{\theta}{2} m_2\}$ which are nonnegative at $x = 0, 1$. Consider the $\sigma < \mu$ case. Using Lemma 3.3, (3.18) and (3.13) and noting for any $x_i$, we have $\alpha'_{\varepsilon}(x_{i+1}) \leq D^+ \alpha_{\varepsilon}(x_i) \leq \alpha'_{\varepsilon}(x_i)$. Then for sufficiently large $N$ we have
\begin{equation}
L^N_{\varepsilon} \psi^\pm(x_i) \leq m_1 (\varepsilon \delta^2 \dot{W} - \alpha_{\varepsilon} D^- \dot{W})(x_i) - \alpha_{\varepsilon}(x_i)m_2 N^{-(1-p)} + ||D^- V_L|||\varphi^-|(x_i)|
\end{equation}
\begin{equation}
\leq -m_1 \frac{h_{i+1}}{2\varepsilon(h_{i+1} + h_i)} (\alpha^2_{\varepsilon} - 2\varepsilon \alpha'_{\varepsilon}) \dot{W}(x_{i-1}) - \alpha_{\varepsilon}(x_i)m_2 N^{-(1-p)} + ||D^- V_L|||\varphi^-|(x_i)|
\end{equation}
\begin{equation}
\leq \left\{ \begin{array}{ll}
-m_1 \frac{\theta}{4} N^{-1} + |\varphi^-|(D^- V_L)||N^{-(1-p)}| & \leq 0, \quad 0 \leq i \leq \frac{N}{4}, \\
-m_1 \frac{\theta}{4} e^{-\frac{\theta}{\pi}(d-x_i)} + |\varphi^-|(D^- V_L)e^{-\frac{\theta}{\pi}(d-x_i)} & \leq 0, \quad \frac{N}{4} < i < Q.
\end{array} \right.
\end{equation}

Hence using Theorem 3.1, we have $|W_L(x_i)| \leq C(\dot{W}(x_i) + N^{-(1-p)})$. The case $\sigma = \mu$ is proved in the same manner as for $\frac{N}{4} < i < Q$ above. The proof of the bound on $W_R$ can be performed in a similar way.

A bound on the error $Y_{\varepsilon} - y_{\varepsilon}$ is given in the following lemma.

**Lemma 3.5.** If $y_{\varepsilon}$ is the solution of $(P_{\varepsilon})$ and $Y_{\varepsilon}$ is the solution of $(P^N_{\varepsilon})$ then
\begin{equation}
|(Y_{\varepsilon} - y_{\varepsilon})(x_i)| \leq C N^{-1}(\ln N)^2 + C N^{-(1-p)}, \quad \forall x_i \in \Omega^N_{\varepsilon},
\end{equation}
where $|d^N - d| < p\sigma$ and $p \leq \frac{1}{2}$.

**Proof.** First, if $\sigma < \mu$ then for $i \leq \frac{N}{4}$, using Lemma 2.2, Lemma 3.4 and (3.1), (A.5) we have
\begin{equation}
|(Y_{\varepsilon} - y_{\varepsilon})(x_i)| \leq |W_L(x_i)| + |w_L(x_i)| + |V_L - v_L| \leq C N^{-(1-p)}.
\end{equation}
Similarly, if $\sigma < \mu$ then for $i \geq \frac{3N}{4}$, we have $|(Y_{\varepsilon} - y_{\varepsilon})(x_i)| \leq C N^{-(1-p)}$.

We now need to examine the error over all mesh points $x_i \in [x_L, x_R]$ where $x_L = 0$ if $\sigma = \mu$ or $x_L = x_N$ if $\sigma < \mu$ and $x_R = 1$ if $\sigma = \mu$ or $x_R = x_{N+1}$ if $\sigma < \mu$. Note the implication that for any $x_i \in (x_L, x_R)$, we have $h_i / \varepsilon \leq C N^{-1} \ln N$. The error $E(x_i) := (Y_{\varepsilon} - y_{\varepsilon})(x_i)$ is defined as a solution of the following
\begin{equation}
L_{\varepsilon}^N E(x_i) = (L_{\varepsilon} - L_{\varepsilon}^N)y_{\varepsilon}, \quad x_i \in \Omega^N_{\varepsilon} \cap (x_L, x_R), \quad |E(x_{L\setminus R})| \leq C N^{-(1-p)},
\end{equation}
Note the following truncation error estimate
\begin{equation}
|\varepsilon(\delta^2 y_{\varepsilon} - y''_{\varepsilon})(x_i)| \leq \frac{2\varepsilon}{h_{i+1} + h_i} \sum_{j=i}^{i+1} \frac{1}{h_j} \int_{t=x_{j-1}}^{x_j} \int_{s=x_i}^{s} y''_{\varepsilon}(s) - y''_{\varepsilon}(x_i) \, ds \, dt.
\end{equation}
For any \( s \in [x_{i-1}, x_{i+1}] \subset [x_L, x_R] \cap \Omega_e^N \), using \((P_\varepsilon)\), \((2.1)\) and Lemma 2.1, we can bound

\[
\varepsilon |y'_e(s) - y''_e(x_i)| \leq (\|a_\varepsilon\| \|y''_e\| + \|b\| + \int_0^1 a_\varepsilon' \, dt \|y'_e\| + \|b'\| \|y_e\| + \|f'\|)|s - x_i| \leq C_{\varepsilon} \frac{b_5}{\varepsilon^2}. \tag{3.21}
\]

Thus using \((3.21)\) with \((3.20)\) we have

\[
|(L_\varepsilon - L_\varepsilon^N)y_e(x_i)| \leq C_{\varepsilon} \varepsilon^{-2} \leq C_{\varepsilon}^{-1} N^{-1} \ln N, \quad x_i \in \Omega_e^N \cap (x_L, x_R). \tag{3.22}
\]

Hence, using the barrier functions in \((3.10)\) we can show that

\[
|E(x_i)| \leq C_{\varepsilon}^{-1} N^{-1} \ln N |(x_i - x_Q)A_\varepsilon(x_i) + \max_{x_i \in \Omega_e^N \cap [x_L, x_R]} (x_i - x_Q)A_\varepsilon(x_i)| + C N^{-1-p} \tag{3.23}
\]

\[
\leq \begin{cases} 
\frac{C_{\varepsilon}}{\varepsilon} N^{-1} \ln N + C N^{-1-p} \leq C N^{-1} (\ln N)^2 + C N^{-1-p}, & \sigma < \mu, \\
\frac{C_{\varepsilon}}{\varepsilon} N^{-1} \ln N + C N^{-1-p} \leq C N^{-1} (\ln N)^2 + C N^{-1-p}, & \sigma = \mu.
\end{cases} \tag{3.24}
\]

Complete the proof using Theorem 3.1. \(\square\)

The nodal bound on the error is easily extended to a global error bound.

**Lemma 3.6.** If \( y_e \) is the solution of \((P_\varepsilon)\) and \( Y_e \) is the solution of \((P^N_\varepsilon)\) then

\[
|\bar{Y}_e - y_e)(t)| \leq C N^{-1} (\ln N)^2 + C N^{-1-p}, \quad t \in [0, 1].
\]

where \( \bar{Y}_e \) is the piecewise linear interpolant of \( Y_e \) on \([0, 1]\).

**Proof.** Proof follows the corresponding proof in [1, Thm 3.12]. \(\square\)

### 4 Numerical Examples

**Example 1**

In this example from the class \((P_1)\) we consider \( a_\varepsilon(x) = (2.25 + x^2) \tanh \left( \frac{1}{\varepsilon} (0.6 - x) \right) \), \( b(x) = e^{-5x} \), \( f(x) = \cos(3x) \), \( A = 2 \), \( B = -3 \), \( d = 0.6 \). We choose \( r = 1 \) and \( \theta = 2.1 \). We consider various values of \( p \).

To generate a value of \( d^N \), we start with a scale factor of \( \kappa = 0.99 \) and reduce by 0.01 until \( d^N = d \pm \kappa p \min \{ \frac{1}{\varepsilon} \min (d, 1 - d), \frac{1}{\sigma} \ln N \} \) satisfies \( |d - d^N| < p \sigma \). \(\tag{4.1}\)

We consider \( p = 0, p = 0.5 \) and \( p = 1 \) and using \((4.1)\) we centre the mesh at \( d^N = d \), \( d^N \approx d - 0.5 \sigma \) and \( d^N \approx d - \sigma \). We compute the approximate errors

\[
E^N_{\varepsilon} = \max_{\Omega_e^N \cup \Omega_e^{8192}} |U^N_{\varepsilon} - \bar{U}^{8192}_{\varepsilon}|
\]

where \( \bar{U}^M_e \) is \( U^M_e \), the numerical solution of \((P^N_\varepsilon)\) using \( N = M \) mesh points, interpolated onto the mesh \( \Omega_e^N \cup \Omega_e^{8192} \). Table 4.1 displays the approximate errors \( E^N_{\varepsilon} \) and the uniform errors \( E^N = \max_{\varepsilon} E^N_{\varepsilon} \), using \((4.2)\). Shifting the mesh off-centre within the limit of \( p \leq \frac{1}{2} \) has little to no effect on the differences. We further test for an effect on the value of \( p \) by producing the computed rates of convergence \( p^N_{\varepsilon} \) and the uniform rates of convergence \( p^N \), computed using the double mesh principle (see [1]), as shown in Table 4.2. The \( N^{-1-p} \) factor established in Lemma 3.5 is not evident for \( p \leq \frac{1}{2} \). However, for \( p = 1 \) we see a collapse in the computed rates of convergence.
## Table 4.1: Approximate errors for Example 1.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$N$</th>
<th>$d^N = d$</th>
<th>$d^N \approx d + 0.5\sigma$</th>
<th>$d^N \approx d + \sigma$</th>
</tr>
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<tr>
<td></td>
<td>32</td>
<td>64</td>
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<td>32</td>
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</tr>
<tr>
<td>$2^{-10}$</td>
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<td>0.99</td>
<td>0.99</td>
<td>1</td>
</tr>
<tr>
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</tr>
<tr>
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<td>0.98</td>
<td>0.99</td>
</tr>
<tr>
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<tr>
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<td>0.96</td>
<td>0.98</td>
</tr>
<tr>
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## Table 4.2: Computed orders of convergence for Example 1.

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<th>$\varepsilon$</th>
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<th>$d^N = d$</th>
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</table>

Table 4.2: Computed orders of convergence for Example 1.
In Figure 4.1, we present comparisons between numerical solutions using $N = 32$ and the mesh centred at $d^N = d$ and $\varepsilon = 2^{-0}, \varepsilon = 2^{-2}, \varepsilon = 2^{-5}$ and $\varepsilon = 2^{-10}$.

In Figure 4.2, we present comparisons between numerical solutions using $N = 32$, $\varepsilon = 2^{-8}$ and the mesh centred at $d^N = d$, $d^N \approx d - 0.5\sigma$ and $d^N \approx d - \sigma$. The location of the meshes for each case are superimposed on the figure below the graphs.

Fig. 4.1: Solution of $(P^N_\varepsilon)$ with $N = 32$, $d^N = d$ for $\varepsilon = 2^{-0}, \varepsilon = 2^{-2}, \varepsilon = 2^{-5}$ and $\varepsilon = 2^{-10}$.

Fig. 4.2: Solution of $(P^N_\varepsilon)$ with $N = 32$, $\varepsilon = 2^{-8}$ and $d^N \approx d - p\sigma$ for $p = 0$, $p = \frac{1}{2}$ and $p = 1$. 
A Properties of the discrete functions $S, C$ and $T$.

Lemma A.1. For sufficiently large $N$, the mesh functions $S$ and $C$ defined in (3.3) satisfy:

\[ S(x_{i-1}) > S(x_i), \forall i; \quad S(x_i) > 0, \ i < Q; \quad S(x_Q) = 0; \quad S(x_i) < 0, \ i > Q; \]  
\[ C(x_{i-1}) > C(x_i), \ i < Q; \quad C(x_{i-1}) < C(x_i), \ i > Q; \quad C(x_i) \geq C(x_Q) = 2, \forall i; \]  
\[ i \leq Q-2, \quad \frac{C(x_i)}{N} \leq C(x_{i+1}) \leq \left\{ \begin{array}{ll} C(x_i), & i < Q \\ \frac{C(x_i)}{1+p}, & Q \leq i \end{array} \right. ; \]  
\[ i \leq Q, \quad \frac{C(x_i)}{Q} \leq C(x_{i-1}) \leq \left\{ \begin{array}{ll} (1+p)C(x_i), & i < Q \\ C(x_i), & Q \leq i \end{array} \right. ; \]  
\[ T(x_i) > 0, \ i < Q; \quad T(x_Q) = 0; \quad T(x_i) < 0, \ i > Q; \]  
\[ |T(x_i)| < 1, |T(x_i) - \tanh(\frac{C(x_i)}{Q}h)| \leq CNp^2, \forall i; \quad T(x_{\frac{Q}{2}}), \quad |T(x_{\frac{Q}{2}})| \geq 1 - Cp > \frac{1}{2}. \]  

Proof. For convenience in this proof, we write any mesh function $Z(x_i)$ as $Z_i$. Clearly $C_i \geq 2, \forall i$ and $S_i \geq 0, i < Q, S_i \leq 0, i \geq Q$. Use $S_i - S_{i-1} = -pC_i$ and $C_i - C_{i-1} = -pS_i$ to establish the bounds in (A.1a) and (A.1b). For (A.1c), we can show $C_{i-1} = C_i + pS_i$. We can use

\[ C_i < S_i < C_i \]  

to show that $(1-p)C_i < C_{i-1} < (1+p)C_i$. Combine these with (A.1b) to establish (A.1c). For (A.1d), we can show that $C_{i+1} = \frac{1}{1+p}(C_i - pS_i)$. Thus from (A.2), we have $C_{i+1} < \frac{1+p}{1-p}C_i$. Similarly $C_{i+1} > \frac{1+p}{1-p}C_i$. Combine these with (A.1b) to establish (A.1d). The expressions in (A.1e) follow from (A.1a) and (A.1b). The first bound in (A.1f) follows from (A.2). Using the inequalities

\[ e^{(1-t/2)t} \leq 1 + t \leq e^t \leq 1 + 2t, \quad t \in [0, 0.5], \]  
\[ e^{-(1+t)t} \leq 1 - t \leq e^{-t} \leq 1 - t/2, \quad t \in [0, 0.5], \]

References

we can show that for any sufficiently small $\rho > 0$ and $0 \leq j \leq N$, we have

\[
(1 - CN_0^2) \sinh \rho j - CN_0^2 \leq \frac{1}{2}[1 + \rho]^j - (1 - \rho)^j \leq (1 + CN_0^2) \sinh \rho j + CN_0^2
\]  

(A.4a)

\[
(1 - CN_0^2) \cosh \rho j \leq \frac{1}{2}[1 + \rho]^j + (1 - \rho)^j \leq \cosh \rho j.
\]  

(A.4b)

With a change of variables, we can prove the second bound in (A.1f) using (A.3), (A.4) with (3.3) for sufficiently large $N$. Note that using (3.1) and (3.2), for sufficiently large $N$ we have

\[
x_Q - x_N > (1 - p)\sigma \quad \Rightarrow \quad Q - \frac{N}{4} > (1 - p)\frac{N}{4}, \tag{A.5a}
\]

\[
x\frac{3N}{4} - x_Q > (1 - p)\sigma \quad \Rightarrow \quad \frac{3N}{4} - Q > (1 - p)\frac{N}{4}. \tag{A.5b}
\]

For $i = \frac{N}{4}$ and for sufficiently large $N$, using (A.5), we can write $C_i \leq (1 + C\rho)(1 + \rho)^{Q - \frac{N}{4}}$ and $S_i \geq (1 - C\rho)(1 + \rho)^{Q - \frac{N}{4}}$. Hence $S_i/C_i \geq 1 - C\rho > \frac{1}{2}$. We can write a similar statement for $i = \frac{3N}{4}$.

\[\Box\]

**Lemma A.2.** Assuming (2.1), then for sufficiently large $N$, the operators in (3.4) and the mesh function $T$ defined in (3.3) satisfy:

\[
D_{-}^h T(x_i) < 0, \quad 0 < i \leq N, \quad D_{+}^h T(x_i) < 0, \quad 0 \leq i < N, \tag{A.6a}
\]

\[
\frac{1 + \rho}{1 - \rho} D_{-}^h T(x_i) \leq D_{+}^h T(x_i) \leq D_{-}^h T(x_i), \quad 0 < i < Q, \tag{A.6b}
\]

\[
D_{-}^h T(x_Q) = D_{-}^h T(x_Q), \tag{A.6c}
\]

\[
\frac{1 + \rho}{1 - \rho} D_{+}^h T(x_i) \leq D_{-}^h T(x_i) \leq D_{+}^h T(x_i), \quad Q < i < N, \tag{A.6d}
\]

\[
|T(x_i) - T(x_{\frac{N}{4}})| \leq C\rho, \quad 0 < i < \frac{N}{4}, \quad |T(x_i) - T(x_{\frac{3N}{4}})| \leq C\rho, \quad i > \frac{3N}{4}. \tag{A.6e}
\]

\[
\varepsilon |D_{+}^h T(x_{\frac{N}{4}})|, |D_{-}^h T(x_{\frac{3N}{4}})| \leq 5\varepsilon N^{-2(1 - p)}, \quad \text{when} \quad \sigma = \frac{\varepsilon}{\rho} \ln N, \tag{A.6f}
\]

\[
\varepsilon \delta_{i}^h T(x_i) - \theta T(x_i)D_{-}^h T(x_i) \geq 0, \quad 0 < i < Q, \tag{A.6g}
\]

\[
\varepsilon \delta_{i}^h T(x_i) - \theta T(x_i)D_{+}^h T(x_i) \leq 0, \quad Q < i < N, \tag{A.6h}
\]

\[
T(x_i)T(x_{i-1}) \geq \frac{\rho}{N}(D_{-}^h + D_{+}^h)T(x_i) \geq 2r, \quad 0 < i < Q, \tag{A.6i}
\]

\[
T(x_i)T(x_{i+1}) \geq \frac{\rho}{N}(D_{-}^h + D_{+}^h)T(x_i) \geq 2r, \quad Q < i < N. \tag{A.6j}
\]

**Proof.** To ease notation, we write $U(x_i)$ as $U_i$ for any mesh function $U$. Using (3.3) with (3.4), we can show that

\[
D_{-}^h T(x_i) = \frac{\varepsilon}{N}(T_iT_{i-1} - 1) = \frac{-4\varepsilon / \rho}{C_iC_{i-1}}, \tag{A.7a}
\]

\[
\delta_{i}^h T(x_i) = \frac{\varepsilon}{N} T_i(D_{+}^h + D_{-}^h)T_i. \tag{A.7b}
\]

Use Lemma A and (A.7) to establish (A.6a), (A.6c) and (A.6e). For (A.6b), using Lemma A we have $C_{i+1}C_i \geq \frac{1}{(1 + \rho)^2}C_{i+1}C_i$. Thus

\[
D_{-}^h T(x_i) = \frac{-4\varepsilon / (1 - \rho)^2 C_{i+1}C_i}{1 - \rho^2} \geq \frac{(1 + \rho)^2 - 4\varepsilon / (1 - \rho)^2 C_{i+1}C_i}{1 - \rho^2} = \frac{1 + \rho}{1 - \rho} D_{-}^h T(x_i). \tag{A.8}
\]

Also, from Lemma A.1 and (A.7) we have $\delta_{i}^h T(x_i) \leq 0$, $i \leq Q$. Thus using (3.4), we have $D_{+}^h T(x_i) \leq D_{-}^h T(x_i)$, $i \leq Q$. Verify (A.6d) in the same manner. For (A.6f), when $i = \frac{N}{4}$ then using Lemma A.1 and (3.3) we have

\[
C_{i+1}C_i \geq C_{i+1}^2 = [(1 + \rho)^Q - (i+1) + (1 - \rho)^Q - (i+1)]^2 \geq (1 + \rho)^2(Q - (i+1)). \tag{A.8}
\]
Using (A.5), (A.8) and the inequality \((1 + t)/(1 - t) \geq e^{2t}, 0 < t \leq 0.5\), assuming \(\sigma = \frac{x}{2} \ln N\), then we have
\[
D_h^+ T(x_i) = \frac{-4r^{-1}e^{-1}(1 - \rho^2)Q^{-((i+1)/2)}}{C_{i+1}C_i} \geq -4r^{-1}e^{-1}(1 + \rho)Q^{-((i+1)/2)(1 - \rho)Q^{-((i+1)/2)}}
\]
\[
= -4r^{-1}e^{-1}Q^{-\frac{N}{2}}
\]
\[
\geq -5r^{-1}e^{-2\rho(Q^{-\frac{N}{2}})} = -5r^{-1}N^{2(1-\rho)}.
\]

Use (A.6a) to bound \(D_h^+ T(x_i)\) from above. Bound \(D^{-}T(x_{\frac{3N}{4}})\) in the same manner. For (A.6g), using (2.1), (A.7) and (A.6b) with Lemma A.1 and (A.6a) for \(i < Q\), we have
\[
\epsilon_0^2 T_i - \theta T_i \geq r(D + D^-)A_i - \theta D_h^+ T_i \geq T_i(\frac{2\rho}{1-\rho} - \theta)D_h^+ T_i \geq 0.
\]
The bound (A.6h) is established in the same manner. For (A.6i) on \(0 < i \leq Q\), using (2.1), Lemma A.1, (A.6a) and (A.7) we have
\[
T_i T_{i-1} - \frac{\epsilon}{2}(D_h^+ + D_h^-)T_i \geq T_i T_{i-1} - \frac{2\epsilon}{\theta}D_h^- T_i = \frac{2\rho}{\theta} + (1 - \frac{2\rho}{\theta})T_i T_{i-1} \geq \frac{2\rho}{\theta}.
\]
The bound (A.6j) is established in the same manner. \(\square\)

**B Proof of Lemma 3.1**

**Proof.** First, using (3.1), (3.2) and the identity \(\tanh(X + Y) = \frac{\tanh X + \tanh Y}{1 + \tanh X \tanh Y}\) we have
\[
\tanh(\frac{x}{2}(xQ - x_i)) = \tanh(\frac{\xi}{d - x_i} + \frac{xQ}{xQ - d}) = \frac{\tanh(\frac{x}{2}(d - x_i)) + \tanh(\frac{xQ}{xQ - d})}{1 + \tanh(\frac{x}{2}(d - x_i)) \tanh(\frac{x}{2}(xQ - d))}
\]
\[
\leq \frac{\tanh(\frac{x}{2}(d - x_i)) + \rho}{1 - \rho} \leq \tanh(\frac{x}{2}(d - x_i)) + C\rho.
\]

Construct a corresponding lower bound to give
\[
| \tanh(\frac{x}{2}(d - x_i)) - \tanh(\frac{x}{2}(xQ - x_i)) | \leq C\rho \leq CN^{-1} \ln N, \quad \forall i.
\]

Consider \(i \in S := \{ \lfloor \frac{N}{4} \frac{3N}{4} \rfloor \} \) if \(\sigma < \mu, \lfloor \frac{N}{4} \frac{3N}{4} \rfloor \) if \(\sigma = \frac{1}{2}(1 - dN), [0, \frac{3N}{4}] \) if \(\sigma = \frac{N}{2} \). Using (3.1), we have \(xQ - x_i = x_{\frac{N}{4}} + h(Q - \frac{3N}{4}) - x_{\frac{N}{4}} - h(i - \frac{3N}{4}) = (Q - i)h\). Now using Lemma A.2 we have
\[
|A_e(x_i) - \theta \tanh(\frac{x}{2}(xQ - d))| \leq CN\rho^2.
\]

For all other \(i \notin S\), using Lemma A.2 and (3.5) we have
\[
|A_e(x_i) - \theta \tanh(\frac{x}{2}(xQ - x_i))|
\]
\[
\leq \theta |T_{\frac{N}{4}} \frac{3N}{4} + CN^{(-1-2p)}| + C|T_i - T_{\frac{N}{4}} \frac{3N}{4}| - \theta \tanh(\frac{x}{2}(xQ - x_i))|
\]
\[
\leq C|T_{\frac{N}{4}} \frac{3N}{4} - 1| + CN^{(-1-2p)} + C\rho + C|1 - \theta \tanh(\frac{x}{2}(xQ - x_i))|
\]
\[
\leq C\rho + CN^{(-1-2p)} + C\rho + C|1 - \theta \tanh(\frac{x}{2}(xQ - x_{\frac{N}{4}}))|
\]
\[
\leq C\rho + CN^{(-1-2p)} + C|1 - \theta \tanh(\frac{x}{2}(Q - (\frac{N}{4} \frac{3N}{4}))h)|
\]
\[
\leq C\rho + CN^{(-1-2p)} + C|1 - T_{\frac{N}{4}} \frac{3N}{4}| + C|T_{\frac{N}{4}} \frac{3N}{4} - \theta \tanh(\frac{x}{2}(Q - (\frac{N}{4} \frac{3N}{4}))h)|
\]
\[
\leq C\rho + CN^{(-1-2p)} + C\rho + CN\rho^2.
\]

Thus, \(\forall i\), we have
\[
|A_e(x_i) - \theta \tanh(\frac{x}{2}(xQ - x_i))| \leq C\rho + CN^{(-1-2p)} + CN\rho^2 \leq CN^{(-1-2p)} + CN^{-1} \ln N)^2. \quad \text{(B.2)}
\]

Combine (B.1) and (B.2) to complete the proof. \(\square\)
C Proof of Lemma 3.3

*Proof.* We can easily bound $\hat{W}$ from below using $1 + t \leq e^t$, $t > 0$ with (2.1), (3.1) and (A.5) as follows

$$\hat{W}(x_i) \geq \prod_{j=i+1}^{Q} e^{-\frac{\alpha(x_j) h_j}{2\epsilon}} = e^{-\frac{1}{2\epsilon} \sum_{j=i+1}^{Q} \alpha(x_j) h_j} \geq e^{-\frac{\theta}{2\epsilon} \sum_{j=i+1}^{Q} h_j} = e^{-\frac{\theta}{2\epsilon} (x_Q - x_i)} \geq \frac{1}{2} e^{-\frac{\theta}{2\epsilon} (d - x_i)}.$$

Note the Rectangle Rule for Numerical Integration: Partition the interval $[a, b]$ into $N^*$ subintervals with width $h^* = 1/N^*$, where $t_j = a + j h^*$. Then

$$\int_{a}^{b} f(s) \, ds = h^* \sum_{j=1}^{N^*} f(t_j) + O((b - a) h^*^2 \|f''\|).$$

Using (A.5) with the rectangle rule on $a \leq i \leq Q$ when $(\sigma \leq \mu)$ or on $0 \leq i \leq Q$ when $(\sigma = \mu)$, we have

$$\frac{1}{2\epsilon} \sum_{j=i+1}^{Q} \alpha(x_j) h_j \geq \frac{1}{2\epsilon} \int_{x_i}^{x_Q} \alpha(t) \, dt \geq \frac{C_{x} \epsilon}{\epsilon} (h/\epsilon)^2 \geq \frac{1}{2\epsilon} \int_{x_i}^{x_Q} \alpha(t) \, dt - CN^{-2}(\ln N)^3. \quad (C.1)$$

Using the inequality $1 + t \geq e^t - t^2/2$, $t > 0$ with (2.1), (2.7), (3.1), (A.5) and (C.1) we have

$$\hat{W}(x_i) \leq \prod_{j=i+1}^{Q} e^{-\frac{\alpha(x_j) h_j}{2\epsilon}} \leq e^{CN(N^{-1} \ln N)^2} e^{-\frac{1}{2\epsilon} \sum_{j=i+1}^{Q} \alpha(x_j) h_j} \leq Ce^{-\frac{1}{2\epsilon} f_{x_i}^{x_Q} \alpha(t) \, dt} \leq Ce^{-\frac{1}{2\epsilon} f_{x_i}^{x_Q} \alpha(t) \, dt} \leq Ce^{-\frac{\theta}{2\epsilon} (d - x_i)}.$$

Finally, we have $\hat{W}(x_i) - \hat{W}(x_{i-1}) = \frac{\alpha(x_i) h_i}{2\epsilon} \hat{W}(x_i) > 0$. Thus for $i \leq \frac{N}{4}$ when $(\sigma < \mu)$, using (2.1) and (A.5) we have

$$\hat{W}(x_i) \leq \hat{W}(x_{\frac{N}{4}}) \leq Ce^{-\frac{\theta}{2\epsilon} (d - x_{\frac{N}{4}})} \leq CN^{-(1-p)}.$$

$\square$