Fuzzy timed Petri nets — analysis and implementation

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Abstract

In [Z. Ding, H. Bunke, M. Schneider, A. Kandel, Fuzzy timed Petri net — definitions, properties and applications, Math. Comput. Modelling (2005) (in press)], we posed two fuzzy timed Petri Net models. Based on the mark changing rate, they can be classified either as a discrete Fuzzy Timed Petri Net model (discrete-FTPN), or as a continuous Fuzzy Timed Petri Net model (continuous-FTPN). In this paper, we present an algorithm developed to compute reachable states for discrete-FTPN models. We also present properties of the continuous-FTPN model, which are used to describe the system’s behavior. From the investigation presented in this paper, we conclude that it is easier to implement a discrete-FTPN model, but for a theoretical study the continuous-FTPN model is better.

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1. Introduction

The earliest timed Petri net models that had been developed are Ramchandani’s \cite{2} and Merlin’s time Petri net models \cite{3}. The first one is derived from Petri nets by associating a firing duration with each transition of the net. It has been used mainly for performance evaluation. The second one associated a crisp time interval with each transition in order to model an uncertain occurrence of the transition. It has been proved very convenient for expressing temporal constraints. Some of these constraints were difficult to express only in terms of firing durations. Hence, the second one is more general than the first one. Zuberek \cite{4} associated a fixed time interval with each transition to model the performance of computer systems. Sifakis \cite{5} associated a fixed time interval with each place so that tokens were considered as unavailable for that time. An interesting presentation of the subject is provided by Pedrycz and Camargo \cite{6}.

By assigning a random variable representing the firing delay of each transition, all enabled transitions are considered to be in the process of firing (incurring delay). The resulting stochastic Petri nets \cite{7}, with exponentially or geometrically distributed delays, are isomorphic to homogeneous Markov chains.

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Incorporating fuzzy time in a Petri net, Valette et al. [8] extended the imprecise enabling duration from Merlin's model to a fuzzy duration defined by a possibility distribution. Also based on possibility distribution, Murata et al. [9], introduced fuzzy time stamp, fuzzy enabling time, fuzzy occurrence time and fuzzy delay. The resulting Fuzzy-timing High-level Petri Net model could be used to simulate communication protocols. By associating with each transition a fuzzy interval, Kunzle et al. [10], tried to develop behavior predictions or explanations, avoiding combinatorial complexity and unmanageable increase of imprecision. In [11] the authors propose another Time Fuzzy Petri Net by attaching a fuzzy enabling duration, delimited by two fuzzy dates, to the transition. The proposed model allows for flexible control of a time schedule and describes causality relations between the events.

In [1], two FTPN models were proposed. There is a fuzzy number associated with each transition during firing. During transition firings, tokens are removed from input places and added to output places. The mark-changing rate and they are included here to make the paper self-contained.

In this paper, we present an algorithm developed for computing reachable states for the constant-FTPN model. We also study the properties of the continuous-FTPN model for describing system behavior. From the investigation we conclude that discrete-FTPN models are easier to implement, while continuous-FTPN models are better for theoretical study.

This paper is organized in four main sections. In Section 2 we present the definitions for fuzzy timed Petri nets. In Section 3 we give some fuzzy differential and integral results. In Section 4, we show how to compute the reachability states by the new developed algorithm for the discrete-FTPN model and study the system behavior based on the eigenvalues of the fuzzy linear system for the continuous-FTPN model.

2. Definitions for fuzzy timed Petri nets

Let \( R^n \) be the finite space with dimension \( n \), and \( N \) be the set of all natural numbers including 0. Let \( \Delta = [0, K] \subset R^1 \), \( K \) can be any arbitrarily large number. We define \( E^n \) as:

\[
E^n = \{ u : R^n \to [0, 1] | u \text{ satisfies (i)–(iv) below} \}
\]

where

(i) \( u \) is normal, i.e. there exists an \( x_0 \in R^n \) such that \( u(x_0) = 1 \),
(ii) \( u \) is fuzzy convex,
(iii) \( u \) is upper semi-continuous,
(iv) \( [u]^0 = \{ x \in R^n : u(x) > 0 \} \) is compact.

If \( n = 1 \), then \( E^1 \) is the set of fuzzy numbers defined in \( R^1 \).

Based on (i), (iii) and (iv) and the extension principle, we have the following result:

For \( \forall u, v \in E^1, 0 \leq \alpha \leq 1 \), the \( \alpha \)-set operations are:

1. \([u + v]^\alpha = [u]^\alpha + [v]^\alpha = \{ a + b : a \in [u]^\alpha, b \in [v]^\alpha \},\]
2. \([u - v]^\alpha = [u]^\alpha - [v]^\alpha = \{ a - b : a \in [u]^\alpha, b \in [v]^\alpha \},\]
3. \([uv]^\alpha = [u]^\alpha [v]^\alpha = \{ ab : a \in [u]^\alpha, b \in [v]^\alpha \},\]
4. \([u/v]^\alpha = [u]^\alpha /[v]^\alpha = \{ ab : a \in [u]^\alpha, b \in [v]^\alpha \}.

Remark 2.1. If either \( u \) or \( v \) is a crisp value, the above definitions also hold.

Since \( u \) is convex, for each \( u \in E^1 \), the \( \alpha \)-set has the form \([u^\alpha](l), u^\alpha(r)\], \([u^\alpha](l), u^\alpha(r)\] \( \in R^1 \).

Definition 2.1. For \( u, v \in E^1 \) with \( \alpha \)-set \([u^\alpha](l), u^\alpha(r)\], \([v^\alpha](l), v^\alpha(r)\] \), we say that \( u \geq v \) if and only if \( u^\alpha(l) \geq v^\alpha(l), \quad u^\alpha(r) \geq v^\alpha(r) \).

Definition 2.2. A fuzzy timed Petri net is a tuple FTPN = \( (P, T, A_{pre}, A_{post}, w, d) \), where
1. \(P = p_1, p_2, \ldots, p_n\) is a finite nonempty set of places,  
2. \(T = t_1, t_2, \ldots, t_m\) is a finite nonempty set of transitions,  
3. \(A_{\text{pre}} = \{p \to t\}\) is a set of directed arcs which connect places with transitions, \(A_{\text{post}} = \{t \to p\}\) a set of directed arcs which connect transitions to places.  
4. \(w : A_{\text{pre}} \cup A_{\text{post}} \to E^1\) is a mapping to assign weight to each arc,  
5. \(d : T \to E^1\) is a mapping to assign firing time to each transition.

For each transition \(t\), let \(I(t)\) and \(O(t)\) be the input and output places, respectively. For each place \(p\), let \(I(p)\) and \(O(p)\) be the input and output transitions, respectively.

**Definition 2.3.** Let \(m_i : \Delta \to E^1, i = 1, 2, \ldots, n\) be a set of mappings. A fuzzy marking of a Fuzzy Timed Petri Net FTPN = \(\langle P, T, A_{\text{pre}}, A_{\text{post}}, w, d\rangle\) is a mapping
\[
m : \Delta \to (m_1, m_2, \ldots, m_n), m(\tau) = (m_1(\tau), m_2(\tau), \ldots, m_n(\tau))
\]
where each \(m_i\) is associated with location \(p_i, i = 1, 2, \ldots, n\).

**Definition 2.4.** A marked Fuzzy Timed Petri Net is a 2-tuple \((N, M_0)\) where  
- \(N\) is a fuzzy timed Petri net,  
- \(M_0 = (m_1(0), m_2(0), \ldots, m_n(0))\) is its initial marking.

**Remark 2.2.** At the very beginning, the marking is precise. In this paper, we assume that the mark in each place could be a fuzzy number in order to study mark distribution. Also, token and mark are different. One mark can be split into many tokens.

**Definition 2.5.** A transition \(t_j\) of FTPN is said to be fuzzy enabled for marking \(m(\tau)\) at time \(\tau\) if and only if \(\forall p_{ij} \in I(t_j), m_{ij}(\tau) \geq w(p_{ij} \to t_j), j = 1, 2, \ldots, k_i\).

Only enabled transitions can be fired. We assume that as soon as a transition gets enabled, it starts to fire. Firing is based on two indivisible primitives: 1. The removal of the tokens from the input places, and their insertion to the output places. Two transitions which do not share any places can be fired independently. 2. Determine the new mark distribution function in both input places and output places while firing. In Section 4, we present a method for computing the new marks.

While a transition is firing, it will meet two situations: a. The input place has no more tokens, and the firing stops. b. A new transition gets enabled, while the old one continues firing. We call the old one a continuing firing transition. In either case, the net will reach a new marking, or a new state. In this state, three cases are possible: 1. Marking, 2. New enabled transition, and 3. Continuing firing transitions. Definitions 2.6 through 2.8 explain this in a formal fashion.

**Definition 2.6.** Let us consider an FTPN. A mark \(M'\) is directly reachable from mark \(M\), if \(\exists \tau, \tau'\) such that \(m(\tau) = M, m(\tau') = M', \tau' - \tau > 0\) and at time \(\tau'\) either a new transition gets enabled or a continuing firing transition stops.

**Definition 2.7.** Let \(t_{\text{new}}\) and \(t_{\text{cont}}\) represent a new enabled and a continuing firing transition respectively. We say that a marking \(M'\) is generally reachable from marking \(M\), if there exists a sequence of new enabled transitions and continuing transitions \(s_1 = \{t_{\text{new}}, t_{\text{cont}}\}, i = 1, 2, \ldots, k + 1\) and a sequence of points in time \(\tau_1, \tau_2, \ldots, \tau_k\) such that \(M = m(\tau_1) \xrightarrow{s_1} m(\tau_2), m(\tau_2) \xrightarrow{s_2} m(\tau_3), \ldots, m(\tau_k) \xrightarrow{s_k} m(\tau_{k+1}) = M'\).

**Definition 2.8.** The reachability set of a fuzzy marked FTPN \(N\) is the set \(R(N, M_0)\) such that \((\forall M \in R(N, M_0)) \leftrightarrow (\exists s = s_1s_2 \ldots s_k, \tau, M_0 \xrightarrow{s} m(\tau) = M)\).

**Definition 2.9.** The reachability graph of a marked FTPN \((N, M_0)\) is a graph \(G(V, C)\) where \(V\) is a set of vertices which is equal to the set of reachable states \(R(N, M_0)\). \(C\) is a set of directed arcs connecting two vertices, such that if \(c_i \in C\) connecting \(M_i\) and \(M_{i+1}\), then \(M_{i+1}\) is directly reachable from \(M_i\). Moreover, this arc contains the time from \(M_i\) to \(M_{i+1}\).
3. Fuzzy differentiation, integral and equations

Let $A$ and $B$ be two nonempty subsets of $R^a$. The distance between $A$ and $B$ is defined by the Hausdorff metric

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a, b\|, \sup_{b \in B} \inf_{a \in A} \|a, b\| \right\}$$

where $\|\|$ denotes the norm in $R^a$. Let

$$P_f(c)(R^a) = \{ A \subset R^a : A \text{ is nonempty, closed (convex)} \}.$$ 

If we endow $P_f(R^a)$ with the distance $d_H$, then $(P_f(R^a), d_H)$ is a complete and separable metric space [3].

Let $u \in E^1$. We assume that $u$ has a left and a right shape, with shape functions $L$ and $R$, respectively [12] and $u$ has a single support point. We use $\tilde{R}^1(L, R)$ to represent all such fuzzy numbers. Denote by $\tilde{R}^a(L, R) = \tilde{R}^1(L_1, R_1) \times \tilde{R}^1(L_2, R_2) \times \cdots \times \tilde{R}^1(L_n, R_n)$. Obviously, $\tilde{R}^a(L, R) \subset E^n$. From the definition, we know that for each $v \in \tilde{R}^a(L, R)$, there is a point $w \in R^a$ such that $\tilde{w} = v$. It is easy to conclude that $w$ is the product of support points of each fuzzy number. Thus for $0 < \alpha \leq 1$, $[v]^\alpha = \{ x \in R^a : v(x) \geq \alpha \} \in P_f(c)(R^a)$. Define $D : \tilde{R}^a(L, R) \times \tilde{R}^a(L, R) \rightarrow R_+ \cup \{0\}$ by

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} d_H([u]^\alpha, [v]^\alpha).$$

**Proposition 3.1.** $(\tilde{R}^a(L, R), D)$ is a metric space.

**Proof.**
(i) $D(u, v) = 0$ implies $d_H([u]^\alpha, [v]^\alpha) = 0$ for each $\alpha > 0$, and this implies $[u]^\alpha = [v]^\alpha, \alpha > 0$, which implies $u = v$. (Recall that $[u]^\alpha$ and $[v]^\alpha$ are compact.)
(ii) Obviously $D(u, v) = D(v, u)$.
(iii) The triangle inequality $D(u, v) \leq D(u, w) + D(w, v)$ follows from the corresponding inequality for $d_H$. \qed

Furthermore, we can prove $(\tilde{R}^a(L, R), D)$ is complete [12,13].

Addition and scalar multiplication in $\tilde{R}^a(L, R)$ are defined as follows:
- addition $\oplus$: $\tilde{u} \oplus \tilde{v} = \tilde{u} + \tilde{v}, \forall \tilde{u}, \tilde{v} \in \tilde{R}^a(L, R)$
- scalar multiplication: $\otimes: \lambda \otimes \tilde{u} = \lambda \tilde{u}, \lambda \in R^a, \tilde{u} \in \tilde{R}^a(L, R)$

With this definition, $\tilde{R}^a(L, R)$ is a vector space. In fact, for $\forall \tilde{u}, \tilde{v} \in \tilde{R}^a(L, R), \lambda \in R^a$, we have

$$\lambda \otimes (\tilde{u} \oplus \tilde{v}) = \lambda \otimes (\tilde{u} + \tilde{v}) = \lambda \tilde{u} + \lambda \tilde{v} = \lambda \tilde{u} \oplus \lambda \tilde{v} = \lambda \otimes \tilde{u} \oplus \lambda \otimes \tilde{v}.$$ 

Based on this definition, we can define subtraction as

$$\tilde{u} \ominus \tilde{v} = \tilde{u} \oplus ((-1) \otimes \tilde{v}).$$

As the direct result of this definition, we can find $\tilde{x} \in \tilde{R}^a(L, R)$ and $y \in R^a$ such that the following are true:

$$\tilde{u} \ominus \tilde{x} = \tilde{v}, \quad y \otimes \tilde{u} \geq \tilde{v}.$$ 

Actually, for the first one, we observe that

$$\tilde{u} \ominus \tilde{x} = \tilde{v} \Rightarrow \tilde{u} + \tilde{x} = \tilde{v} \Rightarrow u + x = v \Rightarrow x = v - u \Rightarrow \tilde{x} = \tilde{v} - \tilde{u}$$

and for the second

$$y \otimes \tilde{u} \geq \tilde{v} \Rightarrow y \tilde{u} \geq \tilde{v} \Rightarrow y \times u \geq v \Rightarrow y \geq \frac{v}{\tilde{u}}.$$ 

Furthermore, we can define multiplication of two fuzzy vectors and the fuzzy unit vector as

$$\tilde{u} \otimes \tilde{v} = \tilde{uv}, \quad \tilde{u}^0 = \tilde{1}, \quad \forall \tilde{u}, \tilde{v} \geq \tilde{0}.$$ 

In analogy to operations on real numbers, we can conclude that

$$\tilde{x}^n = \tilde{x}x, \quad x \in R^1.$$
Proposition 3.2. If \( \tilde{u}, \tilde{v}, \tilde{w} \in \tilde{R}^n(L, R) \) and \( \tilde{u} \oplus \tilde{w} = \tilde{v} \oplus \tilde{w} \), it follows that \( u = v \).

Proof. Since \( \tilde{u} \oplus \tilde{w} = \tilde{u} + \tilde{w} \) and \( \tilde{v} \oplus \tilde{w} = \tilde{v} + \tilde{w} \), we conclude that \( u + w = v + w \). Notice that all fuzzy numbers in \( \tilde{R}^1(L, R) \) have the same shape. Therefore, we get \( u + w = v + w \), which implies \( u = v \) and \( \tilde{u} = \tilde{v} \). \( \square \)

Proposition 3.3. If \( \tilde{u}, \tilde{v}, \tilde{w} \in \tilde{R}^n(L, R) \), then \( D(\tilde{u} \oplus \tilde{w}, \tilde{v} \oplus \tilde{w}) = D(\tilde{u}, \tilde{v}) \).

Proof. From the definition of \( D \), we know that

\[
d_H([\tilde{u} \oplus \tilde{w}]^a, [\tilde{v} \oplus \tilde{w}]^a) = d_H([\tilde{u} + \tilde{w}]^a, [\tilde{v} + \tilde{w}]^a) = d_H([u]^a, [v]^a).
\]

The next theorem gives an embedded result.

Theorem 3.1. There exists a normed space \( X \) such that \( \tilde{R}^n(L, R) \) can be embedded isometrically into \( X \).

Proof. This follows from Propositions 3.2 and 3.3 and from Theorem 1 in [14]. Based on [16,17], and [18], the fuzzy integral will be constructed. \( \square \)

Definition 3.1. Let \( F : \Delta \rightarrow \tilde{R}^n(L, R) \) be a fuzzy mapping, then there is a function \( f : \Delta \rightarrow R^n \) such that \( \tilde{f}(t) = F(t) \). The fuzzy integral of \( F \) over \( \Delta \) denoted by \( \int_{\Delta} F(t) \, dt \), is defined by

\[
\int_{\Delta} F(t) \, dt = \int_{\Delta} \tilde{f}(t) \, dt.
\]

This integral is well defined since \( \tilde{R}^n(L, R) \) is a complete vector space. Let \( \int F \) be a shorthand notation for \( \int_{\Delta} F(t) \, dt \). Then we have the following properties for the fuzzy integral:

Properties. Let \( F, G : \Delta \rightarrow \tilde{R}^n(L, R) \) be two fuzzy mappings which have fuzzy integrals and \( \lambda \in R^n \). Then

1. \( \int F \oplus G = \int F \oplus \int G \),
2. \( \int \lambda \otimes F = \lambda \otimes \int F \),
3. Let \( a, b, c \in \Delta \), then \( \int_a^b F = \int_a^c F \oplus \int_c^b F \).
4. Let \( F_n : \Delta \rightarrow \tilde{R}^n(L, R), n = 1, 2, \ldots \), be fuzzy mappings which have fuzzy integrals and \( F_n \overset{D}{\rightarrow} F \), then \( \int F_n \overset{D}{\rightarrow} \int F \).

Definition 3.2. A fuzzy mapping \( x : \Delta \rightarrow \tilde{R}^n(L, R) \) is differentiable at \( t_0 \in \Delta \), if there exists an \( x'(t_0) \in \tilde{R}^n(L, R) \) such that limits

\[
\lim_{h \to 0^+} \frac{1}{h} \otimes (x(t_0 + h) \ominus x(t_0))
\]

and

\[
\lim_{h \to 0^+} \frac{1}{h} \otimes (x(t_0) \ominus x(t_0 - h))
\]

exist and they are equal to \( x'(t_0) \).

Let \( F : \Delta \times \tilde{R}^n(L, R) \rightarrow \tilde{R}^n(L, R) \). Consider the fuzzy differential equation

\[
x'(\tau, x), \quad x(\tau_0) = x_0 \in \tilde{R}^n(L, R).
\]

(1)

Definition 3.3. A mapping \( x : \Delta \rightarrow \tilde{R}^n(L, R) \) is a weak fuzzy solution to Eq. (1), if it is continuous and satisfies the integral equation

\[
x(\tau) = x_0 \oplus \int_{\tau_0}^{\tau} F(s, x(s)) \, ds, \quad \forall \tau \in \Delta.
\]

It is called a strong solution if it satisfies (1).
Remark 3.1. If \( F(\ldots) \) is continuous, then the weak solution is also the strong solution. In fact,

\[
\begin{align*}
\frac{1}{h} \otimes (x(\tau + h) \odot x(\tau)) &= \frac{1}{h} \left( x(0) \otimes \int_{\tau}^{\tau + h} F(s, x(s)) \, ds \right) \\
&= \frac{1}{h} \left( \int_{\tau}^{\tau + h} F(s, x(s)) \, ds \right)
\end{align*}
\]

From Property 3, we have

\[
\lim_{h \to 0} \frac{1}{h} \otimes (x(\tau + h) \odot x(\tau)) = \left[ \lim_{h \to 0} \frac{1}{h} \int_{\tau}^{\tau + h} f(s, x(s)) \, ds \right] = f(x(\tau)) = F(\tau, x(\tau)).
\]

We need the following proposition.

Proposition 3.4. If \( \tilde{u} \) is a fuzzy set in \( \tilde{R}^n(L, R) \), then \( \int_{\tau_0}^{\tau} \tilde{u} \, ds = (\tau - \tau_0) \otimes \tilde{u} \) [1].

Hence, for the differential equation

\[
x'(\tau) = u, \quad x(\tau_0) = v
\]

where \( u, v \in \tilde{R}^n(L, R) \), the (strong) solution is

\[
x(\tau) = v \oplus \int_{\tau_0}^{\tau} u \, ds = v \oplus ((\tau - \tau_0) \otimes u).
\]

Proposition 3.5. Let \( f : \Delta \to \tilde{R}^n \) and \( \tilde{f} : \Delta \to \tilde{R}^n(L, R) \) be differentiable, then \( (\tilde{f}') = (\tilde{f}). \)

Definition 3.4. A fuzzy matrix is a matrix where every element is a fuzzy number from \( \tilde{R}^1(L, R) \).

Let

\[
A = \begin{pmatrix}
 a_{11} & a_{12} & \ldots & a_{1n} \\
 a_{21} & a_{22} & \ldots & a_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{n1} & a_{n2} & \ldots & a_{nn}
\end{pmatrix}
\]

be a regular matrix and

\[
\tilde{A} = \begin{pmatrix}
 \tilde{a}_{11} & \tilde{a}_{12} & \ldots & \tilde{a}_{1n} \\
 \tilde{a}_{21} & \tilde{a}_{22} & \ldots & \tilde{a}_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 \tilde{a}_{n1} & \tilde{a}_{n2} & \ldots & \tilde{a}_{nn}
\end{pmatrix}
\]

be a fuzzy matrix. Define the determinant of \( \tilde{A} \) as \( |\tilde{A}|_f = |\tilde{A}| \). We define the multiplication of two fuzzy matrices as

\[
\begin{pmatrix}
 \tilde{a}_{11} & \tilde{a}_{12} & \ldots & \tilde{a}_{1n} \\
 \tilde{a}_{21} & \tilde{a}_{22} & \ldots & \tilde{a}_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 \tilde{a}_{n1} & \tilde{a}_{n2} & \ldots & \tilde{a}_{nn}
\end{pmatrix} \cdot \begin{pmatrix}
 \tilde{b}_{11} & \tilde{b}_{12} & \ldots & \tilde{b}_{1n} \\
 \tilde{b}_{21} & \tilde{b}_{22} & \ldots & \tilde{b}_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 \tilde{b}_{n1} & \tilde{b}_{n2} & \ldots & \tilde{b}_{nn}
\end{pmatrix} = \begin{pmatrix}
 \tilde{c}_{11} & \tilde{c}_{12} & \ldots & \tilde{c}_{1n} \\
 \tilde{c}_{21} & \tilde{c}_{22} & \ldots & \tilde{c}_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 \tilde{c}_{n1} & \tilde{c}_{n2} & \ldots & \tilde{c}_{nn}
\end{pmatrix}
\]

where \( c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \). Based on this definition, we derive the next proposition.

Proposition 3.6. Let \( \tilde{A} \) be a matrix, then \( \tilde{A}^i = \tilde{A}^i \).
A special fuzzy matrix is the fuzzy unit matrix

\[ \tilde{I} = \begin{pmatrix} \tilde{1} & 0 & \cdots & 0 \\ 0 & \tilde{1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{1} \end{pmatrix}. \]

For a fuzzy matrix \( \tilde{A} \), define \( e^{\tilde{A}} \) as

\[ e^{\tilde{A}} = \tilde{I} + \frac{1}{1!} \tilde{A} + \frac{1}{2!} \tilde{A}^2 + \cdots + \frac{1}{n!} \tilde{A}^n + \cdots. \]

**Proposition 3.7.** Let \( A \) be a matrix, then \( e^{\tilde{A}} = \tilde{e}^A \).

From Proposition 3.5 and the definition of \( e^{\tilde{A}} \), we have the following

**Proposition 3.8.** \((e^{A\tau})' = \tilde{A}e^{\tilde{A}} \tau = e^{\tilde{A}} \tau \tilde{A}\).

**Theorem 3.2.** Let \( \tilde{A} \) be a fuzzy matrix on \( \tilde{R}^n(L, R) \). Then the solution of the initial value problem

\[ x' = \tilde{A}x, \quad x(\tau_0) = K \in \tilde{R}^n(L, R), \]

is \( e^{\tilde{A} \tau} K \), and there are no other solutions.

We can explicitly solve the system presented by Eq. (2) by introducing the following definition.

**Definition 3.5.** A number \( \lambda \in R^n \) is the eigenvalue of fuzzy matrix \( \tilde{A} \) if

\[ |\tilde{A} - \lambda \tilde{I}|_f = 0. \]

A nonzero vector \( x \in \tilde{R}^n(L, R) \) is called an eigenvector if \( \tilde{A} \otimes x = \lambda \otimes x \) for some \( \lambda \in R^n \).

Since \( |\tilde{A} - \lambda \tilde{I}| = 0 \iff |\tilde{A} - \lambda \tilde{I}| = 0 \) and \((\tilde{A} - \lambda \tilde{I})\tilde{v} = 0 \iff (\tilde{A} - \lambda \tilde{I})\tilde{v} = 0\), we can use the regular methods to find eigenvalues and eigenvectors.

4. Model analysis and implementation

We use \(+, -, \) and \( \times \) to represent fuzzy number addition \( \oplus \), subtraction \( \ominus \), and multiplication \( \otimes \). All the fuzzy numbers are in the space \( \tilde{R}^1(L, R) \).

In [15] David and Alla, studied continuous timed Petri nets. In their work, a timed continuous Petri net, with firing speed associated with transitions, can be obtained from a timed discrete Petri net by means of an approximation. Two kinds of approximation are proposed, the first with constant and the second with variable firing speed. The functional behavior of the model proposed in [15] is decomposed into different phases. Each one is characterized by a set of enabled transitions and a firing speed vector. The evolution of the marking is obtained by computing the balance marking of each place. The changing of firing speed vector occurs when the marking of a place becomes zero. The time evolution is then characterized by a finite number of phases. Motivated by this technique, we studied the time and marking evolution of the fuzzy timed Petri net. The idea behind our approach is that when tokens are removed from input places, then the mark changing rates in the input places are fuzzy constants or variables. The same is true when new tokens are added to output places, i.e. also in this case the mark changing rates can be fuzzy constants or variables.
4.1. Variable speed approach

In this section we investigate a continuous-FTPN model. By analyzing the eigenvalues of the fuzzy linear system we can get the system behavior, without solving any equations. Without losing generality, we use the diagram shown in Fig. 1 as an example, for explaining how to get the desired properties.

In Fig. 1, there are two places \( p_1 \) and \( p_2 \) and two transitions \( t_1 \) and \( t_2 \). Let \( m_1(\tau) \) and \( m_2(\tau) \) be the marks in the places \( p_1 \) and \( p_2 \) at time \( \tau \). The initial marking is \( m(0) = (m_1(0), 0) \). A fuzzy number is associated with each transition. Those numbers are denoted as \( d_1 \) and \( d_2 \) and they represent firing time. For each arc, there is a weight associated with it, denoted as \( w(p_1 \rightarrow t_1), w(p_2 \rightarrow t_2), w(t_2 \rightarrow p_1) \). At the very beginning, if \( m_1(0) \geq w(p_1 \rightarrow t_1) \), then \( t_1 \) is enabled and starts firing. Tokens will be removed from \( p_1 \). The mark change rate is \( m_1'(\tau) \), the derivative of \( m_1(\tau) \), which is equal to a variable \( v_1 = -\frac{1}{d_1} m_1(\tau) \). Hence, we get

\[
\int_0^\tau \frac{m_1'(s)}{m_1(s)} \, ds = \int_0^\tau -\frac{1}{d_1} \, ds.
\]

Let \( m_1(s) = \tilde{g}(s) \). Then from Proposition 3.4, it follows that

\[
\int_0^\tau \frac{g'(s)}{g(s)} \, ds = -\frac{1}{d_1} \tau.
\]

From Proposition 3.7, we get

\[
\tilde{g}(\tau) = m_1(0)e^{-\frac{1}{d_1} \tau} = m_1(0)e^{-\frac{1}{d_1} \tau}.
\]

In the meantime, some new tokens are added to \( p_2 \). The mark change rate in \( p_2 \) is \( m_2'(\tau) \), the derivative of \( m_2(\tau) \), which is equal to

\[
v_2 = \frac{w(t_1 \rightarrow p_2)}{w(p_1 \rightarrow t_1)} v_1 = \frac{w(t_1 \rightarrow p_2)}{w(p_1 \rightarrow t_1)} \frac{1}{d_1} m_1(\tau).
\]

Thus

\[
m_2'(\tau) = v_2 = \frac{w(t_1 \rightarrow p_2)}{w(p_1 \rightarrow t_1)} \frac{1}{d_1} m_1(\tau).
\]

This gives

\[
m_2(\tau) = m_2(0) + \int_0^\tau \frac{w(t_1 \rightarrow p_2)}{w(p_1 \rightarrow t_1)} \frac{1}{d_1} m_1(\tau) \, ds.
\]

Since \( m_1 \) cannot be 0, we have two possible cases here:

1. \( m_2(\tau) \) gets accumulated such that \( m_2(\tau) \geq w(p_2 \rightarrow t_2) \) while \( m_1(\tau) \) is still larger than \( \tilde{0} \). Solving this inequality, we get a solution, say \( \tau_1 \), at this moment, and \( t_2 \) gets enabled and starts firing. Now, in this state,
   - the marking is \( (m_1(\tau_1), m_2(\tau_1), 0) \),
   - the new firing transition is \( t_2 \),
   - the continuing firing transition is \( t_1 \).
2. $m_2(\tau)$ never reaches $w(p_2 \rightarrow t_2)$. The state does not change.

Now assume case 1 occurs. When $t_2$ fires, tokens will be removed from $p_2$ in the rate $v_2 = \frac{1}{d_2}m_2$ while some tokens will be added to $p_1$ at the rate $\frac{w(t_2 \rightarrow p_1)}{w(p_2 \rightarrow t_2)}v_2$. Thus the mark change rates for $p_1$ and $p_2$ are

$$m'_1(\tau) = \frac{w(t_2 \rightarrow p_1)}{w(p_2 \rightarrow t_2)} \frac{1}{d_2} m_2(\tau) - \frac{1}{d_1} m_1(\tau),$$

$$m'_2(\tau) = \frac{w(t_1 \rightarrow p_2)}{w(p_1 \rightarrow t_1)} \frac{1}{d_1} m_1(\tau) - \frac{1}{d_2} m_2(\tau).$$

This can be rewritten as

$$m'(\tau) = Am(\tau),$$

where

$$m(\tau) = \begin{pmatrix} m_1(\tau) \\ m_2(\tau) \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{d_1} & \frac{w(t_2 \rightarrow p_1)}{w(p_2 \rightarrow t_2)} \frac{1}{d_2} \\ \frac{w(t_1 \rightarrow p_2)}{w(p_1 \rightarrow t_1)} \frac{1}{d_1} & -\frac{1}{d_2} \end{pmatrix}.$$

Without finding explicit solutions, we can also obtain important qualitative information about the solutions from the eigenvalues of $A$. We consider the most important cases.

**Case 1.** All eigenvalues have negative real parts. This important case is called a *sink*, see Fig. 2. It has the characteristic property that

$$m(\tau) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty$$

for every solution. In this case all marks will vanish to zero and the system will become idle.

**Case 2.** All eigenvalues have a positive real part. In this case, it is called a *source* (Fig. 3). We have

$$m(\tau) \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty.$$

In this case, the system will get too much overflow and finally will terminate because of limited resources.

**Case 3.** The eigenvalues are purely imaginary. This is called a *center* (Fig. 4). It is characterized by the property that all solutions are periodic with the same period. In this case, the system is working in a stable state.

**Case 4.** The eigenvalues are either negatives or zero. This is called an *equilibrium* (Fig. 5).

$$m(\tau) \rightarrow c \quad \text{as} \quad t \rightarrow \infty.$$

In this case, all places will have a constant marking. The system is working in an asymptotically stable state.
For example, let us assume
\[ d_1 = 0.1, d_2 = 0.2 \]
\[ w(p_1 \rightarrow t_1) = w(p_2 \rightarrow t_2) = 1, \]
\[ w(t_1 \rightarrow p_2) = w(t_2 \rightarrow p_1) = 1. \]

The initial marking is \( m(0) = (2, 0) \). Obviously, at the very beginning, only \( t_1 \) is enabled.
1. When \( \tau = 0 \), the mark change rate for \( p_1 \) is

\[
m'_1(\tau) = -\frac{1}{0.1}m_1(\tau).
\]

Solving this equation, we get

\[
m_1(\tau) = 2e^{-10\tau}.
\]

For \( p_2 \), we have the rate change as

\[
m'_2(\tau) = \frac{w(t_1 \rightarrow p_2)}{w(p_1 \rightarrow t_1)} \frac{1}{0.1}m_1(\tau).
\]

Solving this equation, we get

\[
m_2(\tau) = 2(1 - e^{-10\tau}).
\]

In order to make \( t_2 \) enabled, we need to have \( 2(1 - e^{-10\tau}) \geq \tilde{1} \) or \( \tau \geq 0.069 \). At \( \tau = 0.069 \), \( m_1(0.069) = \tilde{1} \), \( m_2(0.069) = \tilde{1} \). This is the first reachable state. In this state we get:

- new firing transition: \( t_2 \)
- continuing firing transition: \( t_1 \)
- \( m_1(0.069) = \tilde{3} \), \( m_2(0.069) = \tilde{1} \), \( m_3(0.069) = 0 \).

2. Right now at time \( \tau = 0.069 \), all the transitions are firing. Thus the mark changing rates for \( p_1 \) and \( p_2 \) are

\[
m'_1(\tau) = \frac{1}{0.2}m_2(\tau) - \frac{1}{0.1}m_1(\tau),
\]

\[
m'_2(\tau) = \frac{1}{0.1}m_1(\tau) - \frac{1}{0.2}m_2(\tau),
\]

or

\[
m'(\tau) = Am(\tau),
\]

where

\[
A = \begin{pmatrix}
-10 & 5 \\
10 & -5
\end{pmatrix}.
\]

The eigenvalues for \( A \) are 0 and \(-15\). Thus \( m_1(\tau) = c_{11} + c_{12}e^{-15\tau} \) and \( m_2(\tau) = c_{21} + c_{22}e^{-15\tau} \). As \( t \rightarrow \infty \), we will get \( m_1(\tau) \rightarrow c_{11} \) and \( m_2(\tau) \rightarrow c_{21} \). We get an asymptotically stable state.

4.2. Constant speed approach

In this sub-section we introduce an algorithm to compute new markings in order to build a reachability graph. This algorithm is based on the following consideration: Let \( d_1 \) be the fuzzy number associated with transition \( t_1 \). If \( t_1 \) is firing for \( p_1 \in I(t_1) \), then the mark change rate for \( p_1 \) is

\[
m'_1(\tau) = -\frac{1}{d_1},
\]

\[
m_1(\tau) = m_1(0) - \int_0^\tau \frac{1}{d_1} ds = m_1(0) - \frac{1}{d_1}\tau.
\]

Let \( p_2 \in O(t_1), t_2 \in O(p_2) \). If \( t_1 \) and \( t_2 \) are firing at the same time, then the mark change rate for \( p_2 \) is

\[
m'_2(\tau) = \frac{w(t_1 \rightarrow p_2)}{w(p_1 \rightarrow t_1)} \frac{1}{d_1} - \frac{1}{d_2},
\]

\[
m_2(\tau) = m_2(\tau_0) + \int_{\tau_0}^\tau \left( \frac{w(t_1 \rightarrow p_2)}{w(p_1 \rightarrow t_1)} \frac{1}{d_1} - \frac{1}{d_2} \right) ds = m_2(\tau_0) + \left( \frac{w(t_1 \rightarrow p_2)}{w(p_1 \rightarrow t_1)} \frac{1}{d_1} - \frac{1}{d_2} \right) (\tau - \tau_0).
\]
Now consider a continuous-FTPN = \( \langle P, T, A_{\text{pre}}, A_{\text{post}}, w, d \rangle \). We assume that there are \( m \) places \( p_i, i = 1, 2, \ldots, m \) and \( n \) transitions \( t_j, j = 1, 2, \ldots, n \). Then we can compute new reachable states for this model as follows:

1. **State 0.** Initialize all places.
   - **Marking:** \( m_1(0) = m_1, m_2(0) = m_2, \ldots, m_n(0) = m_n \).
   - **new firing transitions:** \( \{ t^0_{\text{new}} \} \)

2. **State \( k \geq 0 \).** In this state
   - **marking:** \( m_1(\tau^k), m_2(\tau^k), \ldots, m_n(\tau^k) \),
   - **new firing transitions:** \( \{ t^k_{\text{new}} \} \),
   - **continuing firing transitions:** \( \{ t^k_{\text{cont}} \} \).

Let \( d_{ij}^k \) be the fuzzy number associated with the transition \( t_j^k \). For each place \( P_i \), the tokens to be moved out can be computed as

\[
m_{i1}(\tau) = m_{i1}(\tau^k) - \sum_{j} \frac{1}{d_{ij}^k} \tau, \quad t_j^k \in O(P_i) \cap (\{ t^k_{\text{new}} \} \cup \{ t^k_{\text{cont}} \}).
\]

For each place \( P_i \), the tokens to be moved in can be computed as

\[
m_{i2}(\tau) = m_{i2}(\tau^k) + \min_{t_j^k \in I(P_i)} \frac{w(t_j^k \to p_i) \cdot 1}{d_{ij}^k} \tau, \quad p_i \in I(t_j^k), \quad t_j^k \in I(P_i).
\]

**Case 1.** If \( \exists \tau^{k+1} \) such that \( m_{i2}(\tau^{k+1}) \geq w(p_{i2} \to t_0), t_0 \in O(P_{i2}) \) and \( m_{i1}(\tau^{k+1}) \geq \tilde{0} \) then get a new firing transition \( t_0 \), which we put into the set \( \{ t^{k+1}_{\text{new}} \} \). If \( m_{i1}(\tau^{k+1}) \) is large enough, so that \( m_{i1}(\tau^{k+1}) \geq w(p_{i1} \to t_j^k) \), then we put \( t_j^k \) into the set \( \{ t^{k+1}_{\text{cont}} \} \). Now the new state is:
   - **marking:** \( m_1(\tau^{k+1}), m_2(\tau^{k+1}), \ldots, m_n(\tau^{k+1}) \),
   - **new firing transitions:** \( \{ t^{k+1}_{\text{new}} \} \),
   - **continuing firing transitions:** \( \{ t^{k+1}_{\text{cont}} \} \).

**Case 2.** If \( \exists \tau^{k+1} \) such that \( m_{i1}(\tau^{k+1}) = \tilde{0} \) and \( m_{i2}(\tau^{k+1}) \leq w(p_{i2} \to t_0), t_0 \in O(P_{i2}) \). In this case, the state does not change.

3. **Compare state \( k + 1 \) with states 1, 2, \ldots, \( k \).** If state \( k + 1 \) is identical to one of them, then stop. Otherwise, repeat the above process to find new states.

For example, Fig. 6 has three places \( p_1, p_2 \) and \( p_3 \) and three transitions \( t_1, t_2 \) and \( t_3 \). Let \( m_1(\tau), m_2(\tau) \) and \( m_3(\tau) \) be the marks in the places \( p_1, p_2 \) and \( p_3 \) at time \( \tau \). The initial marking is \( m(0) = (m_1(0), 0, 0) \). For each transition, there is a fuzzy number associated with it, denoted as \( d_1, d_2 \) and \( d_3 \), representing firing time. For each arc, there is a weight associated with it, denoted as \( w(p_1 \to t_1), w(p_2 \to t_2), w(p_3 \to t_3), w(t_1 \to p_2), w(t_2 \to p_3) \).

Let us assume

\[
\begin{align*}
d_1 &= 0.1, \quad d_2 = 0.2, \quad d_3 = 0.4, \\
w(p_1 \to t_1) &= w(p_2 \to t_2) = w(p_3 \to t_3) = 1, \\
w(t_1 \to p_2) &= w(t_2 \to p_3) = 1.
\end{align*}
\]
The initial marking is $m(0) = (4, 0, 0)$. Obviously, at the very beginning, only $t_1$ is enabled.

1. When $\tau = 0$, the new marking for $p_1$ is
   $$m_1(\tau) = 4 - \tilde{t}_0\tau.$$  

   The new marking for $p_2$ is
   $$m_2(\tau) = \tilde{t}_0\tau.$$  

In order to enable $t_2$, we need to have $\tilde{t}_0\tau \geq \tilde{t}_1$ or $\tau \geq \frac{1}{10}$. At $\tau = \frac{1}{10}$, $m_1(\frac{1}{10}) = 4 - \tilde{t}_0\frac{1}{10} = \tilde{3}$. This is the first reachable state. In this state,
   - new firing transition: $t_2$,
   - continuing firing transition: $t_1$,
   - $m_1(\frac{1}{10}) = \tilde{3}, m_2(\frac{1}{10}) = 1, m_3(\frac{1}{10}) = 0.$

2. When $\tau = \frac{1}{10}$, $t_2$ starts to fire. In the meantime $t_1$ continues to fire. The new marking for $p_2$ is
   $$m_2(\tau) = 1 + 5\left(\tau - \frac{1}{10}\right).$$

$p_3$ starts to get tokens, the new marking for $p_3$ is
   $$m_3(\tau) = 5\left(\tau - \frac{1}{10}\right).$$

We need $5(\tau - \frac{1}{10}) \geq \tilde{t}_1$ to get $t_3$ enabled. We can choose $\tau \geq \frac{3}{10}$ to satisfy this inequality. Now we reach the second reachable state. In this state,
   - new firing transition: $t_3$,
   - continuing firing state: $t_1, t_2$,
   - $m_1(\frac{3}{10}) = \tilde{1}, m_2(\frac{3}{10}) = \tilde{2}, m_3(\frac{3}{10}) = \tilde{1}.$

3. Right now at time $\tau = \frac{3}{10}$, all the transitions are firing, the new markings for $p_1, p_2$ and $p_3$ are
   $$m_1(\tau) = 3.25 - 7.5\tau, \quad m_2(\tau) = 0.5 + 5\tau, \quad m_3(\tau) = 0.25 + 2.5\tau.$$  

At time $\tau = \frac{13}{30}$, we reach a new reachable state:
   - continuing firing: $t_2, t_3$,
   - $m_1(\frac{13}{30}) = \tilde{0}, m_2(\frac{13}{30}) = \tilde{2}, m_3(\frac{13}{30}) = \tilde{1}.$

4. In this state, $p_1$ will receive tokens from $t_3$, while $p_2$ will remove tokens from $t_2$. $p_3$ will receive tokens from $t_2$ and send tokens to $t_3$ at same time. Using the same calculations as above, at time $\tau = \frac{5}{6}$, we come to a new reachable state
   - new firing transition: $t_1$,
   - continuing firing transitions: $t_2, t_3$,
   - $m_1(\frac{5}{6}) = \tilde{1}, m_2(\frac{5}{6}) = \tilde{3}, m_3(\frac{5}{6}) = \tilde{2.3}.$

5. In this state, all transitions are firing. At time $\tau = \frac{29}{30}$, we come to a new reachable state:
   - continuing firing: $t_2, t_3$,
   - $m_1(\frac{29}{30}) = 0, m_2(\frac{29}{30}) = 11; m_3(\frac{29}{30}) = 2\tilde{2}$.  

Continue with this processing, we will get

6. At time $\tau = \frac{37}{30}$, the system comes to a new state:
   - continuing firing: $t_1, t_3$,
   - $m_1(\frac{37}{30}) = \tilde{2}, m_2(\frac{37}{30}) = 0, m_3(\frac{37}{30}) = \tilde{3}.$
7. At time $\tau = \frac{41}{30}$, the system comes to a new state:
   - new firing transition: $t_1$,
   - continuing firing: $t_3$,
   - $m_1(\frac{41}{30}) = \tilde{1}$, $m_2(\frac{41}{30}) = 0$, $m_3(\frac{41}{30}) = \tilde{3}$.

8. At time $\tau = \frac{44}{30}$, the system comes to a new state:
   - new firing transition: $t_2$,
   - continuing firing transition: $t_1, t_3$,
   - $m_1(\frac{44}{30}) = \frac{1}{4}$, $m_2(\frac{44}{30}) = \tilde{1}$, $m_3(\frac{44}{30}) = \frac{23}{4}$.

9. At time $\tau = \frac{45}{30}$, the system comes to a new state:
   - continuing firing transition: $t_2, t_3$,
   - $m_1(\frac{45}{30}) = 0$, $m_2(\frac{45}{30}) = \frac{11}{6}$, $m_3(\frac{45}{30}) = \frac{5}{6}$.

10. At time $\tau = \frac{52}{30}$, the system comes to a new state:
    - continuing firing transitions: $t_1, t_3$,
    - $m_1(\frac{52}{30}) = \frac{7}{12}$, $m_2(\frac{52}{30}) = 0$, $m_3(\frac{52}{30}) = \frac{35}{12}$.

11. At time $\tau = \frac{57}{30}$, the system comes to a new state:
    - new firing transitions: $t_1$,
    - continuing firing transitions: $t_3$,
    - $m_1(\frac{57}{30}) = \tilde{1}$, $m_2(\frac{57}{30}) = 0$, $m_1(\frac{57}{30}) = \tilde{3}$.

12. At time $\tau = \frac{60}{30}$, the system comes to a new state:
    - new firing transitions: $t_2$,
    - continuing firing transitions: $t_1, t_3$,
    - $m_1(\frac{60}{30}) = \frac{1}{4}$, $m_2(\frac{60}{30}) = \tilde{1}$, $m_3(\frac{60}{30}) = \frac{17}{4}$.

13. At time $\tau = \frac{61}{30}$, the system comes to a new state:
    - continuing firing transitions: $t_2, t_3$,
    - $m_1(\frac{61}{30}) = 0$, $m_2(\frac{61}{30}) = \frac{11}{6}$, $m_3(\frac{61}{30}) = \frac{5}{6}$.

14. At time $\tau = \frac{68}{30}$, the system comes to a new state:
    - continuing firing transitions: $t_1, t_3$,
    - $m_1(\frac{68}{30}) = \frac{7}{12}$, $m_2(\frac{68}{30}) = 0$, $m_3(\frac{68}{30}) = \frac{35}{12}$.

15. At time $\tau = \frac{73}{30}$, the system comes to a new state:
    - new firing transition: $t_1$,
    - continuing firing transitions: $t_3$,
    - $m_1(\frac{73}{30}) = \tilde{1}$, $m_2(\frac{73}{30}) = 0$, $m_3(\frac{73}{30}) = \tilde{3}$.

From the calculations above we conclude that
- Marking $(\tilde{1}, 0, \tilde{3})$ will repeat at time $\frac{41}{30}, \frac{57}{30}, \frac{73}{30}, \ldots$. The time step is $\frac{16}{30}$.
- Marking $(\frac{7}{12}, \tilde{1}, \frac{25}{6})$ will repeat at time $\frac{44}{30}, \frac{60}{30}, \frac{76}{30}, \ldots$. The time step is $\frac{16}{30}$.
- Marking $(0, \frac{11}{6}, \frac{5}{6})$ will repeat at time $\frac{45}{30}, \frac{61}{30}, \frac{77}{30}, \ldots$. The time step is $\frac{16}{30}$.
- Marking $(\frac{7}{12}, 0, \tilde{3})$ will repeat at time $\frac{52}{30}, \frac{68}{30}, \frac{84}{30}, \ldots$. The time step is $\frac{16}{30}$.

Fig. 7 shows the reachability graph for the model used in our example.
5. Summary and conclusions

We developed an algorithm to compute markings in order to determine reachable states for a discrete-FTPN model. This provides us with a way of implementing this model of a fuzzy timed system. Performance can be obtained based on the reachability graph. We have also studied a continuous-FTPN model. By studying the properties of this model, we can get a general understanding of the system behavior.

References