May 1996
SNUTP 96-034
hep-th/9605052

Effective Field Theory for Coherent Optical Pulse Propagation

Q-Han Park[1]

and

H. J. Shin[2]

Department of Physics
and
Research Institute of Basic Sciences
Kyunghee University
Seoul, 130-701, Korea

ABSTRACT

Hidden nonabelian symmetries in nonlinear interactions of radiation with matter are clarified. In terms of a nonabelian potential variable, we construct an effective field theory of self-induced transparency, a phenomenon of lossless coherent pulse propagation, in association with Hermitian symmetric spaces G/H. Various new properties of self-induced transparency, e.g. soliton numbers, effective potential energy, gauge symmetry and discrete symmetries, modified pulse area, conserved $U(1)$-charge etc. are addressed and elaborated in the nondegenerate two-level case where $G/H = SU(2)/U(1)$. Using the $U(1)$-charge conservation, a new type of analysis on pulse stability is given which agrees with earlier numerical results.

[1] E-mail address; qpark@nms.kyunghee.ac.kr
[2] E-mail address; hjshin@nms.kyunghee.ac.kr

Contents

1 Introduction

Since the invention of the laser, much progress has been made in understanding nonlinear interactions of radiation with matter which made nonlinear optics a fast developing and independent field of science. Laser light in general is expressed in terms of a macroscopic, classical electric field which interacts with microscopic, quantum mechanical matter. Unlike the case of classical electrodynamics, the electric scalar potential and the magnetic vector potential do not appear to replace electromagnetic fields in nonlinear optics. Instead, the electric field itself, with appropriate restrictions to accomodate specific physical systems, plays the role of a fundamental variable which renders the problem lacking a field theoretic Lagrangian formulation. However, one notable exception is the area theorem introduced by McCall and Hahn [1]. In 1967, McCall and Hahn have discovered a remarkable coherent mode of lossless light pulses which propagate in a resonant, nondegenerate two-level atomic medium with inhomogeneous broadening, and named the phenomenon as self-induced transparency (SIT). In their work, propagation of light pulses is depicted in terms of a potential-like variable $\theta(x)$, the time area of a suitably chosen electric field, which is determined according to the area theorem. Under certain circumstances, the system can be described by an effective potential variable $\varphi(x,t)$ which satisfies the well-known sine-Gordon field theory equation. In this case, the 1-soliton of the sine-Gordon theory is identified with the 2π pulse of McCall and Hahn. The cosine potential term becomes proportional to the microscopic atomic energy and accounts for pulse stability. However, one serious drawback of the sine-Gordon approach is its oversimplification. In the sine-Gordon limit, frequency detuning and frequency modulation effects are all ignored and microscopic atomic motions (inhomogeneous broadening) are not taken into account. Also, the model is limited to the nondegenerate two-level case whereas many interesting physical systems are degenerate and/or multi-level systems. Until now, only the sine-Gordon theory was a known field theory for SIT. Thus, subsequent generalizations of SIT to more complex systems have resorted only to the SIT equation, i.e. the Maxwell-Bloch equation under slowly varying envelope approximation (SVEA), finding soliton type solutions by the inverse scattering method in some cases. It should be noted that even though the SIT equation expressed in terms of a $U-V$ pair in the inverse scattering formalism resembles a nonlinear sigma model equation, it does not provide a field theory like the sine-Gordon theory whose equation of motion becomes the SIT equation [2]. Following the pioneering work of Lamb [3], Ablowitz, Kaup and Newell have extended the inverse scattering formalism to include inhomogeneous broadening and obtained exact solutions [4]. In accordance with the area theorem, these solutions show that an arbitrary initial pulse with sufficient strength decomposes into a finite number of 2π pulses and 0π pulses, plus radiation which decays exponentially. Extensions to the degenerate as well as the multi-level cases have been also found with more complicated soliton solutions [2, 5, 6, 7].

The purpose of this paper is to clarify nonabelian group structures and their physical properties hidden in nonlinear interactions of radiation with matter. These group structures allow us to construct an effective field theory of SIT beyond the sine-Gordon limit. In fact, without inhomogeneous broadening all generalizations of the SIT equation which admit the inverse scattering formalism can be described by group theoretic effective field theories which generalize the sine-Gordon theory to the nonabelian cases. We extend the potential variable φ to a matrix valued scalar potential g and construct a field theory of SIT in terms of a certain two dimensional nonlinear sigma model associated with a particular coset G/H for a pair of Lie groups G and H. In order to do so, we first find a group theoretic formulation of the Bloch equation in terms of the motion of a spinning top moving along the coadjoint orbit which is determined by G/H, and then express the full Maxwell-Bloch equation under SVEA as a field theory generalization of the spinning top. We make such a field theory generalization by modifying the G/H gauged Wess-Zumino-Witten nonlinear sigma model action which is a well-known model in mathematical physics. The $SU(2)/U(1)$ case, being identified with the so-called complex sine-Gordon theory, provides a field theory for the nondegenerate two-level SIT. Frequency detuning and frequency modulation effects are taken into account through the $SU(2)$ valued scalar potential g, or stated differently, through three scalar potential functions. The simplest, sine-Gordon limit arises from the restriction of $SU(2)$ to $U(1) \times U(1)$ and our spinning top interpretation reduces to the well-known physical interpretation of the sine-Gordon equation as the equation of motion for a chain of pendulums. Other generalizations of SIT to the multi-level and the degenerate cases can be made in terms of more complicated coset structures. It is shown that all these generalizations correspond to specific cosets G/H known as Hermitian symmetric spaces. In the presense of inhomogeneous broadening, we still have the concept of effective potential where the matrix scalar potential g is also a function of detuning frequency. That is, we have infinitely many microscopic potential functions g labeled by detuning frequency. This prevents SIT with inhomogeneous broadening from a field theory formulation. Certain field theoretic properties, e.g. conservation laws, no longer hold in this case which nevertheless receive interesting physical interpretations as discussed in Sec. 6. The microscopic potential g also reveals the group structure of the dressing method, a solution finding technique equvialent to the inverse scattering.

The effective field theory description of SIT provides a completely new angle to the SIT problem and reveals various noble aspects of SIT which were not available in the inverse scattering approach. One marked difference is the appearance of a group theoretical potential energy term which clarifies the topological and the nontopological nature of optical pulses. In particular, the periodicity of the potential results in infinitely many degenerate vacua. In order for optical pulses to possess finite energy, they have to reach one of the degenerate vacua asymptotically as $x \to \pm\infty$ which assigns specific soliton numbers, more than one in some cases, to each pulses. We present an explicit form of the potential energy term for various multi-level and degenerate cases of SIT and define soliton numbers for each cases. The full Lagrangian itself also reveals new types of symmetries of SIT. It possesses two kinds of symmetries, first the continuous type; the local H vector gauge symmetry and the global $U(1)$ axial vector gauge symmetry, and secondly the discrete type; the Krammers-Wannier type duality and the chiral symmetry. It is noted that the SIT equation is not invariant under the vector gauge transformation and the specific choice of gauge fixing incorporates frequency detuning effect. This shows that inhomogeneous broadening is equivalent to "averaging" over different gauge fixings. Faraday rotation in the presense of external magnetic field is also identified with the vector gauge transformation. Discrete symmetries generate new solutions from a known one. In particular, the Krammers-Wannier type duality relates the "bright" soliton with the "dark" soliton of SIT. Our group theoretic formulation of SIT allows a systematic understanding of the integrability of SIT. We find infinitely many conserved local integrals of SIT using the properties of Hermitian symmetric spaces. The group structure of SIT also allows us to find exact soliton solutions using the dressing method for SIT.

The topological nature of solitons is particularly useful in understanding the stability of optical pulses. Due to the infinite energy barrier, topological solitons are stable against topological number changing fluctuations. In the case of nontopological solitons, we show that the $U(1)$ axial charge replaces the topological number. By making a perturbative analysis around topological and nontopological solitons and also using the $U(1)$ charge conservation, we find the stability behavior of solitons against small fluctuations. This result agrees well with earlier numerical work and provides a systematic stability analysis based on a field theory formulation which otherwise would not have been possible.

The plan of the paper is the following; in Sec. 2, we introduce a semiclassical Maxwell-Bloch description of the optical pulse propagation problem and the McCall and Hahn's area theorem. The effective potential concept is briefly discussed. In Sec. 3, we construct an effective field theory of SIT using a field theory generalization of the spinning top equation. Various cases of SIT are associated with specific cosets. Inhomogenous broadening is introduced from our field theory point of view and we explain the notion of effective potential energy and its degenerate vacua and associated soliton numbers. In Sec. 4, the dressing method is explained and applied to obtain soliton solutions for the nondegenerate two-level and the degenerate three-level system. In Sec. 5, infinitely many conserved local integrals are obtained using the properties of Hermitian symmetric spaces. Discrete and continuous symmetries of effective theories are aslo explained. The issue of stability is addressed in Sec. 6, particularly in regard to the conserved $U(1)$ charge and inhomogeneous broadening effect is addressed. Finally, Sec. 7 is a discussion.

2 Self-induced transparency

The multi-mode optical pulses propagating in a resonant medium along the x-axis are described by the electric field of the form,

$$\mathbf{E} = \sum_{l=1}^{m} \mathbf{E}_l(x,t) \exp i(k_l x - w_l t) + \text{c.c.} \tag{2.1}$$

where k_l and w_l denote the wave number and the frequency of each mode and the amplitude vector $\mathbf{E}_l$ is in general a complex vector function. The governing equation of propagation is the Maxwell equation,

$$(\frac{\partial^2}{\partial x^2} - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2})\mathbf{E} = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\int dv \text{ tr } \rho\mathbf{d}. \tag{2.2}$$

On the right hand side, electric dipole transitions are treated semiclassically. $\mathbf{d}$ is the atom's dipole moment operator and the density matrix ρ satisfies the quantum-mechanical optical Bloch equation

$$i\hbar(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x})\rho = [(H_0 - \mathbf{E}\cdot\mathbf{d}) \ , \ \rho] \ . \tag{2.3}$$

H_0 denotes the Hamiltonian of a free atom and v is the x-component of the velocity of the atoms. In most cases, SVEA is further made which assumes that the amplitudes $\mathbf{E}_l$ vary slowly compared to the space and time scales determined by k_l and w_l. Under SVEA, the Maxwell-Bloch equation becomes a set of coupled first order partial differential equations for the amplitudes $\mathbf{E}_l$ and the components of the density matrix. Explicit expressions of the Maxwell-Bloch equation for some physically relevant cases are given in Sec. 3. Thus, instead of using quantum electrodynamics for the interaction of radiation with matter, a specific choice of physical systems allows an effective description using the amplitudes $\mathbf{E}_l$ as a reduced set of variables. This makes the use of potential variables quite difficult if not impossible. Also, lack of potential variables causes the physical system to be described only by the equation of motion, not by the action principle. Consequently, a field theoretic formulation is lacking in pulse propagation problems. However, there exists one exceptional case. When pulses propagate in a resonant, nondegenerate two-level atomic medium with inhomogeneous broadening, McCall and Hahn have introduced an effective potential-like variable and shown that an arbitrary pulse evolves into a coherent mode of lossless pulses. This phenomenon, known as self-induced transparency, is also observed in more general, degenerate and/or multi-level cases. Specifically, McCall and Hahn have shown that when the pulse envelope is assumed to be real and the time area of the dimensionless envelope function $2E$,

$$\theta(x) = \int_{-\infty}^{\infty} dt 2E, \tag{2.4}$$

is an integer multiple of 2π ($2n\pi$ pulse), then the pulse propagates without loss of energy. Otherwise, due to inhomogeneous broadening the pulse quickly reshapes to $2n\pi$ pulse according to the area theorem,

$$\frac{d\theta(x)}{dx} = -\alpha \sin\theta(x), \tag{2.5}$$

for some constant α. These $2n\pi$ pulses are another type of optical solitons which arise from the nonlinear response of matter to the radiation field besides the most well-known optical solitons of the nonlinear Schrödinger equation appearing in optical communication problem. The proof of the area theorem follows from inhomogeneous broadening as well as the Maxwell-Bloch equation. The SIT equation for the nondegenerate two-level case is given in a dimensionless form by

$$\begin{aligned} \bar{\partial}E + 2\beta < P > &= 0 \\ \partial D - E^*P - EP^* &= 0 \\ \partial P + 2i\xi P + 2ED &= 0 \end{aligned} \tag{2.6}$$

where β is an arbitrary constant and $\xi = w - w_0$, $\partial \equiv \partial/\partial z$, $\bar{\partial} \equiv \partial/\partial\bar{z}$, $z = t - x/c, \bar{z} = x/c$. The angular bracket signifies an average over the spectrum $f(\xi)$ as given by

$$< \cdots >= \int_{-\infty}^{\infty} (\cdots) f(\xi) d\xi. \tag{2.7}$$

The dimensionless quantities E, P and D correspond to the electric field, the polarization and the population inversion through the relation,

$$\begin{aligned} E &= -i\mathbf{E}\cdot\mathbf{e}t_0 d/\sqrt{6}\hbar \\ P &= -\rho_{12}\exp[-i(kx - \omega t)]/4kt_0N_0f(\xi) \\ D &= -(\rho_{22} - \rho_{11})/8kt_0N_0f(\xi) \end{aligned} \tag{2.8}$$

where $\mathbf{e}$ specifies the linear polarization direction, t_0 is a time constant and N_0 is related to the stationary populations of the levels.[3] In order to understand the structure of $2n\pi$ pulses better, we may impose further restrictions such that the system is on resonance ($\xi = 0$) without frequency modulation (E being real) and inhomogeneous broadening ($f(\xi) = \sigma(\xi)$). Under such circumstances, we can introduce an area function $\varphi(x, t)$,

$$\varphi(x, t) = \int_{-\infty}^{t} E dt' \tag{2.9}$$

[3]For the details of constants, we refer the reader to ref. [2].

which in the limit $t \to \infty$ agrees with $\theta(x)/2$ in (2.4). In terms of φ, the SIT equation reduces to the sine-Gordon equation,

$$\bar{\partial}\partial\varphi - 2\beta \sin 2\varphi = 0, \tag{2.10}$$

together with the identification;

$$E = E^* = \partial\varphi \ \ , \ \ < P >= P = -\sin 2\varphi \ \ , \ \ < D >= D = \cos 2\varphi \ . \tag{2.11}$$

The sine-Gordon equation arises from the Lagrangian

$$S = \frac{1}{2\pi}\int(\partial\varphi\bar{\partial}\varphi - 2\beta\cos 2\varphi). \tag{2.12}$$

The periodic cosine potential term exhibits infinitely many degenerate vacua and gives rise to soliton solutions which interpolate between different vacua. This shows that $2n\pi$ pulse is in fact the topological n-soliton solution of the sine-Gordon equation in this particular limit. The electric field amplitude E, now identified with $\partial\varphi$, receives an interpretation as a topological current. Note that the effective potential variable φ is different from the conventional scalar or vector potentials of the electromagnetic field. Nevertheless, it is remarkable that the potential energy $\cos 2\varphi$ of the sine-Gordon Lagrangian can be identified with the population inversion D which represents the atomic energy. Also the Lorentz invariance, which was broken by SVEA, re-emerges in the sine-Gordon field theory after the redefinition of coordinates. The identification of the atomic energy with the cosine potential term possessing degenerate vacua, consequently topological solitons, underlies the stability of $2n\pi$ pulses.

Unfortunately, the sine-Gordon theory formulation of SIT is too restrictive. In the presense of frequency modulation, E should be complex. Therefore, it can not be replaced by a real scalar field φ and the sine-Gordon limit is no longer valid. Also, frequency modulation effects invalidate the area theorem. By a direct application of the inverse scattering, it has been found, however, that solitons do exist even in the case of complex E [3]. This suggests that a more general field theory of SIT than the sine-Gordon theory could exist in order to account for complex E. Recently, we have shown that this is indeed true and the generalizing theory which includes both frequency detuning and modulation effects is given by the so-called complex sine-Gordon theory in the following way [8]; assume that E is complex and the frequency distribution function of inhomogeneous broadening is sharply peaked at ξ, i.e. $f(\xi') = \delta(\xi' - \xi)$ for some constant ξ. Introduce parametrizations of E, P and D which generalize (2.11) in terms of three scalar fields φ, θ and η,

$$E = e^{i(\theta-2\eta)}(2\partial\eta\frac{\cos\varphi}{\sin\varphi} - i\partial\varphi) \ \ , \ \ P = ie^{i(\theta-2\eta)}\sin 2\varphi \ \ , \ \ D = \cos 2\varphi \ . \tag{2.13}$$

These parametrizations consistently change the SIT equation (2.6) into a couple of second order nonlinear differential equations known as the complex sine-Gordon equation;

$$\bar{\partial}\partial\varphi + 4\frac{\cos\varphi}{\sin^3\varphi}\partial\eta\bar{\partial}\eta - 2\beta\sin 2\varphi = 0 \tag{2.14}$$

$$\bar{\partial}\partial\eta - \frac{2}{\sin 2\varphi}(\bar{\partial}\eta\partial\varphi + \partial\eta\bar{\partial}\varphi) = 0 \tag{2.15}$$

and a couple of first order constraint equations,

$$\begin{aligned} 2\cos^2\varphi\partial\eta - \sin^2\varphi\partial\theta - 2\xi\sin^2\varphi &= 0 \\ 2\cos^2\varphi\bar{\partial}\eta + \sin^2\varphi\bar{\partial}\theta &= 0\ . \end{aligned} \tag{2.16}$$

Note that the complex sine-Gordon equation reduces to the sine-Gordon equation when frequency modulation effect is ignored such that $\eta = 0,\ \theta = \pi/2$ and the system is on resonance ($\xi = 0$). This reduction is consistent with the original equation since solutions of the sine-Gordon equation consists a subspace of the whole solution space. The complex sine-Gordon equation was first introduced by Lund and Regge in 1976 in order to describe the motion of relativistic vortices in a superfluid [9], and also independently by Pohlmeyer in a reduction problem of O(4) nonlinear sigma model [10]. The integrability and various related properties, both classical and quantum, of the complex sine-Gordon equation have been studied since then. In particular, the Lagrangian for the complex sine-Gordon equation in terms of φ and η is given by

$$S = \frac{1}{2\pi}\int \partial\varphi\bar{\partial}\varphi + 4\cot^2\varphi\partial\eta\bar{\partial}\eta - 2\beta\cos 2\varphi. \tag{2.17}$$

This Lagrangian, however, is singular at $\varphi = n\pi$ for integer n which causes difficulties in quantizing the theory. Also, besides the complex sine-Gordon equation, the SIT equation comprises the constraint equation (2.16). Thus the Lagrangian (2.17) does not quite serve for a field theory of two-level SIT. In fact, the singular behavior of the Lagrangian (2.17) is an artifact of neglecting the constraint equation. This fact as well as the rationale of the above parametrizations can be seen most clearly if we reformulate the Lagrangian to include the constraint in the context of a nonlinear sigma model as explained in the next section.

3 Effective field theory

In order to construct a field theory of SIT in terms of potential variables and also find a way to extend to more general multi-level and degenerate cases, we first note that the optical Bloch equation admits an interpretation as a spinning top equation like the corresponding magnetic resonance equations [11]. Denote real and imaginary parts of E and P by $E = E_R + iE_I,\ P = P_R + iP_I$. Then, the Bloch part of the SIT equation (2.6) can be expressed as

$$\partial\vec{S} = \vec{\Omega} \times \vec{S} \tag{3.1}$$

where $\vec{S} = (P_R,\ P_I,\ D)$, $\vec{\Omega} = (2E_I,\ -2E_R,\ -2\xi)$, i.e. it describes a spinning top where the electric dipole "pseudospin" vector $\vec{S}$ precesses about the "torque" vector $\vec{\Omega}$. This clearly shows that the length of the vector $\vec{S}$ is conserved,

$$|\vec{S}|^2 = P_R^2 + P_I^2 + D^2 = 1, \tag{3.2}$$

where the length equals unity due to the conservation of probability. The remaining Maxwell part of the SIT equation determines strength of the torque vector. If $P_I = 0$, we may solve (3.2) by taking $P_R = -\sin 2\varphi$ and $D = \cos 2\varphi$ and also (3.1) by taking $E = \partial\varphi$ as given in (2.11). Then, the Maxwell equation becomes the sine-Gordon equation as before. This picture agrees with the conventional interpretation of the sine-Gordon theory as describing a system of an infinite chain of pendulums. In order to generalize the sine-Gordon limit to the complex E and P case, we make a crucial observation that the constraint (3.2) in general can be solved in terms of an $SU(2)$ matrix potential variable g by

$$\begin{pmatrix} D & P \\ P^* & -D \end{pmatrix} = g^{-1}\sigma_3 g, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.3}$$

The identity equation,

$$\begin{pmatrix} \partial D & \partial P \\ \partial P^* & -\partial D \end{pmatrix} = \partial(g^{-1}\sigma_3 g) = [g^{-1}\sigma_3 g,\ g^{-1}\partial g], \tag{3.4}$$

gives the Bloch equation if we make an identification,

$$g^{-1}\partial g - R = \begin{pmatrix} i\xi & -E \\ E^* & -i\xi \end{pmatrix}, \tag{3.5}$$

where R is an arbitrary matrix commuting with $g^{-1}\sigma_3 g$. The identification (3.3) requires that $\mathrm{Tr} g^{-1}\sigma_3 g = 0$ and $(g^{-1}\sigma_3 g)^T = (g^{-1}\sigma_3 g)^*$. In other words, g is a unitary matrix. Similarly, the tracelessness of $g^{-1}\partial g$ from the identification (3.5) further requires g to be an $SU(2)$ matrix. We will show shortly that R is determined by demanding a Lagrangian for the above equation and is also traceless. Finally, the Maxwell equation becomes

$$\begin{aligned} \bar{\partial}(g^{-1}\partial g - R) &= \begin{pmatrix} 0 & -\bar{\partial}E \\ \bar{\partial}E^* & 0 \end{pmatrix} = -\left[\begin{pmatrix} i\beta & 0 \\ 0 & -i\beta \end{pmatrix},\ i\begin{pmatrix} D & P \\ P^* & -D \end{pmatrix} \right] \\ &= \beta[\sigma_3,\ g^{-1}\sigma_3 g]. \end{aligned} \tag{3.6}$$

Thus, we have successfully expressed the SIT equation in terms of potential variable g up to an undetermined quantity R. Note that the diagonal part of the r.h.s. of (3.5) is fixed to a constant $i\xi$ accounting for frequency detuning. This constrains g such that the diagonal part of $g^{-1}\partial g - R$ is equal to $(i\xi,\ -i\xi)$. Therefore, for a field theory of SIT, we shall construct a Lagrangian in terms of the potential variable g whose equation of motion is (3.6) together with a Lagrange multiplier for the constraint. In order to help understanding, we assume for a moment that $R = 0$ and the system is on resonance ($\xi = 0$). Then, the equation of motion (3.6) arises from a variation of the action

$$S = S_{WZW}(g) - S_{\text{pot}} + S_{\text{const}} \tag{3.7}$$

with the following variational behaviors;

$$\delta_g S_{WZW} = \frac{1}{2\pi}\int \text{Tr}\bar{\partial}(g^{-1}\partial g)g^{-1}\delta g, \quad \delta_g S_{\text{pot}} = \frac{\beta}{2\pi}\int \text{Tr}[\sigma_3,\ g^{-1}\sigma_3 g\ ]g^{-1}\delta g. \tag{3.8}$$

The action $S_{WZW}(g)$ is the well-known $SU(2)$ Wess-Zumino-Witten functional,

$$S_{WZW}(g) = -\frac{1}{4\pi}\int_\Sigma \text{Tr}\ g^{-1}\partial g g^{-1}\bar{\partial}g - \frac{1}{12\pi}\int_B \text{Tr}\ \tilde{g}^{-1}d\tilde{g}\wedge\tilde{g}^{-1}d\tilde{g}\wedge\tilde{g}^{-1}d\tilde{g}\ , \tag{3.9}$$

where the second term on the r.h.s., known as the Wess-Zumino term, is defined on a three-dimensional manifold B with boundary Σ and $\tilde{g}$ is an extension of a map $g : \Sigma \to SU(2)$ to B with $\tilde{g}|_\Sigma = g$ [12]. The potential term S_{pot} can be easily written by

$$S_{\text{pot}} = \frac{\beta}{2\pi}\int \text{Tr} g\sigma_3 g^{-1}\sigma_3. \tag{3.10}$$

Finally, the constraint requires vanishing of the diagonal part of the matrix $g^{-1}\partial g$ which can be imposed by adding a Lagrange multiplying term S_{const} to the action

$$S_{\text{const}} = \frac{1}{2\pi}\int \text{Tr}\Lambda\sigma_3 g^{-1}\partial g. \tag{3.11}$$

The Lagrange multiplier Λ however adds a new term to the equation of motion by

$$\delta_g S_{\text{const}} = \frac{1}{2\pi}\int \text{Tr}(-\partial\Lambda\sigma_3 + [\Lambda\sigma_3,\ g^{-1}\partial g])g^{-1}\delta g, \tag{3.12}$$

which seems to spoil our construction of a SIT field theory. This problem can be resolved beautifully if we lift the action (3.7) to the "vector gauge invariant" one by replacing the constraint term with a "gauging" part of the Wess-Zumino-Witten action,

$$S = S_{WZW}(g) - S_{\rm pot} + S_{\rm gauge} \tag{3.13}$$

$$S_{\rm gauge} = \frac{1}{2\pi}\int {\rm Tr}(-A\bar{\partial}gg^{-1} + \bar{A}g^{-1}\partial g + Ag\bar{A}g^{-1} - A\bar{A}) \tag{3.14}$$

where the connection fields $A, \bar{A}$ gauge the anomaly free subgroup $U(1)$ of $SU(2)$ generated by the Pauli matrix σ_3. They introduce a vector gauge invariance of the action under the transform

$$g \rightarrow h^{-1}gh \ \ , \ \ A \rightarrow h^{-1}Ah + h^{-1}\partial h \ \ , \ \ \bar{A} \rightarrow h^{-1}\bar{A}h + h^{-1}\bar{\partial}h \tag{3.15}$$

where $h : \Sigma \rightarrow U(1)$. Owing to the absence of kinetic terms, $A, \bar{A}$ act as Lagrange multipliers which result in the constraint equations when integrated out. The action (3.7) may be understood as a particular gauge fixing of the vector gauge invariance where $A = 0,\ \bar{A} = \Lambda\sigma_3$. However, a more convenient gauge fixing which manifests the equivalence with the SIT equation (2.6) is where $A = i\xi\sigma_3,\ \bar{A} = 0$. Such a gauge fixing is always possible as a result of (3.23). Before proving this, we discuss about the generalization of the action (3.13) to groups other than $SU(2) \supset U(1)$. We may simply replace the pair $SU(2) \supset U(1)$ by $G \supset H$ for any Lie groups G and H and obtain the G/H gauged Wess-Zumino-Witten action ($S_{WZW} + S_{\rm gauge}$). This action is known to possess conformal symmetry and has been identified with the action for the general G/H coset conformal field theory [13]. The potential term (3.10) breaks conformal symmetry. Nevertheless, it preserves the integrability of the model given by (3.13) where $G/H = SU(2)/U(1)$ and this model has been used in describing integrable perturbation of certain coset conformal field theories [14, 15]. For a general pair of G and H, the expression for the potential which preserves integrability has been also found [15, 16]. It is given by

$$S_{\rm pot} = \frac{\beta}{2\pi}\int {\rm Tr}gTg^{-1}\bar{T} \tag{3.16}$$

where T and $\bar{T}$ are constant matrices which commute with the subgroup H, i.e. $[T, h] = [\bar{T}, h] = 0$, for $h \in H$ so that the potential term is vector gauge invariant. In general, $S_{\rm pot}$ is specified algebraically by a triplet of Lie groups $F \supset G \supset H$ for every symmetric space F/G, where the Lie algebra decomposition $\mathbf{f} = \mathbf{g} \oplus \mathbf{k}$ satisfies the commutation relations,

$$[\mathbf{g}\ ,\ \mathbf{g}] \subset \mathbf{g}\ ,\ [\mathbf{g}\ ,\ \mathbf{k}] \subset \mathbf{k}\ ,\ [\mathbf{k}\ ,\ \mathbf{k}] \subset \mathbf{g}\ . \tag{3.17}$$

We take T and $\bar{T}$ as elements of $\mathbf{k}$ and define $\mathbf{h}$ as the simultaneous centralizer of T and $\bar{T}$, i.e. $\mathbf{h} = C_{\mathbf{g}}(T, \bar{T}) = \{B \in \mathbf{g}\ :\ [B\ ,\ T] = 0 = [B\ ,\ \bar{T}]\}$

with H its associated Lie group. With these specifications, the action (3.13) becomes integrable and generalizes the sine-Gordon model according to each symmetric spaces [17]. For compact symmetric spaces of type II, e.g. symmetric spaces of the form $G \times G/G$, the elements g and T take the form $g \otimes g$ and $T \otimes 1 - 1 \otimes T$ (and similarly for $\bar{T}$). In which case, the model becomes effectively equivalent to the case where $T, \ \bar{T}$ belong to the Lie algebra $\mathbf{g}$. Thus the model is specified by the coset G/H where H is the stability subgroup of T and $\bar{T}$ for $T, \ \bar{T} \in \mathbf{g}$. In this paper, we will restrict ourselves only to this case. As we will see later, physically interesting cases of SIT all correspond to even more specific symmetric spaces, where G/H becomes Hermitian symmetric spaces and the adjoint action of T defines a complex structure on G/H.

Now, we demonstrate the integrability of the model by expressing the equation of motion arising from the action (3.13) in a zero curvature form,

$$\begin{aligned}
\delta_g S &= -\frac{1}{2\pi}\int \mathrm{Tr}([\ \partial + g^{-1}\partial g + g^{-1}Ag, \ \bar{\partial} + \bar{A}] + \beta[T, \ g^{-1}\bar{T}g])g^{-1}\delta g \\
&= -\frac{1}{2\pi}\int \mathrm{Tr}[\ \partial + g^{-1}\partial g + g^{-1}Ag + \beta\lambda T\ , \ \bar{\partial} + \bar{A} + \frac{1}{\lambda}g^{-1}\bar{T}g\ ]g^{-1}\delta g \\
&= 0
\end{aligned} \qquad (3.18)$$

where λ is an arbitrary complex constant. That is, the equation of motion is given by a zero curvature condition in terms of a $U - V$ pair,

$$[\partial - U, \ \bar{\partial} - V] = 0, \qquad (3.19)$$

where

$$U \equiv -g^{-1}\partial g - g^{-1}Ag - \beta\lambda T, \quad V \equiv -\bar{A} - \frac{1}{\lambda}g^{-1}\bar{T}g. \qquad (3.20)$$

This shows that the equation of motion arises as the integrability of the overdetermined linear equations;

$$(\partial + g^{-1}\partial g + g^{-1}Ag + \beta\lambda T)\Psi = 0, \quad (\bar{\partial} + \bar{A} + \frac{1}{\lambda}g^{-1}\bar{T}g)\Psi = 0. \qquad (3.21)$$

The constraint equations coming from the $A, \bar{A}$-variations are

$$\begin{aligned}
\delta_A S &= \frac{1}{2\pi}\int \mathrm{Tr}(\ -\bar{\partial}gg^{-1} + g\bar{A}g^{-1} - \bar{A}\ )\delta A = 0 \\
\delta_{\bar{A}} S &= \frac{1}{2\pi}\int \mathrm{Tr}(\ g^{-1}\partial g + g^{-1}Ag - A\ )\delta\bar{A} = 0\ .
\end{aligned} \qquad (3.22)$$

Note that these constraint equations when combined with (3.18) imply the flatness of the connection A and $\bar{A}$, i.e.

$$F_{z\bar{z}} = [\ \partial + A\ , \ \bar{\partial} + \bar{A}\ ] = 0\ . \qquad (3.23)$$

In the following, we show that (3.18) and (3.22) can be identified with the SIT equation for various cases depending on the choices of the groups G and H, and $T, \bar{T}$ as well as the specific choice of gauge fixing. Thus, we obtain the action principle of SIT equations when inhomogeneous broadening is ignored. The inclusion of inhomogeneous broadening is also obtained in our field theoretic context. Remarkably, in the presense of inhomogeneous broadening, the notion of effective potential still persists and the inhomogeneous broadening effect, i.e. Doppler shifted atomic motions, can be beautifully incorporated into the $U(1)$ vector gauge transformation of the theory.

3.1 Examples

Nondegenerate two-level system

This is the simplest case of SIT which was originally considered by McCall and Hahn. It also accounts for the transitions $1/2 \rightarrow 1/2,\ 1 \leftrightarrow 0,\ 1 \rightarrow 1$ and $3/2 \leftrightarrow 1/2$ for linearly polarized waves and the transitions $1/2 \rightarrow 1/2,\ 1 \leftrightarrow 0$ and $1 \rightarrow 1$ for circularly polarized waves. The SIT equation is given by (2.6) which can be expressed in an equivalent zero cuvature form,

$$\left[\partial + \begin{pmatrix} i\beta\lambda + i\xi & -E \\ E^* & -i\beta\lambda - i\xi \end{pmatrix} \ , \ \bar{\partial} - \frac{i}{\lambda}\begin{pmatrix} D & P \\ P^* & -D \end{pmatrix} \right] = 0 \ . \tag{3.24}$$

In order to show that this SIT equation in fact arises from the effective field theory (3.13), we take $H = U(1) \subset SU(2) = G$ and $T = -\bar{T} = i\sigma_3 = \mathrm{diag}(i, -i)$ for Pauli matrices σ_i. We fix the vector gauge invariance by choosing

$$A = i\xi\sigma_3 \ , \ \bar{A} = 0 \tag{3.25}$$

for a constant ξ. Such a gauge fixing is possible due to the flatness of $A, \bar{A}$. Comparing (3.20) with (3.24), we could identify E, P and D in terms of g such that

$$g^{-1}\partial g + \xi g^{-1}Tg - \xi T = \begin{pmatrix} 0 & -E \\ E^* & 0 \end{pmatrix} \ \ , \ \ g^{-1}\bar{T}g = -i\begin{pmatrix} D & P \\ P^* & -D \end{pmatrix} \tag{3.26}$$

which are consistent with the constraint equation (3.22). Comparing with (3.3)-(3.5), we find that R is fixed to $-i\xi g^{-1}\sigma_3 g$. Then, the zero curvature equation (3.18) agrees precisely with (3.24). Furthermore, if we make an explicit parametrization of the 2×2 $SU(2)$ matrix g by

$$g = e^{i\eta\sigma_3} e^{i\varphi(\cos\theta\sigma_1 - \sin\theta\sigma_2)} e^{i\eta\sigma_3} = \begin{pmatrix} e^{2i\eta}\cos\varphi & i\sin\varphi e^{i\theta} \\ i\sin\varphi e^{-i\theta} & e^{-2i\eta}\cos\varphi \end{pmatrix} \ , \tag{3.27}$$

we recover the parametrizations of E, P and D as given in (2.13) and the SIT equation becomes the complex sine-Gordon equation (2.14)(2.15) and the constraint equation (2.16). The potential term in (3.13) now changes into the population inversion D,

$$S_{\text{pot}} = \int \frac{\beta}{\pi} \cos 2\varphi = \int \frac{\beta}{\pi} D, \tag{3.28}$$

which for $\beta > 0$ possesses degenerate vacua at

$$\varphi = \varphi_n = (n + \frac{1}{2})\pi, \ n \in Z \ \text{ and } \ \theta \ = \theta_0 \ \text{ for } \ \theta_0 \text{ constant} . \tag{3.29}$$

The property of degenerate vacua and the corresponding soliton solutions will be considered in Sec. 4.

Degenerate two-level system

One of the deficiencies of the SIT model of McCall and Hahn is the absence of level degeneracy. Since most atomic systems possess level degeneracy, the analysis of the nondegenerate two-level system does not apply to a more practical system. Moreover, level degeneracy in general breaks the integrability and does not allow exact soliton configurations. For example, propagation of pulses in a two-level medium with the transition $j_b = 2 \rightarrow j_a = 2$ is effectively described by the double sine-Gordon equation

$$\partial\bar{\partial}\varphi = c_1 \sin\varphi + c_2 \sin 2\varphi \tag{3.30}$$

which is not integrable. Nevertheless, there are a few exceptional cases which are completely integrable even in the presense of level degeneracy. It was shown that [5, 6] the SIT equations for the transitions $j_b = 0 \rightarrow j_a = 1, \ j_b = 1 \rightarrow j_a = 0$ and $j_b = 1 \rightarrow j_a = 1$ are integrable in the sense that the SIT equations can be expressed in an inverse scattering form. In the following, we show that these cases correspond to the effective theory with $G = SU(3)$ and $H = U(2) \subset G$. Also, we show that the local vector gauge structure incorporates naturally the effects of frequency detuning and longitudinally applied magnetic field. Consider a monochromatic pulse propagating through a medium of degenerate two-level atoms in the presence of a longitudinal magnetic field. Then, the Maxwell-Bloch equation under SVEA is given by

$$\bar{\partial}\varepsilon^q = i \sum_{\mu m} \langle R_{\mu m} \rangle J^q_{\mu m}$$

$$[\partial + i(2\xi + \Omega_b \mu - \Omega_a m)] R_{\mu m} = i \sum_q \varepsilon^q (\sum_{m'} J^q_{\mu m'} R_{m' m} - \sum_{\mu'} R_{\mu \mu'} J^q_{\mu' m})$$

$$[\partial + i\Omega_a(m-m^{'})]R_{mm'} = i\sum_{q\mu}(\varepsilon^{q*}J^q_{\mu m}R_{\mu m'} - \varepsilon^q J^q_{\mu m'}R_{m\mu})$$

$$[\partial + i\Omega_b(\mu-\mu^{'})]R_{\mu\mu'} = i\sum_{qm}(\varepsilon^{q}J^q_{\mu m}R_{m\mu'} - \varepsilon^{q*} J^q_{\mu' m}R_{\mu m}). \tag{3.31}$$

The dimensionless quantities ε^q and R are propotional to the electric field amplitude E and the density matrix ρ, where q is the polarization index and the subscripts $\mu, \mu^{'}, \ldots$ and $m, m^{'}, \ldots$ denotes projections of the angular momentum on the quantization axis in two-level states $|a>$ and $|b>$ respectively.[4] J denotes the Wigner's $3j$ symbols

$$J^q_{\mu m} = (-1)^{j_b - m}\sqrt{3}\begin{pmatrix} j_a & 1 & j_b \\ -m & q & \mu \end{pmatrix}, \tag{3.32}$$

and $\Omega_a(\Omega_b)$ is a dimensionless coupling constant of an external magnetic field.

In general, (3.31) is not integrable. However, with particular choices j_a and j_b, (3.31) can be recasted into the zero curvature form, or the $U-V$ pair as in (3.19). Specifically, for the transition $j_b = 1/2 \rightarrow j_a = 1/2$, U and V becomes

$$U = \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix}, \; V = \begin{pmatrix} V_+ & 0 \\ 0 & V_- \end{pmatrix} \tag{3.33}$$

where

$$\begin{aligned} U_\pm &= \begin{pmatrix} -i(x+\lambda) & \pm i\varepsilon^{\pm 1} \\ \mp i\varepsilon^{\pm 1*} & i(x+\lambda) \end{pmatrix}, \; V_\pm = -\frac{1}{2\lambda}\begin{pmatrix} R^{(b)}_{\mp\frac{1}{2}\mp\frac{1}{2}} & R^{(ba)}_{\mp\frac{1}{2}\pm\frac{1}{2}} \\ R^{(ba)*}_{\mp\frac{1}{2}\pm\frac{1}{2}} & R^{(a)}_{\pm\frac{1}{2}\pm\frac{1}{2}} \end{pmatrix} \\ x &= \frac{1}{4}(\Omega_a + \Omega_b - 4\xi). \end{aligned} \tag{3.34}$$

In the context of effective field theory, we again identify the $U-V$ pair in terms of g by

$$U = -g^{-1}\partial g - g^{-1}Ag - \beta\lambda T, \quad V = -\frac{1}{\lambda}g^{-1}\bar{T}g \tag{3.35}$$

where the gauge choice is

$$A = \begin{pmatrix} -ix & 0 & 0 & 0 \\ 0 & ix & 0 & 0 \\ 0 & 0 & -ix & 0 \\ 0 & 0 & 0 & ix \end{pmatrix}, \quad \bar{A} = 0, \tag{3.36}$$

and

$$T = -\bar{T} = i\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \tag{3.37}$$

[4] For details of propotionality constants and their physical meanings, we refer the reader to ref. [2].

with the Pauli matrix σ_3. Here we set $\beta = 1$ for convenience. The resulting effective theory is specified by the coset $G/H = (SU(2)\times SU(2))/(U(1)\times U(1))$ such that $g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$ with $g_1, g_2 \subset SU(2)$ and the two $U(1)$ subgroups are generated by $\begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & \sigma_3 \end{pmatrix}$. Note that the specific form of the identification (3.34) requires g_1 and g_2 to be $SU(2)$ as in the case of the nondegenerate two-level system. Thus, this case is simply two sets of the nondegenerate two-level case.

Another integrable cases are the transitions; $j_b = 1 \rightarrow j_a = 0$ and $j_b = 0 \rightarrow j_a = 1$. In each cases, the $U - V$ pair is given by

$$\begin{aligned} U &= \begin{pmatrix} -\frac{4}{3}i\lambda + i(x+y) & -i\varepsilon^{-1} & -i\varepsilon^{1} \\ -i\varepsilon^{-1*} & \frac{2}{3}i\lambda - ix & 0 \\ -i\varepsilon^{1*} & 0 & \frac{2}{3}i\lambda - iy \end{pmatrix} \\ x &= -\Omega_a - \frac{2}{3}\xi, \quad y = \Omega_a - \frac{2}{3}\xi \quad \text{for } j_b = 0 \rightarrow j_a = 1 \\ x &= -\Omega_b - \frac{2}{3}\xi, \quad y = \Omega_b - \frac{2}{3}\xi \quad \text{for } j_b = 1 \rightarrow j_a = 0. \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} V &= \frac{i}{2\lambda} \begin{pmatrix} R^{(b)}_{00} & R^{(ba)}_{0-1} & R^{(ba)}_{01} \\ R^{(ba)*}_{0-1} & R^{(a)}_{-1-1} & R^{(a)}_{-11} \\ R^{(ba)*}_{01} & R^{(a)}_{1-1} & R^{(a)}_{11} \end{pmatrix} \quad \text{for } \; j_b = 0 \rightarrow j_a = 1 \\ V &= \frac{i}{2\lambda} \begin{pmatrix} -R^{(a)}_{00} & R^{(ba)}_{10} & R^{(ba)}_{-10} \\ R^{(ba)*}_{10} & -R^{(b)}_{11} & -R^{(b)}_{-11} \\ R^{(ba)*}_{-10} & -R^{(b)}_{1-1} & -R^{(b)}_{-1-1} \end{pmatrix} \quad \text{for } \; j_b = 1 \rightarrow j_a = 0. \end{aligned} \tag{3.39}$$

The gauge fixing is given by

$$A = \begin{pmatrix} -i(x+y) & 0 & 0 \\ 0 & ix & 0 \\ 0 & 0 & iy \end{pmatrix}, \quad \bar{A} = 0, \tag{3.40}$$

and the effective field theory is specified by $G/H = SU(3)/U(2)$ with

$$T = -\bar{T} = \frac{2i}{3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3.41}$$

and $\beta = 1$. It is interesting to observe that the electric field components $\varepsilon^1, \varepsilon^{-1}$ in (3.38) parametrize the coset $SU(3)/U(2)$ and the vector $(\varepsilon^{-1*}, \varepsilon^{1*})^T$

transforms as a vector under the $U(2)$ action. In particular, since frequency detuning amounts to the global $U(1)(\subset U(2))$ action while longitudinal magnetic field amounts to the global $U(1)\times U(1)(\subset U(2))$ action, the effects of detuning and magnetic field to $\varepsilon^1, \varepsilon^{-1}$ can be easily obtained.

Three level system

The propagation of pulses in a multi-level medium with several carrier frequencies as given in (2.1) is a more complex problem than the two-level SIT and in general is not an integrable system. However, with certain restrictions on the parameters of the medium, it becomes integrable again and reveals much richer structures. Here, we will not exhaust all integrable multi-levels cases but restrict only to the degenerate three-level case and provide an effective field theory formulation. Other multi-level cases can be treated in a similar way. The propagation of two-frequency light interacting with a three-level medium, having either the Λ or V-type resonance configurations, has been studied earlier and its integrability has been demonstrated in the context of the inverse scattering method [7]. We suppress the general Maxwell-Bloch equation formulation for the three-level case and refer the reader to Ref. [7] for details. Here, we extend the Maxwell-Bloch equations of Ref. [7] in order to include a longitudinal magnetic field. Then, the Maxwell-Bloch equation in a dimensionless form, describing the Λ configuration with $j_b = 1, j_a = j_c = 0$, is given by

$$\bar{\partial}\varepsilon_j^q = -ip_j^q, \quad j = 1,2, \quad q = \pm 1 \tag{3.42}$$

and

$$\begin{aligned}
(\partial + it_0(k_1 v - 2\Delta_1 - \Omega_b q))p_1^q &= -i(\sum_{q'} \varepsilon_1^{q'} m_{q'q} - \varepsilon_1^q n_1 - \varepsilon_2^q r) \\
(\partial + it_0(k_2 v - 2\Delta_2 - \Omega_b q))p_2^q &= -i(\sum_{q'} \varepsilon_2^{q'} m_{q'q} - \varepsilon_2^q n_2 - \varepsilon_1^q r^*) \\
(\partial - it_0(k_2 v - k_1 v - 2\Delta_2 + 2\Delta_1))r &= -i\sum_q (\varepsilon_1^q p_2^{q*} - \varepsilon_2^{q*} p_1^q) \\
\partial n_j &= -i\sum_q (\varepsilon_j^q p_j^{q*} - \varepsilon_j^{q*} p_j^q) \\
(\partial + it_0 \Omega_b (q - q'))m_{qq'} &= -i\sum_{j=1,2} (\varepsilon_j^{q*} p_j^{q'} - \varepsilon_j^{q'} p_j^{q*})
\end{aligned} \tag{3.43}$$

where $\varepsilon_j^q, \quad j = 1,2$ is the amplitude of a double-frequency ultrashort pulse and $q = \pm 1$ denote the right(left)-handed polarization. Other variables are proportional to the components of the density matrix

$$p_1^q \;=\; \rho_{-q0}^{(ba)} \exp[-i(k_1 x - w_1 t)]/N_a, \quad p_2^q = \rho_{-q0}^{(bc)} \exp[-i(k_2 x - w_2 t)]/N_a$$

$$
\begin{aligned}
n_1 &= -\rho_{00}^{(a)}/N_a, \quad n_2 = -\rho_{00}^{(c)}/N_a, \quad m_{qq'} = -\rho_{-q'-q}^{(b)}/N_a \\
r &= -\rho_{00}^{(ca)} \exp[i(k_1 - k_2)x - i(w_1 - w_2)t]/N_a
\end{aligned}
\tag{3.44}
$$

and t_0 is a constant with the dimension of time and N_a is the polulation density of the level $|a>$. $2\Delta_1 \equiv w_1 - w_{ba}$, $2\Delta_2 \equiv w_2 - w_{bc}$ measure the amount of detuning from the resonance frequencies. The integrability of (3.43) comes from its equivalent zero curvature formulation with the 4×4 matrix $U - V$ pair,

$$
\begin{aligned}
U &= \begin{pmatrix} -A_1 - i\lambda \mathbf{1}_{2\times 2} & -iE \\ -iE^\dagger & -A_2 + i\lambda \mathbf{1}_{2\times 2} \end{pmatrix} \\
V &= \frac{i}{2\lambda} \begin{pmatrix} -M & P \\ P^\dagger & -N \end{pmatrix}
\end{aligned}
\tag{3.45}
$$

where

$$
\begin{aligned}
E &= \begin{pmatrix} \varepsilon_1^{-1} & \varepsilon_2^{-1} \\ \varepsilon_1^{1} & \varepsilon_2^{1} \end{pmatrix}, \; P = \begin{pmatrix} p_1^{-1} & p_2^{-1} \\ p_1^{1} & p_2^{1} \end{pmatrix}, \; M = \begin{pmatrix} m_{-1-1} & m_{1-1} \\ m_{-11} & m_{11} \end{pmatrix} \\
N &= \begin{pmatrix} n_1 & r^* \\ r & n_2 \end{pmatrix}, \; A_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \; A_2 = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}
\end{aligned}
$$

with

$$
\begin{aligned}
a &= \frac{it_0}{4}(k_1 v + k_2 v - 2\Delta_1 - 2\Delta_2 + 4\Omega_b) \\
b &= \frac{it_0}{4}(k_1 v + k_2 v - 2\Delta_1 - 2\Delta_2 - 4\Omega_b) \\
x &= \frac{it_0}{4}(-3k_1 v + k_2 v + 6\Delta_1 - 2\Delta_2) \\
y &= \frac{it_0}{4}(k_1 v - 3k_2 v - 2\Delta_1 + 6\Delta_2)
\end{aligned}
\tag{3.46}
$$

In the context of effective field theory, this corresponds to the case where $G/H = SU(4)/S(U(2) \times U(2))$ with the gauge fixing

$$
A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \; \bar{A} = 0 \tag{3.47}
$$

and

$$
T = -\bar{T} = i \begin{pmatrix} \mathbf{1}_{2\times 2} & 0 \\ 0 & -\mathbf{1}_{2\times 2} \end{pmatrix}. \tag{3.48}
$$

Similarly, we may repeat an identification for the case [18], $j_a = j_c = 1, j_b = 0$, and can easily verify that it corresponds to the symmetric space $SU(5)/U(4)$.

Thus, we have shown that various cases of SIT all correspond to the effective field theories with specific symmetric spaces, i.e.

$$\begin{aligned}
SU(2)/U(1) &\leftrightarrow \text{ nondegenerate two-level} \\
SU(3)/U(2) &\leftrightarrow \text{ degenerate two-level; } j_b = 0 \rightarrow j_a = 1, \\
&\qquad\qquad j_b = 1 \rightarrow j_a = 0,\ j_b = 1 \rightarrow j_a = 1 \\
(SU(2)/U(1))^2 &\leftrightarrow \text{ degenerate two-level; } j_b = 1/2 \rightarrow j_a = 1/2 \\
SU(4)/S(U(2)\times U(2)) &\leftrightarrow \text{ degenerate three level; } \ j_a = j_c = 0,\ j_b = 1. \\
SU(5)/U(4) &\leftrightarrow \text{ degenerate three level; } \ j_a = j_c = 1,\ j_b = 0.
\end{aligned} \tag{3.49}$$

Each case corresponds to a special type of symmetric spaces known as Hermitian symmetric spaces. Hermitian symmetric space is a symmetric space equipped with a complex structure which in our case is given by the adjoint action of T up to a scaling. In Sec. 5, the characteristic properties of Hermitian symmetric space will be used in obtaining infinitely many conserved local integrals. Our association of various SIT systems with Hermitian symmetric spaces suggests that to each Hermitian symmetric space there may exist a specific SIT with a proper adjusting of physical parameters. Especially, the multi-frequency generalization of SIT in a configuration of the "bouquet" type may correspond to the symmetric space $SU(n)/U(n-1)$ for an integer n. However, for large n, it requires a fine tuning of physical parameters which makes the theory unrealistic.

3.2 Inhomogeneous broadening

So far, we have identified various cases of SIT with specific effective field theories when inhomogeneous broadening is neglected. In the presense of inhomogeneous broadening, due to the microscopic motion of atoms, each atom in a resonant medium responds to the macroscopic incoming light with different Doppler shifts of transition frequencies. Thus, microscopic variables, e.g. the polarization P and the population inversion D are characterized by Doppler shifts and they couple to the macroscopic variable E through an average over the frequency spectrum as given in (2.7). A remarkable property of our effective field theory formulation is that it incorporates inhomogeneous broadening naturally only with minor modifications. The notion of effective potential variable g still persists where the microscopic variable g becomes also a function of frequency ξ, i.e. $g = g(z, \bar{z}, \xi)$. However, the action principle in (3.13) is no longer valid despite the use of the potential variable g. In order to formulate the SIT equation with inhomogeneous broadening in terms of the potential variable g, we relax the constraint equation (3.22) and require only

$$(g^{-1}\partial g + g^{-1}Ag)_{\mathbf{h}} - A = 0 \ , \tag{3.50}$$

where the subscript specifies the projection to the subalgebra **h**. The linear equation is given by

$$\begin{aligned} L_z\Psi &\equiv \left(\partial + g^{-1}\partial g + g^{-1}Ag - \xi T + \tilde{\lambda}T\right)\Psi = 0 \\ L_{\bar{z}}\Psi &\equiv \left(\bar{\partial} + \left\langle \frac{g^{-1}\bar{T}g}{\tilde{\lambda} - \xi'} \right\rangle\right)\Psi = 0 \end{aligned} \tag{3.51}$$

where the constant $\tilde{\lambda}$ is a modified spectral parameter which becomes $\lambda + \xi$ in the absense of inhomogeneous broadening. The angular brackets denote an average over ξ' as in (2.7). As in the previous examples without inhomogeneous broadening, we make the same identification of the matrix $g^{-1}\partial g + g^{-1}Ag - \xi T$ with various components of macroscopic electric fields which are independent of the microscopic quantity ξ. This requires the ξ-dependence of $g(z, \bar{z}, \xi)$ to be such that $g^{-1}\partial g + g^{-1}Ag - \xi T$ is independent of ξ. It is easy to see that this requirement is indeed satisfied by various integrable SIT systems we have considered previously except for the three-level system, where we must take $\xi = -t_0\Delta_1 = -t_0\Delta_2$. It means that in order to preserve the integrability in the presence of inhomogenuous broadening, two detuning parameters of the three-level system must be equal. Note that $\Psi(\tilde{\lambda}, z, \bar{z})$ is not a function of ξ. The integrability of the linear equation (3.51) becomes

$$\begin{aligned} 0 &= \left[\partial + g^{-1}\partial g + g^{-1}Ag - \xi T + \tilde{\lambda}T \ , \ \bar{\partial} + \left\langle \frac{g^{-1}\bar{T}g}{\tilde{\lambda} - \xi'} \right\rangle\right] \\ &= -\bar{\partial}(g^{-1}\partial g + g^{-1}Ag - \xi T) + \left\langle [T, \ g^{-1}\bar{T}g] \right\rangle \end{aligned} \tag{3.52}$$

where we used the fact that $g^{-1}\partial g + g^{-1}Ag - \xi T$ is independent of ξ and also the identity

$$\partial(g^{-1}\bar{T}g) + [g^{-1}\partial g + g^{-1}Ag, \ g^{-1}\bar{T}g] = 0. \tag{3.53}$$

Identifying $g^{-1}\bar{T}g$ with components of the density matrix as in the previous cases, we obtain the SIT equation with inhomogenous broadening. For example, we may identify E, P and D as in (3.26) so that (3.52) and (3.53) become the SIT equation with inhomogeneous broadening for the nondegenerate two-level case given by (2.6). Note that each frequency ξ corresponds to a specific gauge choice of the vector $U(1)$ subgroup. Therefore, inhomogenous broadening is equivalent to averaging over different gauge fixings of $U(1) \subset H$. This implies that inhomogenously broadening can not be treated by a single field theory thereby lacking a Lagrangian formulation. It is remarkable that nevertheless the group theoretical parametrization of various physical variables in terms of an effective potential g is still valid. This group theoretical structure underlies the integrability of the SIT equation as we will show in Sec. 4.

3.3 Effective potentials

One of the main advantage of our effective field theory approach to SIT problems is the introduction of a notion of effective potential energy. This allows us a systematic understanding of stability of solutions as well as the topological nature of soliton solutions. The effective potential energy term in (3.16) reveals a rich structure of the vacuum of the theory. In general, the potential (3.16) is "periodic" according to the coset structure G/H. This results in infinitely many degenerate vacua, which are specified by a set of integers, and also soliton solutions - finite energy solutions which interpolate between different vacua. In the nondegenerate two-level case, the potential term (3.10) becomes a periodic cosine potential (3.28) and each degenerate vacua are labeled by an integer n as in (3.29). Solitons interpolating between two different vacua with labels n_a and n_b as x varies from $-\infty$ to ∞ are characterized by a soliton number $\Delta n = n_b - n_a$. In order to understand the vacuum structure of the potential for a more general G/H case, we first note that the potential term $\mathrm{Tr}(gTg^{-1}\bar{T})$ is invariant under the change $g \rightarrow gh$ for $h \in H$. Consequently, we may express the potential term by a coset element $m \in G/H$ such as $\mathrm{Tr}(mTm^{-1}\bar{T})$, where m is given specifically by

$$m = \exp\begin{pmatrix} 0 & B \\ -B^\dagger & 0 \end{pmatrix} = \begin{pmatrix} \cos\sqrt{BB^\dagger} & B\sqrt{B^\dagger B}^{-1}\sin\sqrt{B^\dagger B} \\ -\sin\sqrt{B^\dagger B}\sqrt{B^\dagger B}^{-1}B^\dagger & \cos\sqrt{B^\dagger B} \end{pmatrix} \tag{3.54}$$

The matrix B parametrizes the tangent space of G/H. This manifests the periodicity of the potential through the cosine and the sine functions. In the case of our interest, B is a complex-valued matrix of dimension 1×1, 1×2 and 2×2 for the coset $SU(2)/U(1)$, $SU(3)/U(2)$ and $SU(4)/S(U(2)\times U(2))$ respectively. Then, due to the relation: $B\sin\sqrt{B^\dagger B}\sqrt{B^\dagger B}^{-1} = \sin\sqrt{BB^\dagger}\sqrt{BB^\dagger}^{-1}B$, the potential term reduces to the form

$$\mathrm{Tr}\left(I - 2\sin^2\sqrt{BB^\dagger}\right) + \mathrm{Tr}\left(I - 2\sin^2\sqrt{B^\dagger B}\right) \tag{3.55}$$

for $SU(2)/U(1)$ and $SU(4)/S(U(2)\times U(2))$ cases and

$$\mathrm{Tr}\left(4I - 6\sin^2\sqrt{BB^\dagger}\right) + \mathrm{Tr}\left(I - 3\sin^2\sqrt{B^\dagger B}\right) \tag{3.56}$$

for $SU(3)/U(2)$ case. In order to reduce further, we define non-zero eigenvalues of $B^\dagger B$ by $\phi_i^2, i = 1, .., r \equiv \mathrm{rank}\{B^\dagger B\}$, which are positive definite and coincide with those of $BB^\dagger$. With these eigenvalues, the potential term takes a particularly simple form,

$$a - b\sum_i \sin^2\phi_i, \tag{3.57}$$

where the positive constants a and b can be read directly from (3.55) and (3.56). This manifests the periodicity of the potential and the infinite degeneracy of the vacuum. The minima of the potential (3.57) occur at $\phi_i = (n_i + 1/2)\pi$ for integer n_i thus the degenerate vacua are specified by a set integers $(n_1, n_2, ..., n_r)$. The rank r of $B^\dagger B$ is one for $SU(2)/U(1)$ and $SU(3)/U(2)$ and two for $SU(4)/S(U(2)\times U(2))$. Thus solitons for the $SU(4)/S(U(2)\times U(2))$ case, interpolating between two vacua (n_{1a}, n_{2a}) and (n_{1b}, n_{2b}), are labeled by two soliton numbers $\Delta n_1 = n_{1b} - n_{1a}$ and $\Delta n_2 = n_{2b} - n_{2a}$. As a specific example, we may write the B matrix of $SU(3)/U(2)$ by

$$B = (-\phi\sin\eta e^{-i\beta} \quad -\phi\cos\eta e^{-i\alpha}) \tag{3.58}$$

so that the m matrix (3.54) is given by

$$m = \begin{pmatrix} \cos\phi & -\sin\phi\sin\eta e^{-i\beta} & -\sin\phi\cos\eta e^{-i\alpha} \\ \sin\phi\sin\eta e^{i\beta} & \cos^2\eta + \cos\phi\sin^2\eta & -\cos\eta\sin\eta e^{i\beta-i\alpha}(1-\cos\phi) \\ \sin\phi\cos\eta e^{i\alpha} & -\cos\eta\sin\eta e^{i\alpha-i\beta}(1-\cos\phi) & \sin^2\eta + \cos\phi\cos^2\eta \end{pmatrix}. \tag{3.59}$$

Then, the potential term becomes

$$\mathrm{Tr}(gTg^{-1}\bar{T}) = \mathrm{Tr}(mTm^{-1}\bar{T}) = 6 - 9\sin^2\phi \tag{3.60}$$

which agrees with (3.56).

4 Dressing and solitons

In order to find exact solutions of the SIT equations with inhomogeneous broadening, we first note that the linear equation (3.51) admits an application of the dressing method. The dressing method is a systematic way to obtain nontrivial solutions from a trivial one. In our case, we take a trivial solution by

$$g = 1 \text{ and } \Psi = \Psi^0 \equiv \exp[-(A - \xi T + \tilde{\lambda}T)z - \left\langle \frac{\bar{T}}{\tilde{\lambda} - \xi'} \right\rangle \bar{z}]. \tag{4.1}$$

Let Γ be a closed contour or a contour extending to infinity on the complex plane of the parameter $\tilde{\lambda}$ and $G(\tilde{\lambda})$ be a matrix function on Γ. Consider the Riemann problem of $\Psi^0 G(\tilde{\lambda})(\Psi^0)^{-1}$ on Γ which consists of the factorization

$$\Psi^0 G(\tilde{\lambda})(\Psi^0)^{-1} = (\Phi_-)^{-1}\Phi_+ \tag{4.2}$$

where the matrix function $\Phi_+(z,\bar{z},\tilde{\lambda})$ is analytic with n simple poles $\mu_1, ..., \mu_n$ inside Γ and $\Phi_-(z,\bar{z},\tilde{\lambda})$ analytic with n simple zeros $\lambda_1, ..., \lambda_n$ outside Γ. We assume that none of these poles and zeros lie on the contour Γ and the factorization is analytically continued to the region where $\lambda \neq \mu_i, \lambda_i$; $i = 1, ..., n$. We normalize Φ_+, Φ_- by $\Phi_+|_{\tilde{\lambda}=\infty} = \Phi_-|_{\tilde{\lambda}=\infty} = 1$. Differentiating (4.2) with respect to z and $\bar{z}$, one can easily show that

$$\begin{aligned}
\partial\Phi_+\Phi_+^{-1} - \Phi_+(A - \xi T + \tilde{\lambda}T)\Phi_+^{-1} &= \partial\Phi_-\Phi_-^{-1} - \Phi_-(A - \xi T + \tilde{\lambda}T)\Phi_-^{-1} \\
\bar{\partial}\Phi_+\Phi_+^{-1} - \left\langle \frac{\Phi_+\bar{T}\Phi_+^{-1}}{\tilde{\lambda}-\xi'} \right\rangle &= \bar{\partial}\Phi_-\Phi_-^{-1} - \left\langle \frac{\Phi_-\bar{T}\Phi_-^{-1}}{\tilde{\lambda}-\xi'} \right\rangle .
\end{aligned} \tag{4.3}$$

Since $\Phi_+(\Phi_-)$ is analytic inside(outside) Γ, we find that the matrix functions $\bar{U}$ and $\bar{V}$, defined by

$$\begin{aligned}
\bar{U} &\equiv -\partial\Phi\Phi^{-1} + \Phi(A - \xi T + \tilde{\lambda}T)\Phi^{-1} - \tilde{\lambda}T \\
\bar{V} &\equiv -(\tilde{\lambda} - \xi)\bar{\partial}\Phi\Phi^{-1} + \Phi\bar{T}\Phi^{-1}
\end{aligned} \tag{4.4}$$

where $\Phi = \Phi_+$ or Φ_- depending on the region, become independent of $\tilde{\lambda}$. Then, $\Psi \equiv \Phi\Psi^0$ satisfies the linear equation;

$$(\partial + \bar{U} + \tilde{\lambda}T)\Psi = 0 \ , \ (\bar{\partial} + \left\langle \frac{\bar{V}}{\tilde{\lambda}-\xi'} \right\rangle)\Psi = 0 \ . \tag{4.5}$$

Since $\bar{U}, \bar{V}$ are independent of $\tilde{\lambda}$, we may fix $\tilde{\lambda}$ by taking $\tilde{\lambda} = \xi$. Define g by $g \equiv H\Phi^{-1}|_{\tilde{\lambda}=\xi}$ where H is an arbitrary constant matrix which commutes with $T, \bar{T}$ and A. Then, $\bar{U}$ and $\bar{V}$ become

$$\bar{U} = g^{-1}\partial g + g^{-1}Ag - \xi T \tag{4.6}$$

$$\bar{V} = g^{-1}\bar{T}g. \tag{4.7}$$

If we further require the constraint condition (3.50) on $\Phi^{-1}|_{\tilde{\lambda}=\xi}$ such that

$$(-\partial\Phi\Phi^{-1} + \Phi A\Phi^{-1})_{\mathbf{h}} - A = 0, \tag{4.8}$$

we obtain a nontrivial solution g and Ψ from a trivial one. The nontrivial solution in general describes n-solitons coupled with radiation mode. If $G(\tilde{\lambda}) = 1$ in (4.2), we obtain exact n-soliton solutions. This formal procedure may be carried out explicitly for each cases of SIT in Sec. 3 and a closed form of n-soliton solutions can be obtained. In the following, we give explicit expressions of 1-soliton solution of two different cases and discuss about their physical properties.

4.1 Nondegenerate two-level case

Here, we set $\beta = 1$ without loss of generality. The 1-soliton solution may be obtained either by using the above dressing method or by applying the Bäcklund transformation directly to the trivial vacuum solution [19]. It is given by

$$\begin{aligned} \cos\varphi &= \frac{b}{\sqrt{(a-\xi)^2+b^2}}\mathrm{sech}(2bz-2bC\bar{z}) \\ \eta &= (a-\xi)z+(a-\xi)C\bar{z} \\ \theta &= -\tan^{-1}[\frac{a-\xi}{b}\coth(2bz-2bC\bar{z})]-2\xi z+2D\bar{z} \end{aligned} \tag{4.9}$$

where a, b are arbitrary constants and

$$C = \left\langle \frac{1}{(a-\xi')^2+b^2} \right\rangle \ , \ D = (a-\xi)\left\langle \frac{1}{(a-\xi')^2+b^2} \right\rangle - \left\langle \frac{a-\xi'}{(a-\xi')^2+b^2} \right\rangle. \tag{4.10}$$

In terms of E as defined in (2.13), 1-soliton is given by

$$E = -2ib \ \mathrm{sech}(2bz-2bC\bar{z})e^{-2i(az-D\bar{z}+(a-\xi)C\bar{z})}. \tag{4.11}$$

In the sharp line limit of the frequency distribution $f(\xi') = \delta(\xi'-\xi)$, 1-soliton retains the same form except for the change of constants C and D,

$$C = \frac{1}{(a-\xi)^2+b^2} \ , \ D = 0. \tag{4.12}$$

The name "n-soliton" is rather ambiguous since even in the sharp line limit the number n does not necessarily mean the topological soliton number. For example, when $a = \xi$ in the above 1-soliton solution, the solution describes a localized pulse configuration which interpolates between two different vacua of (3.29) such that

$$\phi(x=-\infty) = (n+\frac{1}{2})\pi, \quad \phi(x=\infty) = (n+\frac{1}{2}-\frac{b}{|b|}(-1)^n)\pi. \tag{4.13}$$

Thus it carries a topological number $\Delta n = (-1)^{n+1}b/|b|$ and becomes a topological soliton. However, when $a \neq \xi$, the solution interpolates between the same vacuum since the peak of the localized solution does not reach to the point where $\cos\varphi = 1$. Therefore, its topological number is zero. Nevertheless, it shares many important properties, e.g. localization, scattering behavior etc., with the topological soliton so as to deserve the name, a "nontopological soliton". One of the important feature of a nontopological soliton is that instead of a topological charge, it carries a $U(1)$ charge which accounts for stability of a nontopological soliton. The issue of $U(1)$ charge and stability will be addressed in the following sections. It is remarkable that the nontopological soliton can be obtained from the topological one by the local vector transform (3.15). Note that even though the action (3.13) is invariant under the gauge transformation, the physical quatities like E, P and D are not invariant. The gauge choice (3.25) refers to the amount of detuning with a detuning parameter $\Delta\omega = \xi$. Therefore, by choosing a different gauge condition through the vector gauge transformation, we obtain a system with a different amount of detuning. In particular, we can obtain a solution $g(\Delta\omega = \xi)$ of SIT theory with detuning parameter ξ from the solution with zero detuning through the transform

$$g(\Delta\omega = \xi) = h^{-1} g(\Delta\omega = 0) h, \quad h = \exp(\xi z T). \tag{4.14}$$

Or, in case of soliton solution, the topological soliton ($a - \xi = 0$) can be mapped to the nontopological one ($a - \xi \neq 0$). For this reason, in this paper we will simply call n-solitons for both the topological and the nontopological solitons unless otherwise stated explicitly.

4.2 Degenerate three-level case

We first restrict the degenerate three-level system to the resonant case ($\Delta_1 = \Delta_2 = 0,\ v = 0$) without external magnetic field and inhomogeneous broadening. This is equivalent to the case where $A = \bar{A} = 0$ in (3.20) with identifications (3.46) in terms of a 4×4 matrix g. By applying the Bäcklund transformation for the symmetric space sine-Gordon theory in Ref. [17], we obtain the 1-soliton solution with the parametrization as in (3.54),

$$B = -2B_0 \tan^{-1} \exp(2bz + \frac{2}{b}\bar{z} + const.) \equiv \phi B_0 \tag{4.15}$$

where b is a constant and B_0 is a constant 2×2 matrix satisfying

$$B_0 B_0^\dagger B_0 = B_0. \tag{4.16}$$

If the matrix B_0 is degenerate, i.e. $\det B_0 = 0$, B_0 can be given in general by

$$\frac{i}{\sqrt{1 + |\alpha|^2}} \begin{pmatrix} \theta_1 & \theta_2 \\ \alpha\theta_1 & \alpha\theta_2 \end{pmatrix} \tag{4.17}$$

with complex constants $\alpha, \theta_1, \theta_2$ satisfying $|\theta_1|^2 + |\theta_2|^2 = 1$. The eigenvalues of $B_0 B_0^\dagger$ are then zero and one. Therefore, up to a global $SU(2)$ similarity transform of $B_0 B_0^\dagger$, this solution corresponds to the (1,0) or (0,1)-soliton as explained in Sec. 3.3. This solution has been known as a simulton in earlier literatures. Note that the effective potential (3.10) is invariant under the similarity transform. This sounds to be contradictory since the similarity transform mixes two different emission modes of radiation with different resonance frequencies, $w_{ba} \neq w_{bc}$, consequently different transition energies. However, in considering the degenerate three-level case, we have assumed that the oscillator strengths are equal,

$$k_1|d_{ba}|^2 = k_2|d_{bc}|^2, \tag{4.18}$$

for incident wave numbers k_1, k_2 and the reduced dipole momenta d_{ba}, d_{bc} [7]. This assumption, together with a mediatory role of off-diagonal components of the density matrix, the similarity transform in fact becomes an invariance of the potential.

For the nondegenerate B_0, we can take B_0 as an arbitrary $U(2)$ matrix so that $B_0 B_0^\dagger = \mathbf{1}_{2\times 2}$ and the corresponding solution is the (1,1)-soliton in Sec. 3.3. This is different from (1,0) or (0,1)-soliton and can not be reached by the similarity transform since the similarity transform preserves eigenvalues of $B_0 B_0^\dagger$. In fact, they are topologically distinct and separated by an infinite potential energy barrier. Finally, physical quantities can be obtained from g through the identification (3.20). Explicitly, we find E, P and M in (3.46) to be

$$\begin{aligned} E &= iB_0\partial\phi = -2i\eta B_0 \mathrm{sech}(2\eta z + \frac{2}{\eta}\bar{z} + const.) \\ P &= -2B_0 \sin 2\phi, \\ M &= -N = -2\mathbf{1}_{2\times 2}\cos 2\phi \end{aligned} \tag{4.19}$$

respectively. Inclusion of detuning and external magnetic effects can be done easily by a gauge transform;

$$E \to H_1^{-1} E H_2,\ M \to H_1^{-1} M H_1,\ P \to H_1^{-1} P H_2,\ N \to H_2^{-1} N H_2 \tag{4.20}$$

where H_1, H_2 are given by $A_1 = H_1^{-1}\partial H_1,\ A_2 = H_2^{-1}\partial H_2$ for A_1, A_2 in (3.46).

5 Symmetries

The effective field theory action (3.13) reveals various types of symmetries of SIT. As an integrable field theory, at least classically, it possesses infinitely many conserved local integrals. These conservation laws can be extended to the inhomogeneously broadened case without difficulty. Some explicit local integrals for the nondegenerate two-level SIT have been given in earlier literatures [20]. Here, we present a systematic way to find local integrals for the general G/H case and present a few explicit examples. Another important symmetry of (3.13) is the global $U(1)$ axial symmetry; $g \to hgh$ for a constant element $h \in U(1) \subset H$. This type of symmetry has been previously unknown and is one of the outcomes of our effective field theory approach. We show that the $U(1)$ symmetry and its corresponding charge plays a crucial role in pulse stability problem.

Besides the aforementioned continuous symmetries, the action (3.13) also possesses two distinct types of discrete symmetries; the chiral and the dual symmetries. Discrete symmetries relate two different solutions of the SIT equation. In particular, the dual symmetry relates the "bright" soliton with the "dark" soliton of SIT. These discrete symmetries reflect the ubiquitous nature of the action (3.13) without the potential term, as an action for the coset conformal field theory and also a bosonized version of 1+1-dimensional free fermion field theories.

5.1 Conserved local integrals

In the previous section, we have shown that the linear equation with a spectral parameter yields exact soliton solutions through the dressing procedure. The same linear equation can be employed to construct infinitely many conserved local integrals. In this section, using the linear equation as well as the properties of Hermitian symmetric spaces, we present a systematic way of finding such integrals for various cases of SIT introduced earlier.

We first recall some relevant mathematical facts on Hermitian symmetric space [21]. A symmetric space G/H is a coset space with the Lie algebra commutation relations among generators of associated Lie algebras such that

$$[\mathbf{h}\ ,\ \mathbf{h}] \subset \mathbf{h}\ ,\ [\mathbf{h}\ ,\ \mathbf{m}] \subset \mathbf{m}\ ,\ [\mathbf{m}\ ,\ \mathbf{m}] \subset \mathbf{h}\ , \tag{5.1}$$

where $\mathbf{g}$ and $\mathbf{h}$ are Lie algebras of G and H and $\mathbf{m}$ is the vector space complement of $\mathbf{h}$ in $\mathbf{g}$, i.e.

$$\mathbf{g} = \mathbf{h} \oplus \mathbf{m}. \tag{5.2}$$

Hermitian symmetric space is a symmetric space equipped with a complex structure. In general, such a complex structure is given by the adjoint action of T_0 on $\mathbf{m}$ up to a scaling, where T_0 is an element belonging to the Cartan subalgebra of $\mathbf{g}$ whose stability subgroup is H. In our case, T_0 is precisely the T-matrix given in Sec. 3. Namely, with a suitable normalization of T, we have

$$T \in \mathbf{h}\ \ ,\ \ [T\ ,\ \mathbf{h}] = 0 \quad \text{and} \quad [T\ ,\ [T\ ,\ a]] = -a \ \ \text{for any } a \in \mathbf{m}. \tag{5.3}$$

We decompose an algebra element $\psi \in \mathbf{g}$ according to (5.2),

$$\psi = \psi_h + \psi_m. \tag{5.4}$$

Such a decomposition could be extended to a group element $\Psi \in G = SU(n)$ if we substitute the commutator by a direct matrix multiplication and add an identity element $h_0 = I$ to the subalgebra **h**, i.e.

$$\Psi = \Psi_h + \Psi_m, \quad \Psi_h \Psi_h \subset \Psi_h, \quad \Psi_h \Psi_m \subset \Psi_m, \quad \Psi_m \Psi_m \subset \Psi_h. \tag{5.5}$$

In other words, any unitary $n \times n$ matrix can be expressed as a linear combination of $SU(n)$ generators and the identity element h_0 such that

$$\Psi = \Psi_h + \Psi_m = \sum_{a=0}^{\dim \mathbf{h}} \Psi^a h_a + \sum_{b=1}^{\dim \mathbf{m}} \Psi^b m_b. \tag{5.6}$$

In order to solve the linear equation (3.51) recursively, we expand Ψ in terms of a power series in $\tilde{\lambda}$,

$$\Psi \exp(-\tilde{\lambda} T z) = \sum_{i=0}^{\infty} \frac{1}{\tilde{\lambda}^i} \Phi_i, \quad \text{where} \quad \Phi_i = \sum_{a=0}^{\dim \mathbf{h}} \Phi_i^a h_a + \sum_{b=1}^{\dim \mathbf{m}} \Phi_i^b m_b \equiv \mathcal{P}_i + \mathcal{Q}_i. \tag{5.7}$$

We also introduce the abbreviation:

$$\begin{aligned} \mathcal{E} &\equiv \bar{U} = g^{-1}\partial g + g^{-1} A g - \xi T \subset \mathbf{m} \\ \left\langle \bar{V} \right\rangle_l &\equiv \left\langle g^{-1} \bar{T} g \right\rangle_l = \left\langle g^{-1} \bar{T} g (-\xi')^l \right\rangle = D_l + P_l, \quad D_l \subset \mathbf{h}, \quad P_l \subset \mathbf{m} \end{aligned} \tag{5.8}$$

so that the linear equations become

$$(\partial + \mathcal{E})\Phi_i - [T \ , \ \Phi_{i+1}] = 0 \tag{5.9}$$

and

$$\bar{\partial}\Phi_i + \sum_{l=0}^{i-1} (D_{i-l-1} + P_{i-l-1})\Phi_l = 0. \tag{5.10}$$

Then the **m**-component of (5.9) is

$$\partial \mathcal{Q}_{i-1} + \mathcal{E}\mathcal{P}_{i-1} - [T \ , \ \mathcal{Q}_i] = 0, \tag{5.11}$$

which can be solved for $\mathcal{Q}_i$ by applying the adjoint action of T,

$$\mathcal{Q}_i = -[T \ , \ \partial \mathcal{Q}_{i-1}] - [T \ , \ \mathcal{E}]\mathcal{P}_{i-1}. \tag{5.12}$$

$\mathcal{P}_i$ can be solved similarly from the **h**-component of (5.9) and (5.10) such that

$$\mathcal{P}_i = -\int \mathcal{E} \mathcal{Q}_i dz - \sum_{l=0}^{i-1} \int (D_{i-l-1}\mathcal{P}_l + P_{i-l-1}\mathcal{Q}_l) d\bar{z}. \tag{5.13}$$

These recursive solutions of $\mathcal{P}_i$ and $\mathcal{Q}_i$ can be determined completely with appropriate initial conditions. For example, if we choose an initial condition which is consistent with the recursion relation for $i \leq 0$,

$$\mathcal{P}_0 = I \quad , \quad \mathcal{Q}_0 = 0, \tag{5.14}$$

we find for the first few explicit cases in the series,

$$\mathcal{P}_1 = \int \mathcal{E}[T\ ,\ \mathcal{E}]dz - \int D_0 d\bar{z}, \quad \mathcal{Q}_1 = -[T\ ,\ \mathcal{E}] \tag{5.15}$$

and

$$\begin{aligned} \mathcal{P}_2 &= \int(\mathcal{E}\partial\mathcal{E} + \mathcal{E}[T\ ,\ \mathcal{E}]\mathcal{P}_1)dz + \int(-D_1 - D_0\mathcal{P}_1 + P_0[T\ ,\ \mathcal{E}])d\bar{z} \\ \mathcal{Q}_2 &= -\partial\mathcal{E} - [T\ ,\ \mathcal{E}]\mathcal{P}_1. \end{aligned} \tag{5.16}$$

Finally, the consistency condition: $\partial\bar{\partial}\mathcal{P}_i = \bar{\partial}\partial\mathcal{P}_i$ gives rise to infinitely many conserved local currents,

$$J_i \equiv \partial\mathcal{P}_i = -\mathcal{E}\mathcal{Q}_i, \quad \bar{J}_i \equiv \bar{\partial}\mathcal{P}_i = -\sum_{l=0}^{i-1}(D_{i-l-1}\mathcal{P}_l + P_{i-l-1}\mathcal{Q}_l), \tag{5.17}$$

satisfying $\partial\bar{J}_i = \bar{\partial}J_i$. A few examples are

$$J_1 = \mathcal{E}[T\ ,\ \mathcal{E}] \quad , \quad \bar{J}_1 = -D_0 \tag{5.18}$$

$$J_2 = \mathcal{E}\partial\mathcal{E} + \mathcal{E}[T\ ,\ \mathcal{E}]\mathcal{P}_1 \quad , \quad \bar{J}_2 = -D_1 - D_0\mathcal{P}_1 + P_0[T\ ,\ \mathcal{E}]. \tag{5.19}$$

The first current J_1, $\bar{J}_1$ gives rise to the energy conservation law. With the repetitive use of the properties of the Hermitian symmetric space, it can be easily checked that these conservation laws are indeed consistent with the equations of motion (3.52)(3.53), which in the present convention take a particularly simple form:

$$\begin{aligned} \bar{\partial}\mathcal{E} - [T\ ,\ P_0] &= 0 \\ \partial D_i + [\mathcal{E}\ ,\ P_i] &= 0 \\ \partial P_i + [\mathcal{E}\ ,\ D_i] - [T\ ,\ P_{i+1}] &= 0. \end{aligned} \tag{5.20}$$

In general, the conserved current contains nonlocal terms. These nonlocal terms may be dropped out by taking the T-component of the currents. For instance, the T-component of the "spin-2" current conservation is

$$\bar{\partial}\mathrm{Tr}(T\mathcal{E}\partial\mathcal{E}) = \partial\mathrm{Tr}(TP_0[T\ ,\ \mathcal{E}] - TD_1) \tag{5.21}$$

which obviously does not contain nonlocal terms. In order to demonstrate the above procedure more explicitly, we take the $SU(2)/U(1)$ case as an example. In this case, relevant 2×2 matrices are given by

$$\mathcal{E}=\begin{pmatrix}0 & -E\\ E^* & 0\end{pmatrix},\quad T=\frac{1}{2}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} \tag{5.22}$$

and we introduce the notation

$$\begin{aligned}\left\langle g^{-1}\bar{T}g\right\rangle_l &= \left\langle g^{-1}\bar{T}g(-\xi')^l\right\rangle = -i\begin{pmatrix}\left\langle D(\xi')(-\xi')^l\right\rangle & \left\langle P(\xi')(-\xi')^l\right\rangle\\ \left\langle P^*(\xi')(-\xi')^l\right\rangle & -\left\langle D(\xi')(-\xi')^l\right\rangle\end{pmatrix}\\ &\equiv -i\begin{pmatrix}D_l & P_l\\ P_l^* & -D_l\end{pmatrix}\end{aligned} \tag{5.23}$$

Let

$$\Psi\exp(-\tilde{\lambda}Tz)=\sum_{i=0}^{\infty}\frac{\Phi_i}{\tilde{\lambda}^i}\ ;\ \Phi_i\equiv\begin{pmatrix}p_i & q_i\\ r_i & s_i\end{pmatrix} \tag{5.24}$$

so that the linear equation changes into

$$(\partial+\begin{pmatrix}0 & -E\\ E^* & 0\end{pmatrix})\Phi_i-[T\ ,\ \Phi_{i+1}]=0 \tag{5.25}$$

and

$$\bar{\partial}\Phi_i+\sum_{l=0}^{i-1}\left\langle g^{-1}\bar{T}g\right\rangle_{i-l-1}\Phi_l=0. \tag{5.26}$$

These equations can be solved iteratively in component,

$$q_i = \frac{1}{2i}(\partial q_{i-1}-Es_{i-1}) \tag{5.27}$$

$$r_i = -\frac{1}{2i}(\partial r_{i-1}+E^*p_{i-1}) \tag{5.28}$$

$$p_i = \int Er_i dz+i\sum_{l=0}^{i-1}\int(D_{i-l-1}p_l+P_{i-l-1}r_l)d\bar{z} \tag{5.29}$$

$$s_i = -\int E^*q_i dz+i\sum_{l=0}^{i-1}\int(-D_{i-l-1}s_l+P^*_{i-l-1}q_l)d\bar{z} \tag{5.30}$$

together with the initial conditions:

$$p_0=s_0=-2i,\ r_0=q_0=0. \tag{5.31}$$

The consistency: $\partial\bar{\partial}p_m-\bar{\partial}\partial p_m=0$ leads to the infinite current conservation laws, $\partial\bar{J}_i+\bar{\partial}J_i=0$, or

$$i\partial\sum_{l=0}^{i-1}(D_{i-l-1}p_l+P_{i-l-1}r_l)-\bar{\partial}(Er_i)=0. \tag{5.32}$$

The consistency condition, $\partial\bar{\partial}s_i - \bar{\partial}\partial s_i = 0$, gives rise to the complex conjugate pair of (5.32). A few explicit examples of conserved currents are

$$\begin{aligned} \bar{J}_1 &= -2D_0 \\ J_1 &= EE^* \end{aligned} \tag{5.33}$$

$$\begin{aligned} \bar{J}_2 &= 4iD_1 - 2P_0E^* \\ J_2 &= E\partial E^* \end{aligned} \tag{5.34}$$

$$\begin{aligned} \bar{J}_3 &= -2P_0\partial E^* + 8D_2 + 4iE^*P_1 \\ J_3 &= E\partial^2 E^* + (EE^*)^2 \end{aligned} \tag{5.35}$$

$$\begin{aligned} \bar{J}_4 &= -16iD_3 + 8E^*P_2 + 4iP_1\partial E^* - 2P_0\partial^2 E^* - 2P_0E^*|E|^2 \\ J_4 &= E\partial^3 E^* + |E|^2\partial|E|^2 + 2E|E|^2\partial E^* \end{aligned} \tag{5.36}$$

5.2 $U(1)$ symmetry

One of the properties of the Wess-Zumino-Witten action (3.9) is the global axial vector gauge symmetry, i.e.

$$S_{WZW}(fgf) = S_{WZW}(g) \tag{5.37}$$

for a constant $f \in G$. The extra terms in the deformed action (3.13) break this global axial symmetry in general. Nevertheless, there remains at least an unbroken $U(1)$-axial symmetry given by $g \to hgh$ for $h = \exp(\gamma T) \in U(1)$. This $U(1)$-invariance results in a conserved charge according to the Noether method. Even though a general expression for the conserved charge should be possible, in practice it requires an explicit parametrization of the group variable g. Thus, for the sake of brevity, here we will restrict ourselves only to the $SU(2)/U(1)$ case. In this case, the global axial transform is given by

$$\eta \to \eta + \gamma \quad \text{for } \gamma \text{ constant.} \tag{5.38}$$

The corresponding Noether currents and the associated axial charge are

$$\begin{aligned} J &= \frac{\cos^2\varphi}{\sin^2\varphi}\partial\eta \quad , \quad \bar{J} = \frac{\cos^2\varphi}{\sin^2\varphi}\bar{\partial}\eta \\ Q &= \int_{-\infty}^{\infty} dx(J + \bar{J}) \end{aligned} \tag{5.39}$$

where, owing to the complex sine-Gordon equation (2.14)-(2.16), J and $\bar{J}$ can be shown to satisfy the conservation law: $\partial\bar{J} + \bar{\partial}J = 0$. In particular, the axial charge of the nontopological 1-soliton in (4.9) is

$$Q_{\text{1-sol}} = c\tan^{-1}\frac{|b|}{a-\xi}. \tag{5.40}$$

The charge of the topological soliton is not well defined [19]. Stability of nontopological solitons can be proved either by using conservation laws in terms of charge and energy as given in [19], or by studying the behavior against small fluctuations which will be explained in Sec. 6 [22].

5.3 Discrete symmetries

Besides continuous symmetries, the action (3.13) also reveals discrete symmetries of SIT, *the chiral symmetry* and *the dual symmetry*. They are manifested most easily in the gauge where $A = \bar{A} = 0$. Extensions to different gauges, e.g. the off-resonant case which requires a different gauge fixing as in (3.25), can be made by the vector gauge transform in (3.15).

One peculiar property of the action (3.13) is its asymmetry under the change of parity $z \leftrightarrow \bar{z}$. This is because the Wess-Zumino-Witten action (3.9) is a sum of the parity even kinetic term and the parity odd Wess-Zumino term thereby breaking parity invariance. In the SIT context, broken parity is due to the slowly varying enveloping approximation which breaks the apparent parity invariance of the Maxwell-Bloch equation. Nevertheless, the action (3.13) is invariant under the chiral transform

$$z \leftrightarrow \bar{z} \ , \ g \leftrightarrow g^{-1} \ (\text{ or } \eta \leftrightarrow -\eta \ , \ \varphi \leftrightarrow -\varphi) \tag{5.41}$$

which may be compared with the CP invariance in the context of particle physics. Thus, parity invariance is in fact not lost but appears in a different guise, namely the chiral invariance. This chiral symmetry relates two distinct solutions, or itgenerates a new solution from a known one. For example, under the chiral transform (5.41), the 1-soliton solution (4.9) in the resonant case ($\xi = 0$) becomes again a soliton but with the replacement of constants a, b by

$$a \rightarrow -\frac{a}{a^2 + b^2} \ , \ b \rightarrow \frac{b}{a^2 + b^2}. \tag{5.42}$$

This implies the change of pulse shape and the change of pulse velocity by $v \rightarrow c - v$. The current and the charge also change into

$$J \rightarrow -\bar{J} \ , \ \bar{J} \rightarrow -J \ , \ Q \rightarrow -Q. \tag{5.43}$$

It is remarkable that the velocity changes from v to $c - v$ unlike the usual parity change $v \rightarrow -v$.

The other type of discrete symmetry of the action (3.13) is the dual symmetry of the Krammers-Wannier type: [19]

$$\beta \leftrightarrow -\beta \ , \ g \leftrightarrow i\sigma g \tag{5.44}$$

where σ is a constant matrix with a property $\sigma T + T\sigma = 0$. For example, $\sigma = \sigma_1$ of Pauli matrices for the $SU(2)/U(1)$ case. This rather unconventional symmetry, also the name, stems from the ubiquitous nature of the action (3.13), i.e. it also arises as a large level limit of parafermions in statistical physics and the above transform is an interchange between the spin and the dual spin variables [15]. In general, the change of the sign of β makes the potential upside down so that the degenerate vacua becomes maxima of the potential and vice versa. Therefore, the dual transformed solutions are no longer stable solutions. This allows us to find a localized solution which approches to the maximum of the potential asymptotically (so called a "dark" soliton). In practice, the dark soliton for positive β can be obtained by replacing $\beta \to -\beta$, $\bar{z} \to -\bar{z}$ in the "bright" soliton (usual one) of the negative β case. For example, we obtain the dark 1-soliton for the $SU(2)/U(1)$ case as follows:

$$\begin{aligned} \cos\varphi e^{2i\eta} &= -\frac{b}{\sqrt{(a-\xi)^2+b^2}}\tanh(2bz+2bC\bar{z}) - i\frac{a-\xi}{\sqrt{(a-\xi)^2+b^2}} \\ \theta &= -2(a-\xi)(z-C\bar{z}) - 2\xi z. \end{aligned} \tag{5.45}$$

6 Stability

The physical relevance of a soliton number is that it accounts for the stability of solitons against "topological" (soliton number changing) fluctuations. Note that any finite energy solution must approach to one of the degenerate vacua at both ends of the x-axis thus carries a specific soliton number. In general, solutions are made of soliton, antisoliton, breather and radiation modes where each modes carry soliton number 1, -1, 0 and 0 respectively. The overall soliton number is a conserved topological charge which satisfies the so-called superselection rule, i.e. it cannot be changed during any physical processes due to the infinite potential energy barrier between any two finite energy solutions with different topological charges. This infinite energy barrier results from the infinite length of the x-axis despite the finite potential energy density per unit length. In the case of sine-Gordon solitons, the "area" of the McCall and Hahn's theorem just enumerates the soliton number, i.e.

$$\begin{aligned} \int_{-\infty}^{\infty} 2E dt &= \int_{-\infty}^{\infty} 2\partial\varphi dt = 2\varphi(t=\infty) - 2\varphi(t=-\infty) \\ &= 2\varphi(x=-\infty) - 2\varphi(x=\infty)(= -2\pi\Delta n) \end{aligned} \tag{6.1}$$

where the equality in the last line holds for localized solutions. However, this coincidence is only for a specific case where E is real and inhomogeneous broadening is ignored. As was shown in Sec. 2, the inclusion of a phase degree of freedom into E makes E complex so that the time area of $2E$ becomes complex too, thus it loses its meaning as a soliton number.

Inhomogeneous broadening also spoils the notion of a soliton number. In Sec. 3, we have noted that the effective potential variable g becomes a microscopic variable depending on the frequency ξ and atomic variables P and D also depend on ξ. This implies that the soliton number is also a ξ-dependent microscopic variable. However, the macroscopic field E, also given in terms of g, is independent of ξ. In other words, the microscopic variable g arranges the ξ-dependence in such a way that the resulting E field becomes independent of ξ. This is exemplified by the 1-soliton solution in (4.9)-(4.11). Thus, inhomogeneous broadening in general requires E to be a function of "frequency ξ averaged" coefficients so that it does not carry a topological soliton number. In this regard, the McCall and Hahn's area theorem is different from the topological stability argument and $2n\pi$ pulses should be distinguished from n-solitons. It is remarkable that the area theorem provides a stability argument even in the absence of the topological argument. In fact, the proof of the area theorem relied crucially on the averaging over the frequency ξ of detuning associated with inhomogeneous broadening. However, one serious drawback of the area theorem is that it applies only to a very restricted case, i.e. the case of real E (neglecting frequency modulation effect) and the symmetric frequency distribution. Presently, a more general area theorem including frequency modulation is not known and finding such a theorem would be one of the most important problem in the theory of self-induced transparency.

For the rest of the section, we consider a restricted case which however generalizes the area theorem to a certain extent. We consider the $SU(2)/U(1)$ case without inhomogeneous broadening and prove the area stability around the 1-soliton solution (topological or nontopological) using a perturbative argument similar to the one given in [20]. This shows that the reshaping of a pulse also occurs in the off resonant case ($a-\xi \neq 0$) but slower than in the resonant case. We argue that the $U(1)$ charge of optical pulses introduced in earlier sections could be used also in generalizing the area theorem. In particular, we prove the $U(1)$ charge stability using the local conservation law. These results are in good agreement with numerical results obtained earlier [23].

Consider the bright soliton (4.9) which describes a coherent pulse propagating in the attenuator. In order to overcome the difficulty with the complexity of the usual area function for complex E, we regard φ of the complex sine-Gordon equation as a "modified" area function. Without loss of generality, we may take the asymptotic time behavior of φ by

$$\begin{aligned} \varphi(t=-\infty,x) = -\pi/2, \quad \varphi(t=\infty,x) = \pi/2 \quad \text{for} \quad a=\xi \\ \varphi(t=-\infty,x) = \varphi(t=\infty,x) = -\pi/2, \quad \text{for} \quad a \neq \xi, \end{aligned} \tag{6.2}$$

so that the modified area

$$A = \int_{-\infty}^{\infty} 2\partial\varphi dt \tag{6.3}$$

of the topological soliton ($a-\xi=0$) is 2π while that of the nontopological soliton is zero. Now assume that the system was initially in the vacuum state ($\varphi(t=-\infty,x)=-\pi/2$). If the solution is perturbed around a soliton so that near the trailing edge of the pulse ($t >> 1$), the modified area function changes by $\delta\varphi=\epsilon$ for small ϵ, i.e. $\varphi(t>>1) = \pm\pi/2+\epsilon$. Then, the complex sine-Gordon equation for $t >> 1$ becomes

$$\bar{\partial}\partial\varphi + 4\frac{b^2}{(a-\xi)^2+b^2}\epsilon = 0 \tag{6.4}$$

where we have neglected the variation of η which contributes only to the order ϵ^2. This shows that if the modified area is greater than 2π (or zero) by the amount $\epsilon > 0$, then $\bar{\partial}\partial\varphi < 0$ so that the field $\partial\varphi$ at the trailing edge tends to decrease along the $\bar{z} = x/c$ axis to recover a total modified area 2π (or zero). On the other hand, if the perturbation is such that $\epsilon < 0$, then $\bar{\partial}\partial\varphi > 0$ and the field at the trailing edge increases. Therefore, the total modified area tends to remain 2π or zero. Moreover, (6.4) shows that the recovery of area is faster in the resonant case ($a=\xi$) than in the off-resonant case ($a \neq \xi$) which agrees precisely with the numerical result [23]. In fact, the recovery of the modified area is acompanied by a stronger pulse reshaping recovery to that of a soliton. Here, we only point out that the stability of a soliton shape against fluctuations which preserve the modified area may be shown by making a modification of the Lamb's proof using the Liapunov function [20], and also by proving the stability of higher order conserved charges as discussed below. In case of the dark soliton (5.45) which describes a coherent pulse in the amplifier, the system is initially in the upper level ($\varphi(t=-\infty,x)=0$). In such a case, one can easily see that the above consideration of perturbation results in the instability of a soliton configuration.

It was suggested previously that pulse reshaping arises due to the stability of all higher order conserved integrals against small fluctuations and each integrals behaves like an "area" [4]. In order to explain the stability of conserved integrals, we apply the method introduced in [4] to the $U(1)$-charge conservation law:

$$\frac{\partial}{\partial t}[\cot^2\phi(\partial+\bar{\partial})\eta] + c\frac{\partial}{\partial x}[\cot^2\phi\partial\eta] = 0. \tag{6.5}$$

Introduce the $U(1)$ charge in terms of a "time area",

$$A(x) \equiv \int_{-\infty}^{+\infty} [c \cot^2 \varphi \partial\eta] dt, \tag{6.6}$$

so that (6.5) becomes

$$\frac{dA}{dx} = -\cot^2 \varphi (\partial + \bar{\partial})\eta|_{t=-\infty}^{t=+\infty}. \tag{6.7}$$

The boundary contribution from the 1-soliton is zero thus the $U(1)$-charge $A(x)$ is conserved in space, i.e. $dA/dx = 0$. The physical meaning of the $U(1)$ charge is clear; since $\partial\eta = a - \xi = a + w_o - w$ for the 1-soliton which expresses frequency detuning and $\cot^2 \varphi$ is peaked around the soliton, the $U(1)$-charge measures precisely the amount of detuning. We caution that, as already noted by Lamb [3], the carrier frequency w_o shifts to $w_o + a$ in the presense of a soliton so that the detuning parameter is given by $a + w_o - w$. Now, if the solution is perturbed around the soliton such that near the trailing edge of the pulse,

$$\begin{aligned} \phi(t >> 1, x) &= \pm\pi/2 + \epsilon(x) \\ \eta(t >> 1, x) &= (a - \xi)(t - \frac{x}{c} - \frac{1}{(a-\xi)^2 + b^2}\frac{x}{c}) + \delta(x) \end{aligned} \tag{6.8}$$

for small parametric functions $\epsilon(x)$ and $\delta(x)$. To the leading order, the variation of the $U(1)$-charge then becomes

$$\frac{d\delta A}{dx} = -(a - \xi)\left(1 + \frac{1}{(a-\xi)^2 + b^2}\right)\epsilon^2. \tag{6.9}$$

This shows that the detuning by a higher frequency, i.e. $a - \xi > 0$ reduces A for increasing x while the lower frequency detuning does exactly the opposite. Since the conserved charge A of the 1-soliton is $c \tan^{-1}[|b|/(a - \xi)]$, it can be concluded that the absolute value of $A(x)$ decreases monotonically along the x-axis and converges to a constant charge of the soliton. Note that the recovery of $|A|$ value to that of the soliton is slower than the area case since it is of the order ϵ^2. The decreasing behavior of $|A|$ is in good agreement with the numerical work [23] which showed that the frequency of the optical pulse is pulled towards the transition frequency and reaches to a constant value along the x-axis. Thus, the $U(1)$-charge stability provides a generalization of the area theorem in the presense of frequency detuning. This type of stability argument might be extended to the case of other conserved integrals.

In the presence of inhomogeneous broadening, the $U(1)$ conservation breaks down unlike the local conservation laws in Sec. 5.1. It introduces an anomaly term M in the $U(1)$-current conservation such that $\partial\bar{J} + \bar{\partial}J = M$ for $J, \bar{J}$ in (5.39) and

$$
\begin{aligned}
M &= 2\cot\varphi[\ \cos(\theta-2\eta) < \sin(\theta-2\eta)\sin 2\varphi > \\
&- \sin(\theta-2\eta) < \cos(\theta-2\eta)\sin 2\varphi > -(\cot^2\varphi\bar{\partial}\eta + \frac{1}{2}\bar{\partial}\theta)\partial\varphi\ ]
\end{aligned}
\tag{6.10}
$$

This anomaly vanishes in the sharp line limit due to the constraint (2.16). It also vanishes in the case of 1-soliton and the charge remains conserved. This behavior may be compared with the conserved area of topological solitons in the presence of inhomogenous broadening. The area theorem of McCall and Hahn proves that inhomogeneous broadening changes the pulse area until it reaches to those of $2n\pi$ pulses. It remains as an important open question whether one could prove a generalized area theorem of pulse stability including frequency modulation by making use of $U(1)$ charge and anomaly.

7 Discussion

In this paper, we have presented a field theoretic formulation of SIT. Various cases of SIT have been associated with specific symmetric spaces G/H and a systematic group theory approaches to the SIT problem has been made. In doing so, the introduction of a matrix potential variable g was an essential step. One immediate question is about the generality of such a group variable in the description of nonlinear optics problems. In the nondegenerate two-level case, the $SU(2)$ variable g was introduced by solving the constraint which expresses the conservation of probability. In more general multi-level cases, the complexity of atomic states requires bigger groups G than $SU(2)$. However, not all degrees of freedom associated with bigger groups become physical degrees of freedom, they are constrained by the Bloch equation which requires a restriction of g e.g. as in (3.26). That is, the microscopic atomic variables couple to the macroscopic electric field selectively through dipole transitions which suppresses the excitation of certain degrees of freedom. This reduction of variables has been accomplished by considering the G/H coset structure. It should be emphasized that the effective field theory we introduced is not an ordinary sigma model where g takes value in the coset G/H. In the present case, a coset structure is imposed only through a nontrivial Lagrange multiplier.[5] Unfortunately, this coset theoretic description is not valid for all cases of SIT. For example, the particular way electric field interacts with two-level degenerate states in the $2 \to 2$ transition does not allow a nice coset structure, but only described by the double sine-Gordon

[5] More correctly, it corresponds to the coset LG/LH of loop groups LG and LH rather than G/H, see Ref. [24]

equation when further restrictions are made. Thus, a systematic application of group theory to the general SIT problems still remains as an interesting open question.

On the other hand, the group theoretical approach is not restricted to the SIT problem only. The nonlinear Schrödinger equation, which is the governing equation for optical soliton communication systems, can be also generalized according to each Hermitian symmetric spaces [25]. In fact, both SIT and the nonlinear Schrödinger equation share the same Hamiltonian structure and can be combined together. We will consider this case and its physical applications in a separate paper.

Finally, our approach provides a vantage point to the quantum SIT problem as well as the quantum optics itself. A direct quantization of SIT using the quantum inverse scattering has been made by Rupasov and a localized multiparticle state has been found and compared with a quantum soliton [26]. Our group theory formulation of SIT suggests an alternative, yet more systematic way of quantizing the SIT theory according to each specific coset structures. More generally, our approach can be extended to other quantum optical systems. Thes works are in progress and will be reported elsewhere.

ACKNOWLEDGEMENT

This work was supported in part by the program of Basic Science Research, Ministry of Education BSRI-96-2442, and by Korea Science and Engineering Foundation through CTP/SNU.

References

[1] S. L. McCall and E. L. Hahn, Phys. Rev. Lett. **18**, 908 (1967); Phys. Rev. **183**, 457 (1969).

[2] A. I. Maimistov, A. M. Basharov, S. O. Elyutin and Yu. M. Sklyarov, Phys. Rep. **191**, No. 1 , 1-108 (1990).

[3] G. L. Lamb, Jr., Phys. Rev. Lett. **31**, 196 (1973).

[4] M. J. Ablowitz, D. J. Kaup and A. C. Newell, J. Math. Phys. **15**, 1852 (1974).

[5] A.I. Maimistov, Opt. Spektrosk. **57**, 564 (1984).

[6] A.M. Bashrov and A.M.Maimstov, Sov. Phys. JETP **60**, 913 (1984).

[7] A.M. Bashrov and A.M.Maimstov, Sov. Phys. JETP **67**, 2426 (1988).

[8] Q-H. Park and H. J. Shin, Complex sine-Gordon theory for coherent optical pulse propagation, hep-th/9509120.

[9] F. Lund and T. Regge, Phys. Rev. **D14**, 1524 (1976).

[10] K. Pohlmeyer, Commun. Math. Phys. **46**, 207 (1976).

[11] F. Bloch, Phys. Rev. **70**, 460 (1946).

[12] E. Witten, Commun. Math. Phys. **92**, 455 (1984).

[13] D.Karabali, Q-H.Park, H.J.Schnitzer and Z.Yang, Phys. Lett. **B216**, 307 (1989); K.Gawedski and A.Kupiainen,, Nucl. Phys. **B320**, 625 (1989).

[14] I. Bakas, Int. J. Mod. Phys. **A9**, 3443 (1994).

[15] Q-H. Park, Phys. Lett. **B328**, 329 (1994).

[16] Q-H. Park and H. J. Shin, Phys. Lett. **B347**, 73 (1995); Nucl. Phys. **B458**, 327 (1996); T. J. Hollowood, J. L. Miramontes and Q-H. Park, Nucl. Phys. **B445**, 451 (1995).

[17] I. Bakas, Q-Han Park and H. J. Shin, Phys. Lett. **B372**, 45 (1996).

[18] A.M. Basharov, Sov. Phys. JETP **70**, 94 (1990).

[19] Q-H. Park and H. J. Shin, Phys. Lett. **B359**, 125 (1995).

[20] G.L. Lamb, Jr., Rev. Mod. Phys. **43**, 99 (1971).

[21] S. Helgason, Differential geometry, Lie groups and symmetric spaces. 2nd ed. New York; Academic Press 1978.

[22] Q-H. Park and H. J. Shin, to appear.

[23] J.C. Diels, Phys. Lett. A **31**, 111 (1970).

[24] Q-H. Park, Phys. Lett. **B223**, 175 (1989).

[25] A.P. Fordy and P.P. Kulish, Commum. Math. Phys. **89**, 427 (1983); P. Oh and Q-H. Park, More on generalized Heisenberg ferromagnet models, hep-th/9604031.

[26] V.I. Rupasov, Pis'ma Zh. Eksp. Teor. Fiz. **36**, 115 (1982); Sov. Phys. JETP **56**, 989 (1982).