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Edge-pancyclicity of Möbius cubes ☆

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Abstract

The Möbius cube M_n is a variant of the hypercube Q_n and has better properties than Q_n with the same number of links and processors. It has been shown by Fan [J. Fan, Hamilton-connectivity and cycle-embedding of Möbius cubes, Inform. Process. Lett. 82 (2002) 113–117] and Huang et al. [W.-T. Huang, W.-K. Chen, C.-H. Chen, Pancyclicity of Möbius cubes, in: Proc. 9th Internat. Conf. on Parallel and Distributed Systems (ICPADS'02), 17–20 Dec. 2002, pp. 591–596], independently, that M_n contains a cycle of every length from 4 to 2^n. In this paper, we improve this result by showing that every edge of M_n lies on a cycle of every length from 4 to 2^n inclusive.

Keywords: Combinatorial problems; Cycles; Möbius cubes; Hypercubes; Pancyclicity; Edge-pancyclicity

1. Introduction

The hypercube network has proved to be one of the most popular interconnection networks. The Möbius cubes, proposed first by Cull and Larson [1–3], form a class of hypercube variants. Like hypercubes, Möbius cubes are expansible, have a simple routing algorithm, and have a high fault tolerance. The Möbius cubes are superior to the hypercube in having about half of the diameter of the hypercube, about two-thirds of the average distance of hypercube. Various properties of Möbius cubes have been extensively investigated in the literature, see, for example, [1–5,7–9,13].

The cycle embedding problem is to find a cycle of given length in graph, which is of practical importance in interconnection networks (see Section 1.3.2 in [11]). Recently, it has been shown by several authors that every edge of n-dimensional crossed cube, another variants of the hypercube, lies on a cycle of every length from 4 to 2^n inclusive for $n \geqslant 2$ (see [6, 10,12,14]).

Fan [5] and Huang et al. [9] have proved that the n-dimensional Möbius cube contains a cycle of length

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from 4 to 2^n. In this paper, we improve this result by showing the following theorem.

Theorem. *Every edge of n-dimensional Möbius cube lies on a cycle of every length from 4 to 2^n inclusive for $n \geqslant 2$.*

Corollary (Fan [5], Huang et al. [9]). *Every n-dimensional Möbius cube M_n contains a cycle of every length from 4 to 2^n inclusive for $n \geqslant 2$.*

The proof of the theorem is in Section 3. In Section 2, the definition and basic properties of the n-dimensional Möbius cube M_n are given.

2. Möbius cubes

The architecture of an interconnection work is usually represented by a connected simple graph $G = (V, E)$, where the vertex-set V is the set of processors and the edge-set E is the set of communication links in the network. The edge connecting two vertices x and y is denoted by (x, y). We follow [11] for graph-theoretical terminology and notation not defined here.

An n-dimensional Möbius cube, denoted by M_n, has 2^n vertices. Each vertex has a unique n-component binary vector on $\{0, 1\}$ for an address, also called an n-bit string. A vertex $X = x_1x_2 \cdots x_n$ connects to n neighbors $Y_1, Y_2, \ldots, Y_n$, where each Y_i satisfies one of the following rules:

$$Y_i = x_1 \cdots x_{i-1}\bar{x}_i x_{i+1} \cdots x_n \quad \text{if } x_{i-1} = 0, \tag{1}$$

$$Y_i = x_1 \cdots x_{i-1}\bar{x}_i \bar{x}_{i+1} \cdots \bar{x}_n \quad \text{if } x_{i-1} = 1, \tag{2}$$

where $\bar{x}_i$ is the complement of the bit x_i in $\{0, 1\}$.

More informally, a vertex X connects to a neighbor that differs in a bit x_i if $x_{i-1} = 0$, and to a neighbor that differs in bits x_i through x_n if $x_{i-1} = 1$. The connection between X and Y is undefined when $i = 1$, so we can assume x_0 is either equal to 0 or equal to 1, which gives us slightly different network topologies. If we assume $x_0 = 0$, we call the network a "0-Möbius cube", denoted by M_n^0; and if we assume $x_0 = 1$, we call the network a "1-Möbius cube", denoted by M_n^1. Figs. 1 and 2 show the 0-Möbius cube M_4^0 and the 1-Möbius cube M_4^1.

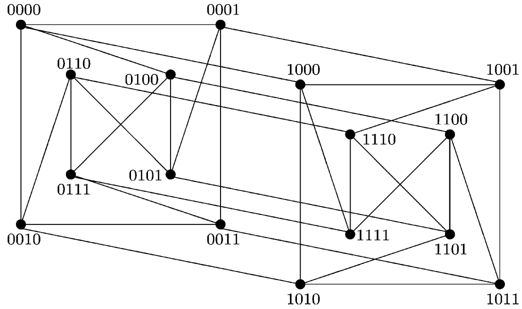

Fig. 1. 0-Möbius cube M_4^0.

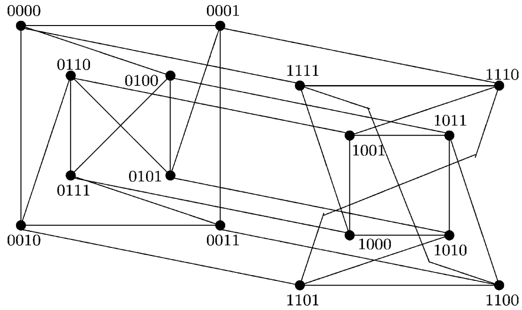

Fig. 2. 1-Möbius cube M_4^1.

According to the above definition, it is not difficult to see that M_n^0 (respectively, M_n^1) can be recursively constructed from M_{n-1}^0 and M_{n-1}^1 by adding 2^{n-1} edges. For any vertex $X = x_1x_2 \cdots x_{n-1}$ in M_{n-1}^0 or M_{n-1}^1, we construct a new vertex $X' = x'_1x'_2 \cdots x'_n$, where $x'_2 = x_1, x'_3 = x_2, \ldots, x'_n = x_{n-1}$, then assigning $x'_1 = 0$ if X is in M_{n-1}^0, or $x'_1 = 1$ if X is in M_{n-1}^1. So M_n^0 can be constructed by connecting all pairs of vertices that differ only in the first bit, and M_n^1 can be constructed by connecting all pairs of vertices that differ in the first through the nth bits. For short, we denote $M_n = L \oplus R$, where $L \cong M_{n-1}^0$ and $R \cong M_{n-1}^1$, and call edges between L and R *cross edges*. Moreover, we write a cross edge as (u_L, u_R), where $u_L \in L$ and $u_R \in R$. An edge in M_n is called a *critical edge* if its end-vertices differ in only the last bit x_n.

Note that if $M_n = L \oplus R$ then, for any two adjacent u_L and v_L in L, two vertices u_R and v_R in R are not always adjacent in R, and vice versa. However, it is clear from the rules (1) and (2) that if (u_L, v_L) is a critical edge, then two vertices u_R and v_R in R must be adjacent in R, and vice versa. Critical edges play an important role in the proof of our theorem. A cycle in M_n is called a 2-*critical* if it contains at least two critical edges. It is easy to see that every vertex in M_n is incident with a critical edge and every cross edge lies on a 2-critical cycle of length four.

Lemma. *Every edge of M_n lies on a 2-critical cycle of length 2^n for $n \geqslant 2$.*

Proof. We prove the lemma by induction on $n \geqslant 2$. Clearly the result is true for $n = 2$ since M_2 is a cycle of length 4. Assume that the lemma is true for every k with $2 \leqslant k < n$. Let $M_n = L \oplus R$ and e be any edge in M_n. There are two cases according as e is in M_n^0 or M_n^1.

Case 1. The edge e is in M_n^0.

Subcase 1.1. The edge e is in L.

Since $L \cong M_{n-1}^0$, by the induction hypothesis, there exists a 2-critical cycle C of length 2^{n-1} in L that contains e. Choose a critical edge (u_L, v_L) in C different from e and let $P = C - (u_L, v_L)$. Obviously, e is in P. From the definition of L, we can write $u_L = 0B0$ and $v_L = 0B1$, where B is an $(n-2)$-bit string. Then $u_R = 1B0$ and $v_R = 1B1$ are adjacent in R. By the induction hypothesis, there exists a 2-critical cycle C' of length 2^{n-1} in R that contains the edge (u_R, v_R). Let $P' = C' - (u_R, v_R)$. Then $P + (v_L, v_R) + P' + (u_R, u_L)$ is a 2-critical cycle of length 2^n in M_n^0 that contains the edge e.

Subcase 1.2. The edge e is in R. The proof is similar to Subcase 1.1. The details are here omitted.

Subcase 1.3. The edge e is a cross edge between L and R.

Let $e = (u_L, u_R)$, $u_L = 0B0$ in L and $u_R = 1B0$ in R, where u is an $(n-2)$-bit string. Let $v_L = 0B1$ and $v_R = 1B1$. Then $\langle u_L, v_L, v_R, u_R, u_L\rangle$ is a cycle of length four in M_n^0 and contains e.

By the induction hypothesis, there exist a 2-critical cycle C of length 2^{n-1} in L that contains the edge (u_L, v_L) and a 2-critical cycle C' of length 2^{n-1} in R that contains the edge (u_R, v_R). Let $P = C - (u_L, v_L)$ and $P' = C' - (u_R, v_R)$. Then $P + (v_L, v_R) + P' + (u_R, u_L)$ is a 2-critical cycle of length 2^n in M_n^0 that contains the edge e.

Case 2. The edge e is in M_n^1. By the same argument as that used in Case 1, we can prove that the lemma is true for this case, and the details are here omitted. □

3. Proof of theorem

In this section, we give the proof of theorem stated in Introduction.

Proof. We prove the theorem by induction on $n \geqslant 2$. The theorem is true for $n = 2$.

Since $M_3^0 \cong M_3^1$ from Fig. 3, we only need to prove that every edge of M_3^0 lies on a cycle of every length from 4 to 8 inclusive.

The union of the following four cycles of length four covers all edges of M_3^0.

$\langle 000, 001, 011, 010, 000\rangle$,

$\langle 100, 101, 110, 111, 100\rangle$,

$\langle 000, 001, 101, 100, 000\rangle$,

$\langle 010, 011, 111, 110, 010\rangle$.

The union of the following four cycles of length five covers all edges of M_3^0.

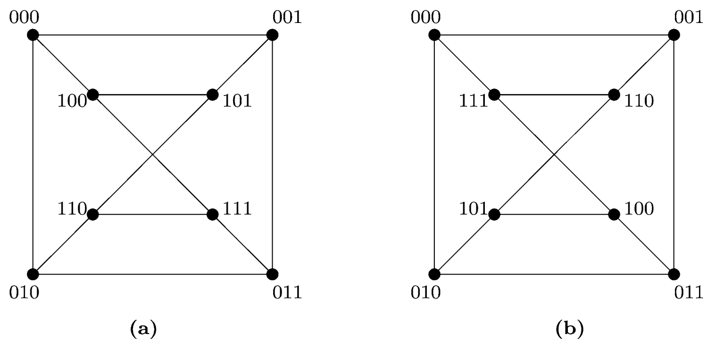

Fig. 3. (a) M_3^0 and (b) M_3^1.

$\langle 000, 001, 011, 111, 100, 000\rangle$,

$\langle 000, 010, 011, 111, 100, 000\rangle$,

$\langle 000, 100, 101, 110, 010, 000\rangle$,

$\langle 001, 011, 111, 110, 101, 001\rangle$.

The union of the following three cycles of length six covers all edges of M_3^0.

$\langle 000, 001, 011, 111, 110, 010, 000\rangle$,

$\langle 000, 010, 011, 001, 101, 100, 000\rangle$,

$\langle 100, 111, 110, 101, 001, 000, 100\rangle$.

The union of the following three cycles of length seven covers all edges of M_3^0.

$\langle 000, 100, 101, 110, 111, 011, 010, 000\rangle$,

$\langle 001, 101, 100, 111, 110, 010, 011, 001\rangle$,

$\langle 000, 100, 111, 110, 010, 011, 001, 000\rangle$.

The union of the following two cycles of length eight covers all edges of M_3^0.

$\langle 000, 001, 101, 110, 010, 011, 111, 100, 000\rangle$,

$\langle 000, 100, 101, 001, 011, 111, 110, 010, 000\rangle$.

Thus the theorem is true for $n = 3$.

Assume now that the theorem is true for all $3 \leqslant k < n$. Let e be any edge of M_n and let ℓ be any integer with $4 \leqslant \ell \leqslant 2^n$, where $n \geqslant 4$. To complete the proof of the theorem, we need to show that e is contained in a cycle of length ℓ by considering two cases according as e is in M_n^0 or in M_n^1.

Case 1. The edge e is in M_n^0. Let $M_n^0 = L \oplus R$.

Subcase 1.1. The edge e is in L. Since $L \cong M_{n-1}^0$, we can express $L = L_0 \oplus R_0$, where $L_0 \cong M_{n-2}^0$ and $R_0 \cong M_{n-2}^1$.

If $4 \leqslant \ell \leqslant 2^{n-1}$, by the induction hypothesis, there exists a cycle of length ℓ in $L \subset M_n^0$ that contains e.

Suppose that $2^{n-1} + 1 \leqslant \ell \leqslant 2^{n-1} + 3$. By the induction hypothesis, there exists a cycle C of length $\ell - 3$ in L containing e. For $n \geqslant 4$, we have $\ell - 3 \geqslant 2^{n-1} - 2 > 2^{n-2}$, and so C contains at least two cross edges between L_0 and R_0. Thus, we can choose a cross edge (u_L, v_L) in C different from e. Let $(u_L, v_L) = (00B, 01B)$, where B is an $(n-2)$-bit string. Then $u_R = 10B$, $w_R = 11\bar{B}$, $v_R = 11B$ are in R with $(u_R, w_R), (w_R, v_R) \in E(R)$ by the rule (2) in the definition of M_n, i.e., $P' = \langle v_R, w_R, u_R\rangle$ is a path between v_R and u_R in R. Let $P = C - (u_L, v_L)$. Then P contains e and $P + (v_L, v_R) + P' + (u_R, u_L)$ is a cycle of length ℓ in M_n^0 containing e.

Suppose that $2^{n-1} + 4 \leqslant \ell \leqslant 2^n$. Let $\ell' = \ell - 2^{n-1}$. Then $4 \leqslant \ell' \leqslant 2^{n-1}$. By lemma, there exists a 2-critical cycle C of length 2^{n-1} in L containing e. We can choose a critical edge (u_L, v_L) different from e. Without loss of generality, let $u_L = 0B0$ and $v_L = 0B1$, where B is an $(n-2)$-bit string. Then $u_R = 1B0$ and $v_R = 1B1$ are adjacent in R. Let $P = C - (u_L, v_L)$. Obviously e lies on P. By the induction hypothesis there exists a cycle C' of length ℓ' in R that contains (u_R, v_R). Let $P' = C' - (v_R, u_L)$. Then $P + (v_L, v_R) + P' + (u_R, u_L)$ is a cycle of length ℓ in M_n^0 and contains e.

Subcase 1.2. The edge e is in R. Since $R \cong M_{n-1}^1$, we can express $R = L_1 \oplus R_1$, where $L_1 \cong M_{n-2}^0$ and $R_1 \cong M_{n-2}^1$. The proof is similar to Subcase 1.1. The details are here omitted.

Subcase 1.3. The edge e is a cross edge between L and R.

Let $e = (u_L, u_R) = (0x_2x_3\cdots x_n, 1x_2x_3\cdots x_n)$. Let $v_L = 0x_2x_3\cdots x_{n-1}\bar{x}_n$ and $v_R = 1x_2x_3\cdots x_{n-1}\bar{x}_n$. Obviously, $\langle u_L, v_L, v_R, u_R, u_L\rangle$ is a cycle of length four in M_n^0 containing e. And

$$\langle 0x_2x_3\cdots x_n, 1x_2x_3\cdots x_n, 1\bar{x}_2\bar{x}_3\cdots\bar{x}_n, 0\bar{x}_2\bar{x}_3\cdots\bar{x}_n, 0\bar{x}_2x_3x_4\cdots x_n, 0x_2x_3\cdots x_n\rangle$$

is a cycle of length five in M_n^0 containing e for $x_2 = 0$;

$$\langle 0x_2x_3\cdots x_n,\ 1x_2x_3\cdots x_n, 1\bar{x}_2\bar{x}_3\cdots\bar{x}_n, 0\bar{x}_2\bar{x}_3\cdots\bar{x}_n, 0x_2\bar{x}_3\bar{x}_4\cdots\bar{x}_n, 0x_2x_3\cdots x_n\rangle$$

is a cycle of length five in M_n^0 containing e for $x_2 = 1$.

For $\ell \geqslant 6$, we can write $\ell = \ell_1 + \ell_2$ where $\ell_1 = 2$, $4 \leqslant \ell_2 \leqslant 2^{n-1}$ or $4 \leqslant \ell_1 \leqslant 2^{n-1}, 4 \leqslant \ell_2 \leqslant 2^{n-1}$. Consider the cycle $\langle u_L, v_L, v_R, u_R, u_L\rangle$ of length four in M_n^0 containing e. By the induction hypothesis, there exists a cycle C of length ℓ_1 in L containing (u_L, v_L) if $\ell_1 \geqslant 4$ and exists a cycle C' of length ℓ_2 in R containing (u_R, v_R). Let $P = (u_L, v_L)$ if $\ell_1 = 2$ or $P = C - (u_L, v_L)$ if $\ell_1 \geqslant 4$; $P' = C' - (v_R, u_R)$. Then $P + (v_L, v_R) + P' + (u_R, u_L)$ is a cycle of length ℓ in M_n^0 and contains e.

Case 2. The edge e is in M_n^1. By the same argument as that used in Case 1, we can prove that the theorem is true for this case, and the details are here omitted. □

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