THICKNESS FORMULA AND C^1-COMPACTNESS FOR $C^{1,1}$ RIEMANNIAN SUBMANIFOLDS

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ABSTRACT. The properties of normal injectivity radius $i(K, M)$ (thickness), of $C^{1,1}$ submanifolds K of complete Riemannian manifolds M are studied. We introduce the notion of geometric focal distance for $C^{1,1}$ submanifolds by using metric balls. A formula for $i(K, M)$ in terms of the double critical points and the geometric focal distance is proved. The thickness of knots and ideal knots relate to the study of DNA molecules and other knotted polymers. We prove that the set of all $C^{1,1}$ submanifolds K of a fixed manifold M contained in a compact subset $D \subset M$ and $i(K, M) \geq c > 0$ is C^1-compact and this collection has finitely many diffeomorphism and isotopy types. Estimates on upper bounds for the number of such types are constructible, and we calculate them for submanifolds of $\mathbf{R}^n$. C^1-compactness is related to Gromov's compactness theorem, but it is an extrinsic and isometric embedding type theorem.

1. Introduction

Let M^n denote a complete connected n-dimensional Riemannian manifold. For a compact k-dimensional C^1 submanifold K^k ($\partial K = \emptyset$) of M^n, the normal exponential map, $\exp^N$ on the normal bundle of K in M and its normal injectivity radius $i(K, M)$ are well defined. If K is $C^{1,1}$, then $i(K, M) > 0$. We will introduce the notion of "Geometric Focal Distance" by using metric balls, which naturally extends the notion of the focal distance of smooth category to C^1 category in Riemannian manifolds. We prove a formula for $i(K, M)$ in terms of geometric focal distance and double critical points for $C^{1,1}$ submanifolds, and that the set of all submanifolds K of a fixed manifold M contained in a compact subset $D \subset M$ and $i(K, M)$ bounded away from zero is C^1-compact. These results are essential to the study of the maximization of $i(K, M)$. The motivation for the maximization of $i(K, M)$ comes from two directions- the ideal knots and the history of maximization of the intrinsic injectivity radius.

The thickness of a knotted curve is the radius of the largest tubular neighborhood around the curve without intersections of normal discs, that is $i(K, M)$. The ideal knots are the embeddings of S^1 into $\mathbf{R}^3$, maximizing $i(K, M)$ in a fixed isotopy (knot) class of fixed length. As noted in [Ka], "...the average shape of knotted polymeric chains in thermal equilibrium is closely related to the ideal representation of the corresponding knot type". "Knotted DNA molecules placed in certain solutions follow paths of random closed walks and the ideal trajectories are good predictors of time averaged properties of knotted polymers" as a biologist referee pointed out to the author. The analytical properties ideal knots will be tools in the research

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on the physics of knotted polymers. Theorem I and the methods developed in this article are used extensively in [D6] where we study the local structure ideal knots in $\mathbf{R}^3$.

Studying ideal knots in $\mathbf{R}^3$ corresponds to placing molecules in homogenous solutions with uniform conditions. Studying ideal knots in Riemannian manifolds, i.e. varying metrics, may bring new possibilities with varying conditions, such as inhomogeneous solutions.

For a compact Riemannian manifold M, let $d(M)$, $v(M)$ and $i(M)$ denote its (intrinsic) diameter, volume, and injectivity radius of its exponential map, respectively. Maximization of $i(M)$ for fixed $d(M)$ or $v(M)$ has a long history. $i(M) \leq d(M)$ and equality holds if and only if M is a Blaschke manifold, Warner [Wa], Besse [Be]. It is conjectured that a Blaschke manifold is isometric to a sphere or a projective space with the standard metrics up to rescaling of the metric. Berger proved that $v(M)/i(M)^n \geq v(S^n(1))/i(S^n(1))^n$, and equality holds if and only if M^n is isometric to a standard sphere $S^n(r)$, by using an inequality proved by Kazdan. This resolves the S^n and $\mathbf{RP}^n$ cases of the Blaschke conjecture. See Besse [Be] and Berger [B] for the literature on Blaschke manifolds as well as proofs by Berger and Kazdan.

As well as finding these ideal metrics, we examine the topological restrictions imposed by large injectivity radii. The class defined by $i(M)^n/v(M) \geq c_1 > 0$ is Hausdorff-Gromov precompact, see [Gr, Prop 5.2], [GWY] and [Cr, prop 14]. However, the condition $i(M)/d(M) \geq c_2 > 0$ does not provide precompactness without curvature restrictions. By the author's work [D4], and [Y], one can estimate a priori upper bounds for the number of possible homotopy types for M and Betti numbers [D4], and fundamental group [D5] in terms of c_1. The author also studied manifolds with large injectivity radii: $c_2 \approx 1$ in [D1, D2, D3].

In this article, we approach the normal injectivity radius $i(K, M)$ from a more general point of view. Let $\mathcal{D}^\infty(k, \varepsilon, D; M) = \{(K, M) : K \in C^\infty, \dim K = k, K \subset D, \text{and } i(K, M) \geq \varepsilon\}$ where D is a compact subset of M. The behavior of focal points and $i(K, M)$ are better understood in the smooth category, but $\mathcal{D}^\infty$ is not complete under C^1 topology. Since $i(K, M) \geq \varepsilon$ restricts curvature in a sense, the completion must include $C^{1,1}$ submanifolds. By Proposition 2, $i(K, M)$ is an upper semi-continuous function of K in C^1 topology: $\limsup_m i(K_m, M) \leq i(K_\infty, M)$. The extremal cases are more likely not to occur in $\mathcal{D}^\infty$. Very few ideal knots in $\mathbf{R}^3$ are expected to be C^2, and possibly the unknotted standard circles are the only ones. This requires the study of $i(K, M)$ in $C^{1,1}$ category.

The formal definitions will be given in section 2. $F_g(K)$ is the geometric focal distance defined in terms of local intersections with metric balls, $MDC(K)$ is the length of the shortest geodesic normal to K at both of its endpoints on K, and the "rolling bead/ball radius, $R_O(K, M)$" is the largest radius of open metric balls which are tangent to K without intersecting K elsewhere. We prove the following expected formula for $i(K, M)$.

Theorem 1. *For every complete connected smooth Riemannian manifold M and every compact $C^{1,1}$ submanifold K ($\partial K = \emptyset$) of M,*

$$i(K, M) = R_O(K, M) = \min\{F_g(K), \frac{1}{2}MDC(K)\}.$$

For a $C^{1,1}$ curve γ, γ'' and the curvature $\kappa\gamma$ of γ exist almost everywhere by Rademacher's Theorem. The supremum of $\kappa\gamma$ is taken on the set of all points where

$\kappa\gamma$ exists. See [D6], Lemma 2 for a proof of $F_k(\gamma) = F_g(K^1)$ in $\mathbf{R}^n$. We prove the following corollary for any dimensions $n > k \geq 1$, for $K^k \subset \mathbf{R}^n$ in Proposition 12.

Corollary 1. *(Thickness Formula for Curves in $\mathbf{R}^n$) For every simple, $C^{1,1}-$closed curve γ in $\mathbf{R}^n$ and $K = image(\gamma)$,*

$$i(K,M) = R_O(K,M) = \min\{F_k(\gamma), \frac{1}{2}MDC(K)\},$$

where $F_k(\gamma) = (\sup \kappa\gamma)^{-1}$.

The formula $i(K,M) = \min\{F_g(K), MDC(K)/2\}$ was proved for C^2-knots in $\mathbf{R}^3$ in [LSDR], and for $C^{1,1}$-knots in $\mathbf{R}^3$ by Litherland in [L]. Nabutovsky [N] extensively studied $C^{1,1}$ hypersurfaces K^{n-1} in $\mathbf{R}^n$ and their injectivity radii. Some of our results overlap with [N] in this special case. [N] proves the upper semicontinuity of $i(K^{n-1}, \mathbf{R}^n)$, the lower semicontinuity of $v(K)/i(K,\mathbf{R}^n)^{n-1}$ in C^1 topology, and the compactness of the class of hypersurfaces with $i(K,\mathbf{R}^n)$ bounded from below. The analogous (codimension 2) results were obtained by Litherland in [L] for $C^{1,1}$ knots in $\mathbf{R}^3$. Their proofs use $\varepsilon-$approximations or curvature, while ours use intersections with metric balls. The relations between curvature and $F_g(K)$ are simple in all spaces of constant curvature. The equality $i(K,M) = R_O(K,M)$, a rolling ball/bead description of the injectivity radius in $\mathbf{R}^n$, was known by Nabutowsky for hypersurfaces, and by Buck and Simon for C^2 curves, [BS]. The notion of the global radius of curvature developed by Gonzales and Maddocks for smooth curves in $\mathbf{R}^3$ defined by using circles passing through 3 points of the curve in [GM] is a different characterization of $i(K,\mathbf{R}^3)$ from R_O due to positioning of the circles and metric balls.

Gromov's Compactness Theorem was first stated in [Gr] and some of its details were clarified by Katsuda in [K]. The $C^{1,\alpha}$ estimates of the metrics of bounded curvature in harmonic coordinates by Jost and Karcher [JK] were used to complete the proof by Peters [P] and Greene and Wu [GW]. [P] and [GW] proved the optimal a priori $C^{1,\alpha}$ regularity of the limit metric and obtained the Lipschitz convergence intrinsically by studying the transition functions. Gromov's proof, the clarifications by Katsuda, and the version by Pugh [Pu] rely on Whitney type, non-isometric embeddings into $\mathbf{R}^N$ (large N) to show Lipschitz closeness of manifolds.

Let $\mathcal{D}(k,\varepsilon,D;M) = \{(K,M) : K \in C^{1,1}, \dim K = k, K \text{ is connected}, K \subset D, \text{ and } i(K,M) \geq \varepsilon\}$, where (K,M) denotes an embedding $e : K \to (M, g_0)$ with the induced submanifold metric e^*g_0 on K.

Theorem 2. *For every complete connected Riemannian manifold (M^n, g_0), every compact subset $D \subset M$, $1 \leq k \leq n-1$, and $\varepsilon > 0$, the following holds.*

i. $\mathcal{D}(k,\varepsilon,D;M)$ has finitely many diffeomorphism and isotopy classes.

ii. $\mathcal{D}(k,\varepsilon,D;M)$ is sequentially compact in C^1-topology, i.e. every sequence $\{(K_m,M)\}_{m=1}^{\infty}$ in $\mathcal{D}(k,\varepsilon,D;M)$ has C^1-convergent subsequence whose limit (K_0,M) is in $\mathcal{D}(k,\varepsilon,D;M)$.

iii. For every $(K,M) \in \mathcal{D}(k,\varepsilon,D;M)$, the induced submanifold metric e^*g_0 is a priori $C^{0,1}$. However, there exists an isometric $C^{1,1}$ embedding of (K, g_∞) onto (K, e^*g_0) in M such that g_∞ is $C^{1,\alpha}$ $(\alpha < 1)$ in harmonic coordinates of K where (K, g_∞) is a limit of C^∞ Riemannian metrics of bounded curvature and injectivity radii with respect to Lipschitz distance and it is a $C^{1,\alpha}$ Alexandrov space with a well defined exponential map.

In *Theorem 1* and *Section 3*, K is not assumed to be connected. However, *Theorem 2* is proved for connected K, and the general case is discussed as *Corollary 3* of *Section 4.2*. Theorem 2 is an extrinsic and isometric embedding type Gromov compactness theorem, but it differs from the versions above in several aspects. Its proof uses the intrinsic versions [P] and [GW], and the harmonic coordinates of [JK] to secure the isometric embedding of the intrinsic limit. We will also show that there exists $\rho_0(k, \varepsilon, D, M) > 0$ such that for all $K, L \in \mathcal{D}(k, \varepsilon, D; M)$ satisfying $L \subset B(K, \rho_0; M)$ there exists a continuous isotopy between K and L through $C^{1,1}$ embeddings of L. Thus, if the submanifolds are close in the Hausdorff topology in M, then they are isotopic and close in C^1-topology. By Theorems 1 and 2, and Proposition 2, every isotopy class must have a thickest-$i(K, M)$ maximizing submanifold.

The method using the embeddings into $\mathbf{R}^N$ could only obtain a priori $C^{0,1}$ regularity of the limit metric, see [Pu], in contrast to $C^{1,\alpha}$ ($\alpha < 1$) regularity obtained by the intrinsic proofs of [P] and [GW]. Any C^1 simple closed curve γ of length 2π in a Riemannian manifold has a C^0 metric induced by the embedding, while γ is intrinsically isometric to standard smooth S^1, and the regularity is lost in the embedding. Part (iii) of Theorem 2 emphasizes the recovery the possible loss of regularity coming from the embedding. We know more about the geometry of $C^{1,\alpha}$ Alexandrov spaces [Ni] than $C^{0,1}$ metrics which do not even admit an exponential map a priori. Note that not all $C^{0,1}$ Riemannian metrics can be $C^{1,1}$ embeddable into some smooth M with positive thickness.

Section 3 contains the proof of Theorem 1. For a $C^{1,1}$ submanifold K, the normal exponential map $\exp^N$ of K is of class $C^{0,1}$, a priori differentiable almost everywhere. Hence, the Inverse Function Theorem can not be used to obtain local diffeomorphisms around regular points, and the property that the focal points being the singular points of $\exp^N$ fails. In general, the set of focal points may not be closed, and F_g is not semi-continuous, see Example 1. We prove a lower-semicontinuity of the normal cut value in a certain case in Proposition 7 which is sufficient for Theorem 1. Our main tool is the distance functions from the submanifolds. Despite the similarities of the main theme to the smooth case, our proof contains many technical details which are not derivable from the classical lemmas of the smooth cases.

The compactness is discussed in Section 4. The technical details differ from the previous section. The thickness controls the directional derivatives $\|f_{\mathrm{u}\,\mathrm{u}}\|$ a priori for a graph of a function f locally representing K. For smooth f, the Hessian is symmetric and one can control $\|f_{uv}\|$ by the polarization identities. For a $C^{1,1}$ function f, f_{uv} are defined a.e., and $f_{uv} = f_{vu}$ a.e. $\|f_{uv}\|$ are not necessarily uniformly bounded in terms of $\|f_{\mathrm{u}\,\mathrm{u}}\|$ at a point, see Example 1. To apply Arzela-Ascoli Theorem to a family of such graphs requires equicontinuity of f_u, for which one may wish to use uniform boundedness of f_{uv} on the family. Hence, using mollifiers is a good way to proceed, and one can obtain that smooth $\mathcal{D}^\infty(k, \varepsilon, D; \mathbf{R}^n)$ is "almost" dense in $\mathcal{D}(k, \varepsilon, D; \mathbf{R}^n)$. For a Riemannian manifold M, $F_g(p, K)$ depends on the behavior of the metric of M in the given normal direction as well as the normal curvature. If *codimension*$(K) \geq 2$, it is possible to have high normal curvatures (if defined) in low ambient curvature directions and low normal curvatures in high ambient curvature directions at a given point achieving high $F_g(p, K)$. For $C^{1,1}$ K in general, the "second fundamental form" $II^K_w(v)$ is not defined everywhere, not a quadratic form (Example 1), and not continuous in v even at a point. Normal curvatures do not satisfy Euler's formula. An averaging procedure in small neighborhoods with a C^1 convergence may not be able to control the change of normal curvatures in different directions, especially if the limit is discontinuous. We were able to show the existence of $\delta(\varepsilon) > 0$ for which $\mathcal{D}(k, \varepsilon, D; M)$ is in the closure of $\mathcal{D}^\infty(k, \delta, D_\varepsilon; M)$.

The remaining parts of the proof of Theorem 2 are straightforward. We show that the collection of the submanifolds in $\mathcal{D}^\infty(k, \delta, D; M)$ as a collection of manifolds satisfy the conditions of Gromov's Compactness Theorem. By following Peter's version [P], every subsequence has a convergent subsequence whose limit is an (intrinsic) $C^{1,\alpha}$ Riemannian manifold. We use the harmonic coordinates of [JK] to apply Arzela-Ascoli Theorem and positive thickness $\delta(\varepsilon)$ to secure the isometric embedding of the (intrinsic) limit into M. All constants introduced are constructible in terms of n, k, ε, D, and M. In the last section, we calculate some estimates for upper bounds of isotopy and diffeomorphism types for submanifolds of $\mathbf{R}^n$ with thickness bounded away from 0.

2. Basic Definitions

In this section M^n always denotes a complete and smooth Riemannian manifold, and K^k denotes a C^1 submanifold of M^n. We refer to [CE], [GKM] and [DoC] for basic Riemannian geometry. TK, UTK, NK and UNK denote the tangent, unit tangent, normal and unit normal bundle to K in M. $\exp^N : NK \to M$ denotes the normal exponential map.

Definition 1. *i. For any metric space X with a distance function d, $B(p,r) = \{x \in X : d(x,p) < r\}$ and $\bar{B}(p,r) = \{x \in X : d(x,p) \leq r\}$. For $A \subset X$ and $x \in X$ define $d(x,A) = \inf\{d(x,a) : a \in A\}$ and $B(A,r) = \{x \in X : d(x,A) < r\}$. The diameter $d(X)$ of X is defined to be $\sup\{d(x,y) : x, y \in X\}$. If there is ambiguity, we will use d_X and $B(p,r;X)$.*

ii. For $A \subset M^n$ and any curve γ in A, the length $\ell(\gamma)$ is defined with respect to the metric space structure of M^n. For any one-to-one curve $\gamma, \ell_{ab}(\gamma)$ and $\ell_{pq}(\gamma)$ both denote the length of γ between $\gamma(a) = p$ and $\gamma(b) = q$.

iii. $v(M)$ denotes the volume of a C^1 Riemannian manifold M.

Definition 2. *Let K be a C^1 submanifold of M.*

i. Define the thickness of K in M or the normal injectivity radius of $\exp^N$ to be

$$i(K, M) = \sup(\{0\} \cup \{r > 0 : \exp^N : \{v \in NK : \|v\| < r\} \to M \text{ is one-to-one}\}).$$

ii. For any $w \in UNK_p$, define the normal cut value in the direction w with respect to K to be $r_w = \sup\{r : d(\exp^N rw, K) = r\}$.

Definition 3. *For a smooth and complete Riemannian manifold M and $p \in M$, define pointwise injectivity radius*

$$i(p, M) = \sup\{r > 0 : \exp : \{v \in TK_p : \|v\| < r\} \to M \text{ is one-to-one}\}.$$

For a compact subset D of M, define $i(D) = \min_{p \in D} i(p, M) > 0$.

Definition 4. *Let K be a C^1 submanifold of M. K and M will be suppressed, if there is no ambiguity. For any $v \in UNK_p$ and any r, define* *i.* $O_p(v,r;M) = \bigcup_{w\in v^\perp(1)} B(\exp_p rw, r)$, *where* $v^\perp(1) = \{w \in UTM_p : \langle v,w\rangle = 0\}$

ii. $O_p(r;K) = \bigcap_{v\in UTK_p} O(v,r;K) = \bigcup_{w\in UNK_p} B(\exp_p rw, r)$

iii. $O(r;K) = \bigcup_{p\in K} O_p(r;K)$

iv. $O_p^c(v,r;K) = \mathbf{M} - O_p(v,r;K)$.

Definition 5. *Let K be a C^1 submanifold of M. Define*

i. The ball radius of K in M to be $R_O(K,M) = \inf\{r > 0 : O(r;K) \cap K \neq \emptyset\}$

ii. The pointwise geometric focal distance $F_g(p) = \inf\{r > 0 : p \in \overline{O_p(r;K) \cap K}\}$ for any $p \in K$ and the geometric focal distance $F_g(K) = \inf_{p\in K} F_g(p)$.

Definition 6. *Let K be a C^1 submanifold of M. A pair of points p and q in K are called a double critical pair for K, if there is a geodesic γ_{pq} of positive length from p to q, normal to K at both p and q, and minimal up to its midpoint from both p and q. Define the minimal double critical distance*

$MDC(K) = \inf\{\ell(\gamma_{pq}) : \{p,q\}$ *is a double critical pair for* $K\}$.

Definition 7. *By furnishing the Grassmanian bundle $G_{n,k}(TM)$ with a fixed Riemannian metric, for C^1-diffeomorphic compact k-dimensional submanifolds K and L of M, one defines $d_{C^1}(K,L)$ to be*

$\inf\{\sup_{x\in K}\left(d_M(x,\psi(x)) + d_G(TK_x, TL_{\psi(x)})\right) : \forall C^1-$*diffeomorphisms* $\psi : K \to L\}$.

3. Thickness Formula

Throughout this section, we assume that K is a compact $C^{1,1}$ submanifold of a complete connected smooth Riemannian manifold M and $\partial K = \emptyset$, unless stated otherwise. K is not assumed to be connected.

Proposition 1. $i(K,M) = R_O(K,M)$.

Proof. i. Choose any $R > i(K,M)$. There exists $p_i \in K$ and $v_i \in NK_{p_i}$ for $i = 1,2$ such that $v_1 \neq v_2$, $q = \exp^N_{p_1} v_1 = \exp^N_{p_2} v_2$, and $\|v_1\| \leq \|v_2\| < R$, by definition of $i(K,M)$. Let $q_2 = \exp^N_{p_2} v_2 R/\|v_2\|$. Since $\gamma_1(t) = \exp^N_{p_1} tv_1$ and $\gamma_2(t) = \exp^N_{p_2} tv_2$ are distinct geodesics, $\pi > \measuredangle(-\gamma_1'(q), -\gamma_2'(q))$. By First Variation, [CE, GKM], $d(q_2,p_1) < d(q_2,q) + d(q,p_1) \leq R - \|v_2\| + \|v_1\| \leq R$. Thus, $p_1 \in B(q_2,R) \subset O_{q_2}(R;K)$, $O(R) \cap K \neq \emptyset$ and this concludes $R \geq R_O(K,M)$. We have shown that $\forall R > i(K,M)$, $R \geq R_O(K,M)$. This proves that $i(K,M) \geq R_O(K,M)$.

ii. Choose any $R > R_O(K,M)$ so that $O(R) \cap K \neq \emptyset$. There exists $p \in K$, $v \in UNK_p$, $q = \exp^N_p vR$ such that $B(q,R) \cap K \neq \emptyset$. Let $p_1 \in B(q,R) \cap K$ be any point and $d(q,p_1) = R - \delta$, for some $\delta > 0$. Let $q_1 = \exp^N_p v(R - \frac{\delta}{3})$, and p_2 be any closest point of K to q_1.

$$R' = d(q_1,p_2) \leq d(q_1,p_1) \leq d(q_1,q) + d(q,p_1) \leq \frac{\delta}{3} + R - \delta = R - \frac{2\delta}{3}$$

For any normal minimal geodesic γ from p_2 to q_1, $w = \gamma'(p_2) \in UNK_{p_2}$. So, $q_1 = \exp^N_p v(R - \frac{\delta}{3}) = \exp^N_{p_2} wR'$, but $v(R - \frac{\delta}{3}) \neq wR'$ since $\|w\| = \|v\| = 1$. Consequently, $\exp^N|\{v \in NK : \|v\| < R\}$ is not injective and $R \geq i(K,M)$. We have shown that $\forall R > R_O(K,M)$, $R \geq i(K,M)$. This proves that $i(K,M) \leq R_O(K,M)$. ∎

Proposition 2. *Let $K, K_j, j \in \mathbf{N}$, be compact C^1 submanifolds of a complete Riemannian manifold M, such that $K_j \to K$ in C^1 sense. Then $\limsup_{j\to\infty} R_O(K_j, M) \leq R_O(K, M)$.*

Proof. Choose arbitrary $R > R_O(K, M)$, that is $O(R; K) \cap K \neq \emptyset$.

$$\exists p \in K\ \exists v \in UNK_p\ \exists \varepsilon > 0\ \exists q \in B(\exp^N_p Rv, R-\varepsilon) \cap K \neq \emptyset$$

$$\forall j\ \exists p_j \in K_j\ \exists v_j \in UN(K_j)_{p_j} \text{ such that } (p_j, v_j) \to (p, v),\ \text{as } j \to \infty$$

$$\exists j_0 \forall j \geq j_0,\ d(\exp^N_{p_j} Rv_j, \exp^N_p Rv) < \frac{\varepsilon}{2} \text{ and } \exists q_j \in K_j \text{ with } d(q, q_j) < \frac{\varepsilon}{2}$$

$$\forall j \geq j_0,\ d(\exp^N_{p_j} Rv_j, q_j) \leq d(\exp^N_{p_j} Rv_j, \exp^N_p Rv) + d(\exp^N_p Rv, q) + d(q, q_j) < R$$

$$\forall j \geq j_0,\ O(R; K_j) \cap K_j \neq \emptyset, \text{that is: } R_O(K_j, M) \leq R$$

$$R \geq \limsup_{j\to\infty} R_O(K_j, M)$$

We have shown that if $R > R_O(K, M)$ then $R \geq \limsup_{j\to\infty} R_O(K_j, M)$. Hence, $R_O(K, M) \geq \limsup_{j\to\infty} R_O(K_j, M)$. □

Proposition 3. *Let $K_j, j \in \mathbf{N}$, be a sequence of C^1 k-dimensional submanifolds of a complete Riemannian manifold M^n, such that $K_j \to K$ in C^1 sense, where K is compact. If $\liminf_j MDC(K_j) > 0$ then $\liminf_j MDC(K_j) \geq MDC(K)$.*

Proof. We will use the same indices for subsequences. Let $a = \liminf_j MDC(K_j)$, and choose a subsequence with $a = \lim_j MDC(K_j)$ and $\forall j, MDC(K_j) > 0$. By compactness of K_j and positivity of $MDC(K_j)$, there exists a minimal double critical pair $\{p_j, q_j\}$ for K_j, $\ell(\gamma_{p_j q_j}) = MDC(K_j)$. Since K is compact and $a > 0$, there exist subsequences $p_j \to p_0 \in K$, $q_j \to q_0 \in K$, and $\gamma_{p_j q_j} \to \gamma_{p_0 q_0}$ in C^1 sense. Geodesics converge to geodesics, and normality to submanifolds is preserved under C^1 limits. $\{p_j, q_j\}$ is a double critical pair for K.

$$MDC(K) \leq \ell(\gamma_{p_0 q_0}) = \lim_j \ell(\gamma_{p_j q_j}) = \lim_j MDC(K_j) = a$$

□

Proposition 4. $R_O(K, M) \leq \min\{F_g(K), \frac{1}{2} MDC(K)\}$.

Proof. This is am immediate consequence of definitions of $F_g(K)$ and $MDC(K)$. □

Proposition 5. *Let $v \in UNK_p$ be such that $0 \leq r_v < F_g(K)$. Then there are finitely many and at least two minimal geodesics between $q = \exp^N_p r_v v$ and K. Hence, $r_v > 0$.*

Proof. Any geodesic that is a shortest curve between a point of $M - K$ and K is normal to K. We assume that all geodesics are unit speed and start at K when $s = 0$. Let $\gamma_0(s) = \exp^N_p sv$. $\forall j \in \mathbf{N}^+$, there exists a minimal geodesic γ_j between $q_j = \exp^N_p (r_v + \frac{1}{j})v$ and K. Since γ_j is not minimal between p and q_j, $\gamma_0 \neq \gamma_j, \forall j$. By compactness, and taking a subsequence and using the same subindices, we can assume that $\gamma_j \to \gamma_\infty$, a minimal geodesic between q and K.

Suppose that $\gamma_0 = \gamma_\infty$ or $r_v = 0$. Choose j_0 sufficiently large such that $a = r_v + \frac{1}{j_0} < F_g(K)$. $\forall j > j_0$, $d(q_{j_0}, \gamma_j(0)) < d(q_{j_0}, q_j) + d(q_j, \gamma_j(0)) \le a$, since $\gamma_j'(q_j) \neq \gamma_0'(q_j)$ and First Variation. $\gamma_j(0) \in B_a(q_{j_0}) \cap K$. $\gamma_j(0) \neq p, \forall j$, but $\gamma_j(0) \to p$. Hence, $p \in \overline{K \cap O_p(a)}$ which contradicts $a < F_g(K)$. This shows that $r_v > 0$ and $\gamma_0 \neq \gamma_\infty$, that is there are at least two geodesics between q and K.

Suppose that there are infinitely many minimal geodesics θ_j between K and q. By compactness, there exists a convergent subsequence of distinct geodesics $\theta_j \to \theta_0$ which is also minimal between K and q. Then one uses a proof similar to above, with $\theta_j(0) \to \theta_0(0) = p'$, to show $p' \in \overline{K \cap O_{p'}(a')}$ where a' is chosen similar to above $a' = r_{\theta_0'(0)} + \frac{1}{j_0} < F_g(K)$. Hence, there are finitely such geodesics. ☐

Lemma 1. *Let $p \in K$ be such that $F_g(p) > 0$. $\forall v \in UNK_p, \forall r < F_g(p)$, $q = \exp_p^N rv$, there exists an open disc D of K, such that $p \in D$ and $\forall x \in D - \{p\}$, $d(x,q) > r$.*

Proof. Choose a such that $r < a < F_g(p)$. Let $\gamma(s) = \exp_p^N sv$, and $q = \gamma(a)$. $p \notin \overline{K \cap O_p(a)}$, since $a < F_g(p)$. Hence, there exists an open disc D of K such that $p \in D$ and $D \cap O_p(a) = \emptyset$. $B(q,r) \subset B(q',a)$ and $\forall x \in D$, $d(x,q) \ge r$ and $d(x,q') \ge a$. Let $x \in D$ be such that $d(x,q) = r$. If γ_x is any normal minimal geodesic from x to q, distinct from γ, that is $\gamma'(q) \neq \gamma_x'(q)$, then by First Variation, $d(x,q') < d(x,q) + d(q,q') \le a$. This contradicts $D \cap O_p(a) = \emptyset$. Finally, γ and γ_x must follow the same minimal geodesic and $x = p$. ☐

Proposition 6. *Let $v \in UNK_p$ be such that $r_v < F_g(K)$ and there are two distinct minimal geodesics γ_1 and γ_2 between $q = \exp_p^N r_v v$ and K. Then either $\measuredangle(\gamma_1'(q), \gamma_2'(q)) = \pi$ or $R_O(K,M) < r_v$.*

Proof. Assume that $\measuredangle(\gamma_1'(q), \gamma_2'(q)) < \pi$, to show that $R_O(K,M) < r_v$. Let $\gamma_j(0) = p_j \in K$, for $j = 1,2$. $\gamma_j(r_v) = q$, $d(q,p_j) = d(q,p) = r_v$, for $j = 1,2$. Choose $\varepsilon > 0$ small enough that

1. $D_j = B(p_j, \varepsilon) \cap K$ is a small open disc in K and $\overline{D_j}$ is compact, for $j = 1,2$,
2. $\overline{D_1} \cap \overline{D_2} = \emptyset$, and
3. $\forall x \in \overline{D_1} \cup \overline{D_2} - \{p_1, p_2\}$, $d(x,q) > r_v$ by the previous lemma.

Let $\delta = \min\{d(x,q) - r_v : x \in \partial D_1 \cup \partial D_2\} > 0$. Choose $w \in UTM_q$ such that $\measuredangle(w, \gamma_j'(q)) > \frac{\pi}{2}$, for $j = 1,2$. By the First Variation, $d(p_j, \exp_q tw)$ decreases strictly, for small $t > 0$ and for $j = 1,2$. If q is on *cutlocus*(p_j), then one can use Toponogov's Theorem, see [CE], [GKM]. There exists $t_0 \in (0, \frac{\delta}{3})$ and $q_0 = \exp_q t_0 w$ such that $r_v - \frac{\delta}{3} < d(p_j, q_0) < d(p_j, q) = r_v$, for $j = 1,2$. Let m_j be the closest point of $\overline{D_j}$ to q_0, for $j = 1,2$.

Suppose that $m_j \in \partial D_j$, for $j = 1$ or 2. Then, we obtain a contradiction as follows:

$$d(m_j, q_0) \ge d(m_j, q) - d(q, q_0) \ge r_v + \delta - \frac{\delta}{3} = r_v + \frac{2\delta}{3}$$

$$d(m_j, q_0) \le d(p_j, q_0) \le d(p_j, q) + d(q, q_0) \le r_v + \frac{\delta}{3}$$

Hence, m_j are interior points of D_j, for $j = 1$ and 2. The minimal geodesics from m_j to q_0 are normal to K at m_j. $m_1 \neq m_2$, since $\overline{D_1} \cap \overline{D_2} = \emptyset$. Finally, $\exp_K^N$ fails to be injective on the closed disc bundle of radius $\max(d(p_1,q_0), d(p_2,q_0)) < r_v$. $R_O(K,M) = i(K,M) < r_v$. ☐

Proposition 7. *If $r_v < F_g(K)$, then $\liminf_{w\to v} r_w \geq r_v$. That is, r_v is lower semi-continuous in v on UNK when $r_v < F_g(K)$.*

Proof. Suppose not, and choose $v_j \to v$ such that $\lim_{v_j\to v} r_{v_j} = L < r_v < F_g(K)$, where $v \in UNK_p$ and $v_j \in UNK_{p_j}, \forall j \in \mathbf{N}$, and $p_j \to p$. We will obtain a contradiction in both cases below.

Case 1. $L > 0$. By Proposition 6, $\forall j \in \mathbf{N}$, there exists $p'_j \in K$, $u_j \in UNK_{p'_j}$, $q_j \in M$ such that $u_j \neq v_j$, $r_{u_j} = r_{v_j}$ and $\exp^N_{p_j} v_j r_{v_j} = \exp^N_{p'_j} u_j r_{u_j} := q_j$. By taking subsequences and using same indices, we may assume that $p'_j \to p'$, $u_j \to u$ and $q_j \to q = \exp^N_p Lv = \exp^N_{p'} Lu$. The case of $u \neq v$ cannot occur, since $L < r_v$. Hence, we need to study the case of $p = p'$ and $v = u$. Let $c_v = \sup\{t : d(p, \exp_p tv) = t\}$ be the cut value of the exponential map $\exp : TM \to M$ in the direction of v from p. Obviously, $c_v \geq r_v > L = d(p,q)$, from the definition of the normal cut value.

See [CE, p.93, 95] or [DoC, p267-276], for the C^∞ Riemannian manifolds M, to conclude that

i. q is not conjugate to p along the unique minimal geodesic $\exp_p tv$, and $q \notin$ *cutlocus*(p) and hence,

ii. p is not conjugate to q along the unique minimal geodesic $\exp_p(L-t)v = \exp_q tw$, $p \notin$ *cutlocus*(q) and $c_w > L$. By [DoC, p276 or CE p.94], the cut value function $c_{(.)} : UM \to [0,\infty]$ is continuous and the tangential cutlocus is a closed subset of UM.

Hence, there exists $\varepsilon > 0$ satisfying:

1. $0 < \varepsilon < \frac{1}{2}\min(F_g(K) - L, L)$, and
2. p is the unique closest point of K to q in $K \cap B(p, 2\varepsilon)$, by Lemma 1, and
3. $\forall x \in B(p,\varepsilon), \forall y \in B(q,\varepsilon)$, $x \notin$ *cutlocus*(y) and unique minimal geodesics γ_{xy} vary continuously on $B(p,\varepsilon) \times B(q,\varepsilon)$.

Let $K' = K \cap B(p,\varepsilon)$, and consider $\partial K'$ with respect to the topology of K. Define $\delta := \frac{1}{4}\min\{d_M(x,q) - L : x \in \partial K'\}$. Observe that $0 < \delta \leq \frac{\varepsilon}{4}$, since p is the unique closest point of $\overline{K'}$ to q and triangle inequality. Choose and fix sufficiently large j_0 with $q_{j_0} \in B(q,\delta; M)$, $p_{j_0}, p'_{j_0} \in B(p, \frac{\delta}{2}; K)$ and $r_{u_{j_0}} = r_{v_{j_0}} < F_g(K)$. There exists a curve γ of length $\leq \delta$ in K between p_{j_0} and p'_{j_0}. Define $f(x) = d_M(x, q_{j_0}) : K' \to \mathbf{R}$ and set $m = \min f = d_M(p_{j_0}, q_{j_0}) = d_M(p'_{j_0}, q_{j_0})$. By triangle inequality, we have:

$$0 < L - 2\delta \leq m \leq f(x) \leq L + \delta + \varepsilon < F_g(K), \forall x \in K'$$

$$m \leq f(\gamma(t)) \leq m + \frac{\delta}{2} \leq L + \frac{5\delta}{2} < F_g(K) \text{ and}$$

$$\min_{x\in\partial K'} f(x) \geq \min_{x\in\partial K'} d(x,q) - \delta = L + 3\delta$$

$f \in C^1$, since $K' \cap (\{q_{j_0}\} \cup cutlocus(q_{j_0})) = \emptyset$ and K is $C^{1,1}$. A point $x \in K'$ is a critical point of f if and only if the minimal geodesic from q_{j_0} to x is normal to K. All of the critical points of f are isolated strict local minima by $f(x) < F_g(K)$ and Lemma 1. For an isolated local strict minimum point x_0, $x_0 \notin \overline{f^{-1}((0, f(x_0))}$. As b increases, $f^{-1}((0,b))$ will gain new components at each critical point x_0. Away from critical points $f^{-1}(b)$ is a codimension 1 submanifold with a normal ∇f pointing away from $f^{-1}((0,b))$. Hence, as b increases, the number of components of $f^{-1}((0,b))$ will not decrease at regular points. By Milnor [M, p.12], for $m < b < L + 3\delta$, $f^{-1}((0,b))$ is a disjoint union of open sets where each component is away from $\partial K'$ and contains exactly one local minimum. However,

$\gamma \subset f^{-1}((0, L+\frac{11\delta}{4})) \subset int(K')$ and the end points of γ, p_{j_0} and p'_{j_0} are the absolute minima of f. This gives a contradiction. Consequently, the case of $p = p'$ and $v = u$ can't occur either.

Case 2. $L = 0$. Let $\eta > 0$ be the infimum of the pointwise injectivity radius of $\exp_p : TM_p \to M$ where p ranges over $d(K)$ neighborhood of K in M. Let $p_j \to p$, $p'_j \to p'$ and $q_j \to q$ be chosen as in $L > 0$ case. $p = p' = q$ since $L = 0$. Choose j sufficiently large so that $max(d(p_j, q_j), d(p'_j, q_j)) < \frac{\eta}{4}$. Suppose that there exists a curve γ in K between p_j and p'_j of length $\leq \frac{\eta}{2}$. Apply the method of in Case 1 to the critical points of C^1-function $f(x) = d(x, q_j) : K \cap B(q_j, \eta) \to (0, \eta)$, which are strict local minima to obtain a contradiction. Hence, all curves in K between p_j and p'_j have length $> \frac{\eta}{2}$, and hence $d_K(p_j, p'_j) \geq \frac{\eta}{2}$ for sufficiently large j. That is not possible since $p_j \to p$ and $p'_j \to p' = p$.

Finally, we obtained contradictions in both cases. We can conclude that there exists no subsequence $v_j \to v$ such that $\lim_{v_j \to v} r_{v_j} = L < r_v$. □

3.1. **Proof of Theorem 1.**

Proof. By Propositions 1 and 4: $i(K, M) = R_O(K, M) \leq \min\{F_g(K), \frac{1}{2}MDC(K)\}$.

Claim. $\inf_{v \in UNK} r_v = i(K, M)$.

Let $\inf_{v \in UNK} r_v = r$. Choose any $\rho > r$ and let $x = \frac{\rho + r}{2}$. There exists $v_0 \in UNK_{p_0}$ such that $d(\exp^N_{p_0} xv_0, K) < x$, hence $\exp^N_{p_0} tv_0$ and any minimal geodesic $(\neq \exp^N_{p_0} tv_0)$ between $\exp^N_{p_0} xv_0$ and K will have a common point, $\exp^N_{p_0} xv_0$. Hence, injectivity of $\exp^N$ fails in $\rho-$neighborhood and $\rho > i(K, M)$. Thus, $r \geq i(K, M)$.

Choose any $\rho > i(K, M)$ and let $x = \frac{\rho + i(K,M)}{2}$. $\exp^N$ fails to be injective in the $x-$neighborhood. $\exp^N_{p_1} t_1 v_1 = \exp^N_{p_2} t_2 v_2$ for $t_j < x$ and $v_1 \neq v_2$. Then $\exp^N_{p_2} tv_1$ is not minimal to K for $t > t_1$, by First Variation. Hence, $r_{v_1} \leq t_1 < x < \rho$. Thus $r < \rho$, to conclude $r \leq i(K, M)$. This proves the claim.

If $i(K, M) = F_g(K)$, then there is nothing to prove. Hence, assume that $\inf_{v \in UNK} r_v = i(K, M) < F_g(K)$. Choose $v_j \in UNK_{p_j}, \forall j \in \mathbf{N}$, and $\lim_j r_{v_j} = \inf_{v \in UNK} r_v$. By compactness of K, there exists $v_\infty \in UNK_{p_\infty}$ and a subsequence which we will denote with the same indices such that $v_j \to v_\infty$ and $p_j \to p_\infty$. By Proposition 7, $\lim_j r_{v_j} \geq r_{v_\infty} \geq \inf_{v \in UNK} r_v$. Hence $R_O(K, M) = i(K, M) = \inf_{v \in UNK} r_v = r_{v_\infty} < F_g(K)$. By Propositions 5 and 6, there are two distinct minimal geodesics γ_1 and γ_2 between $q = \exp^N_{p_\infty} r_{v_\infty} v_\infty$ and K with $\measuredangle(\gamma'_1(q), \gamma'_2(q)) = \pi$. This means that $\frac{1}{2}MDC(K) = i(K, M)$. □

Example 1. *Let $h(\theta) : \mathbf{R} \to \mathbf{R}$ be smooth with $h(\theta + \pi) = h(\theta), \forall\theta$. Consider the graph K of $z = f(x, y) = \frac{1}{2}r^2 h(\theta)$ in $\mathbf{R}^3$ where (r, θ) is the polar coordinates. $f \in C^{1,1}$. Obviously, $f_{\mathrm{u\,u}}(0, 0) = h(\theta)$, if $u = (\cos\theta, \sin\theta)$, and $f_{yx}(0, 0) = \frac{1}{2}h'(0)$. Away from $(0, 0)$, f is smooth and*

$$f_{xx}(x, y) = h(\theta) - \frac{xy}{x^2 + y^2} h'(\theta) + \frac{y^2}{2(x^2 + y^2)} h''(\theta).$$

a) Choose h such that h is identically 0 on an open subset of $\mathbf{R}$ near $\theta = 0$, $\|h\|_\infty = 1$, and $h''(\frac{\pi}{2}) \approx n$, large n. $F_g(\mathbf{0}, K) = 1$, and every neighborhood of $\mathbf{0}$ contains planar points $\mathbf{a}$ with $F_g(\mathbf{a}, K) = \infty$, as well as points $\mathbf{b}$ with $F_g(\mathbf{b}, K) \approx \frac{2}{n}$.

Hence, F_g is not a semicontinuous function into $[0,\infty]$. Same is true for the normal cutvalue if F_g is the controlling factor.

b) Choose h such that $\|h\|_\infty = 1$, and $h'(0) \approx n$, large n, to observe that $\forall u, \|f_{\mathrm{u\,u}}(0,0)\| \leq 1$, and $F_g(\mathbf{0},K) = 1$ does not control $\|f_{yx}(0,0)\|$.

4. Compactness

Throughout this section we will assume the following. M denotes a smooth connected complete n-dimensional Riemannian manifold, $D \subset M$ denotes a compact subset, and K denotes a k-dimensional compact **connected** (except in Section 4.2) $C^{1,1}$ manifold. (K,M) denotes that K is a Riemannian submanifold with a particular embedding and furnished with the induced submanifold metric. (K,g) denotes a manifold with a metric g without any indication of any embedding. We refer to DoCarmo [DoC], for basic submanifold theory for Riemannian manifolds.

Definition 8.
$\mathcal{A}(k,\varepsilon,D;M) = \{(K,M) : K \in C^{1,1}, \dim K = k,\ K \subset D,\ \textit{and}\ i(K,M) > \varepsilon\}$
$\mathcal{A}^\infty(k,\varepsilon,D;M) = \{(K,M) : K \in C^{\infty}, \dim K = k,\ K \subset D,\ \textit{and}\ i(K,M) > \varepsilon\}$
$\mathcal{D}(k,\varepsilon,D;M) = \{(K,M) : K \in C^{1,1}, \dim K = k,\ K \subset D,\ \textit{and}\ i(K,M) \geq \varepsilon\}$
$\mathcal{D}^\infty(k,\varepsilon,D;M) = \{(K,M) : K \in C^{\infty}, \dim K = k,\ K \subset D,\ \textit{and}\ i(K,M) \geq \varepsilon\}$

Definition 9. *For $K \in \mathcal{A}^\infty(k,\varepsilon,D;M)$,*
i. $Sect(K)$ denotes the sectional curvatures of K and
ii. II^K_w denotes the second fundamental form of K with respect to a normal vector w in M.
Define $\|II^K\|(p) = \max\{\|II^K_w(v)\| : \forall w \in UNK_p$ and $v \in UTK_p\}$ and $\|II^K\| = \sup_{p\in K}\|II^K\|(p)$.

Proposition 8. *Let $\mathcal{A}^\infty(k,\varepsilon,D;M)$ be given and g_0 be the Riemannian metric of M. There exists positive constants C_0, C_1, d_0, v_0, i_0 depending on n, k, ε, D and M such that $\forall K \in \mathcal{A}^\infty(k,\varepsilon,D;M)$, $e : K \hookrightarrow M$, the Riemannian manifold (K, e^*g_0) satisfies the following intrinsically:*
i. $\|II^K\| \leq C_0$ and $|Sect(K)| \leq C_1$,
ii. $v(K) \geq v_0$,
iii. $d(K) \leq d_0$, and
iv. consequently, $i(K) \geq i_0$ by Cheeger [Ch], [CE].

Proof. Let $D = \overline{B}(D,\varepsilon)$ and $\varepsilon_0 = \min(\varepsilon, \frac{1}{2}i(D'))$.

i. $\forall q \in D'$, $\partial B(q,\varepsilon_0)$ are smooth submanifolds of M, which are diffeomorphic to S^{n-1}. By compactness, there exists $C_0 > 0$ such that $\|II^{\partial B(q,\varepsilon_0)}\| \leq C_0$, for all $q \in D'$.

Let $K \in \mathcal{A}^\infty(k,\varepsilon,D;M)$ be arbitrarily chosen. Choose any $p \in K$, $v \in UK_p$, and $w \in UNK_p$. Let $q = \exp^M_p \varepsilon_0 w$. Since $\varepsilon_0 \leq \varepsilon < R_O(K,M)$, $B(q,\varepsilon_0) \cap K = \emptyset$. Let S denote $\partial B(q,\varepsilon_0)$ below in this proof. S is smooth and $v \in UTS_p$. Define $\alpha_1(s) = \exp^K_p sv$ and $\alpha_2(s) = \exp^S_p sv$, $f_j(s) = d_M(\alpha_j(s), q)$, for $j = 1,2$, and

$W = -grad\ d_M(.,q)$. f_1 has a local minimum at $s = 0$, hence $f_1''(0) \geq 0 = f_2''(0)$.

$$f_j''(0) = -\left(\left\langle \nabla^M_{\alpha_j'} \alpha_j', w\right\rangle + \left\langle \nabla^M_v W, v\right\rangle\right)$$
$$\left\langle \nabla^M_{\alpha_1'} \alpha_1', w\right\rangle \leq \left\langle \nabla^M_{\alpha_2'} \alpha_2', w\right\rangle$$
$$II^K_w(v) \leq II^S_w(v)$$
$$-II^K_w(v) = II^K_{-w}(v) \leq II^{S'}_{-w}(v)$$

where $S' = \partial B\left(\exp_p^M(-\varepsilon_0 w), \varepsilon_0\right)$. Hence, $\left\|II^K\right\| \leq C_0$. By Gauss's Theorem [DoC, p130] relating the second fundamental form and the sectional curvatures of K and M, and the polarization identities, there exists $C_1(C_0, |Sect(M)|)$ such that $|Sect(K)| \leq C_1$.

ii. There exists $v_1 > 0$ such that $\forall p \in D', vol_n(B(q, \varepsilon; M)) \geq v_1$ by compactness of D'. Furthermore, v_1 can be chosen only depending on the dimension n, ε and $i(D')$ but not on M, by using the estimates of the lower bounds for the volumes of the balls of radius less than $i(D')/2$ by Croke [Cr, Prop.14]. Let $K \in \mathcal{A}^\infty(k, \varepsilon, D; M)$ and $p \in K$ be arbitrarily chosen. By *Theorem 2.1* and *Remark 2*, page 453 of Heintze & Karcher [HK]:

$$0 < v_1 \leq vol_n(B(p, \varepsilon; M)) \leq vol_n(B(K, \varepsilon; M)) \leq v(K) \cdot C(k, n, C_0, C_1, \varepsilon)$$

where $v(K)$ is the k-dimensional volume of the K with the induced submanifold metric.

iii. Choose $d_2 = \min(\varepsilon_0, d_1)$ with d_1 of Lemma 2 below. There exists $v_2 > 0$ such that $\forall p \in D', vol_n(B(q, \frac{1}{4}d_2; M)) \geq v_2$ as in part (ii). Let $K \in \mathcal{A}^\infty(k, \varepsilon, D; M)$ be arbitrarily chosen. Since all geodesics γ of K satisfy $\left\|\nabla^M_{\gamma'}\gamma'\right\| \leq C_0$ by part (i), one can conclude that

$$\forall p \in K,\ B(p, \frac{1}{2}d_2; M) \cap K \subset B(p, d_2; K)$$

by using the Lemma 2(i) and $d_2 \leq i(K, M)$. Let p and q be a pair of intrinsically furthest apart points in K, that $d_K(p,q) = d(K, e^*g_0)$. Choose a normal minimal geodesic γ of K form p to q, $\gamma(0) = p, \gamma(d(K)) = q$, and $\|\gamma'\| = 1$. Suppose that $B(\gamma(ad_2), \frac{1}{4}d_2; M) \cap B(\gamma(bd_2), \frac{1}{4}d_2; M) \neq \emptyset$, for some integers $a, b \in \mathbf{N} \cap [0, \frac{d(K)}{d_2}]$. Then $d_M(\gamma(ad_2), \gamma(bd_2)) < \frac{1}{2}d_2$ which implies $d_K(\gamma(ad_2), \gamma(bd_2)) < d_2$. Thus, $a = b$. Hence, the balls $B(\gamma(ad_2), \frac{1}{4}d_2; M)$, for $a \in \mathbf{N} \cap [0, \frac{d(K)}{d_2}]$, are disjoint in D'.

$$v_2 \cdot \frac{d(K)}{d_2} \leq vol_n(D').$$

iv. This follows Cheeger [Ch] and parts (i-iii). □

Lemma 2. *Let D' be a compact subset of M. Given C_0, there exist $0 < d_1 \leq \frac{1}{2}i(D')$ such that any C^2 curve $\alpha : [0, d_1] \to D'$ with $\|\alpha'(s)\| = 1$ and $\|\nabla_{\alpha'}\alpha'\| \leq C_0$, must satisfy*

i. $d_M(\alpha(0), \alpha(s)) \geq \frac{3s}{4}, \forall s \in [0, d_1]$ and

ii. $d_M(\gamma(s), \alpha(s)) \leq \frac{s}{4}, \forall s \in [0, d_1]$ where $\gamma(s) = \exp_{\alpha(0)} s\alpha'(0)$.

Proof. i. Suppose that such $d_1 > 0$ does not exist. Then, $\forall m \in \mathbf{N}^+, \exists \alpha_m : [0,1] \to D'$ with $d(\alpha_m(0), \alpha_m(s_m)) < \frac{3s_m}{4}$ for some $s_m \in (0, \frac{1}{m}]$, $\|\alpha_m'(s)\| = 1$ and $\left\|\nabla_{\alpha_m'}\alpha_m'\right\| \leq C_0$. Then by compactness of D', there exists a subsequence which we denote with the same subindices, $\alpha_m(0) \to p_0$ and $\alpha_m'(0) \to v_0 \in UTM_{p_0}$. Since

$d\exp_{p_0}(0) = Id$, for a given $\delta > 0$, there are sufficiently small $\eta > 0$, $\sigma > 0$ and sufficiently large m such that $\tilde{\alpha}_m(s) = \left(\exp_{p_0} |B(0,\eta,TM_p)\right)^{-1} \alpha_m(s)$ are defined for $0 \le s \le \sigma$, $|\|\tilde{\alpha}'_m(s)\| - 1| < \delta$, $\|\tilde{\alpha}_m(0) - \tilde{\alpha}_m(s_m)\| < \frac{3s_m}{4}(1+\delta)$, and $\|\tilde{\alpha}''_m(s)\| \le C_2$. η, σ and C_2 depend on δ, C_0, the metric g_0 locally and the derivatives of $\exp_{p_0}$ near 0, but not on m. This contradicts Schur's Theorem in $\mathbf{R}^n$, [Cn] or the fact that all C^2 curves γ in $\mathbf{R}^n$, with $\|\gamma'(s)\| = 1$ and $\|\gamma''(s)\| \le C_2$ satisfy $\|\gamma(s) - \gamma(0)\| \ge \frac{\sin sC_2}{C_2}$ for $s \in (0, \frac{\pi}{2C_2}]$, (see [D6, proof of Proposition 2a] for a proof). Consequently, $\exists d_1 > 0$ as indicated.

ii. This is an immediate consequence of (i) since $\gamma(0) = \alpha(0)$ and $d(\gamma(s), \gamma(0)) = s$ for $0 \le s \le d_1 \le i(D')$. □

Proposition 9. *Let $\mathcal{A}^\infty(k,\varepsilon,D;M)$ be given and g_0 be the Riemannian metric of M. Consider*

$$\mathcal{C}^\infty(k,\varepsilon,D;M) = \{(K, e^*g_0) : \forall K \in \mathcal{A}^\infty(k,\varepsilon,D;M), e : K \hookrightarrow D \subset M\}$$

as a collection of Riemannian manifolds, not as submanifolds of M. By Gromov's (pre)Compactness Theorem, $\mathcal{C}^\infty(k,\varepsilon,D;M)$ has finitely many diffeomorphism types, and any sequence (K_m, g_m) in $\mathcal{C}^\infty(k,\varepsilon,D;M)$ has a Cauchy subsequence (K_{m_j}, g_{m_j}) in $\mathcal{C}^\infty(k,\varepsilon,D;M)$ with respect to Lipschitz distance. As it was stated in [Pe], all K_{m_j} are diffeomorphic to a fixed C^∞ manifold K, and $g_{m_j} \to g_\infty$ in C^1 sense on K with respect to some harmonic coordinates, in which g_∞ is a $C^{1,\alpha}$ Riemannian metric of K. (K, g_∞) is an Alexandrov space of bounded curvature by [Ni].

Proof. In Proposition 8, we proved all necessary conditions for hypothesis of Gromov's Compactness Theorem, see [Gr], [Ni], [Pe], [GW] and [D3]. □

Remark 1. *A priori, (K, g_∞) is not a Riemannian submanifold of (M, g_0). We will prove in Theorem 2 that there exists an isometric embedding $(K, g_\infty) \hookrightarrow (M, g_0)$. If one starts with an arbitrary $C^{1,1}$ $K \in \mathcal{A}(k,\varepsilon,D;M)$, $e : K \hookrightarrow D \subset M$, then e^*g_0 is a priori $C^{0,1}$, which is of too low regularity to have any sense of curvature. It is not a priori necessary that smooth approximations of e^*g_0 are of uniformly bounded curvature or smooth approximations of the embedding $e : K \hookrightarrow D \subset M$ have thickness close to ε.*

Notation 1. *For C^1 $f : \mathbf{R}^k \to \mathbf{R}^m, v \in U\mathbf{R}^k$ the directional derivative of f in the direction v is $f_v(p) = \frac{d}{dt} f(p+tv)\big|_{t=0}$. The Jacobian $f'(p)$ is an $m \times k$ matrix and $\|f'(p)\|$ is its norm in $\mathbf{R}^{km}$, and $\|f'\| = \sup_p \|f'(p)\|$. For $p \in \mathbf{R}^k$, $v \in U\mathbf{R}^k_p$, $f_{vv}(p) = \frac{d^2}{dt^2} f(p+tv)\Big|_{t=0}$ which is defined almost everywhere in p, when $f \in C^{1,1}$.*

Lemma 3. *Let $f : \mathbf{R}^k \to \mathbf{R}^m$ be $C^{1,1}$, $p \in \mathbf{R}^k$, $v \in U\mathbf{R}^k_p$ and let G be the graph of f in $\mathbf{R}^{k+m}$. Define*

$$I(f,p,v,w) = \left(1 + \|f_v(p)\|^2\right)\left(1 + \|\nabla(f \cdot w)(p)\|^2\right)^{\frac{1}{2}} \ge 1,$$
$$\forall x \in \mathbf{R}^k, \forall v \in U\mathbf{R}^k, w \in U\mathbf{R}^{n-k}.$$

If $F_g((p, f(p)), G) \geq R$ and $f_{vv}(p)$ exists, then

$$\forall w \in U\mathbf{R}^m, \|f_{vv}(p) \cdot w\| \leq \frac{1}{R} I(f, p, v, w).$$

Conversely, if $f_{vv}(p)$ exists and $\exists w \in U\mathbf{R}^m$ such that $\|f_{vv}(p) \cdot w\| > \frac{1}{R} I(f, p, v, w)$, then $F_g((p, f(p)), G) < R$ and particularly, $B((p, f(p)) + Rn, R) \cap G \neq \emptyset$ where $n = (-\nabla(f \cdot w)(p), w) \left(1 + \|\nabla(f \cdot w)(p)\|^2\right)^{-\frac{1}{2}}$.

Proof. Let $\phi(u) = (u, f(u)) = (x, y) \in \mathbf{R}^k \times \mathbf{R}^m$ be a parametrization of G. For $(w_1, w) \in \mathbf{R}^k \times \mathbf{R}^m$ to be normal to G at p, $(u, f_u(p)) \cdot (w_1, w) = 0$ should be true for all $u \in U\mathbf{R}^k_p$, that is $(w_1, w) \in nullspace([I_k \,| f'(p)^T])$.

$$(w_1, w) = \sum_{j=1}^{m} (-\nabla f_j(p), e_j) w_j = (-\nabla(f \cdot w)(p), w)$$

Let $n = (w_1, w) \|(w_1, w)\|^{-1} = (w_1, w) \left(\|w\|^2 + \|\nabla(f \cdot w)(p)\|^2\right)^{-\frac{1}{2}}$

and define $\sigma(t) = \frac{1}{2} \|\phi(p) + Rn - \phi(p + tv)\|^2$.

$$\sigma'(0) = -(v, f_v(p)) \cdot Rn = 0$$

$$\sigma''(0) = -(0, f_{vv}(p)) \cdot Rn + \|(v, f_v(p))\|^2$$

$$f_{vv}(p) \cdot w = \frac{1}{R} \left(1 + \|f_v(p)\|^2 - \sigma''(0)\right) \left(1 + \|\nabla(f \cdot w)(p)\|^2\right)^{\frac{1}{2}}, \forall w \in U\mathbf{R}^m$$

If $F_g(p, G) \geq R$, then $B(\phi(p) + Rn, R) \cap G = \emptyset$ and $\sigma(t)$ has a local minimum at $t = 0,$ that is $\sigma''(0) \geq 0$, since it exists.

$$f_{vv}(p) \cdot w \leq \frac{1}{R} \left(1 + \|f_v(p)\|^2\right) \left(1 + \|\nabla(f \cdot w)(p)\|^2\right)^{\frac{1}{2}}, \forall w \in U\mathbf{R}^m$$

Using $-w$ gives the inequality with the absolute value. For the converse, choose w or $-w$ for positive $f_{vv}(p) \cdot w$. $\sigma''(0) < 0$ implies $\sigma(t) < \sigma(0)$ for small $t \neq 0$, even for a $C^{1,1}$ function. Hence, $B(\phi(p) + Rn, R) \cap G \neq \emptyset$ and $F_g(p, G) < R$. □

Lemma 4. *Let $f : \mathbf{R}^k \to \mathbf{R}^m$ be $C^{1,1}$ and satisfy*

a. $\|f'\| \leq A$,

b. $\|f'(x) - f'(y)\| \leq B \|x - y\|, \forall x, y \in \mathbf{R}^k$, and

c.$\|f_{vv}(x) \cdot w\| \leq C$ for a.e. $x \in \mathbf{R}^k$, for fixed $v \in U\mathbf{R}^k$ and $w \in U\mathbf{R}^m$.

Then $\forall \delta, \rho > 0, \exists$ a $C^{1,1}$ function $h : \mathbf{R}^k \to \mathbf{R}^m$ such that

i. h is C^∞ on $B(0, \rho)$ and $h = f$ outside $B(0, 2\rho)$,

ii. $\|h - f\| \leq A\delta$,

iii. $\|h' - f'\| \leq (aA + B)\delta$,

iv. $\|h'(x) - h'(y)\| \leq \|x - y\| (B + \delta(2aB + bA))$, and

v. $\|h_{vv}(x) \cdot w\| \leq C + \delta(2aB + bA))$.

where a and b are constants depending on $\frac{1}{\rho}$ but not on f.

Proof. Choose $\eta : [0, \infty) \to [0, 1]$ smooth with $\sup p(\eta) \subset [0, 1], \eta^{-1}(1) = [0, \frac{1}{2}]$, $-2.25 \leq \eta' \leq 0$, and $\|\eta''\| \leq 10$.

Define $g : \mathbf{R}^k \to \mathbf{R}^m$ and $h : \mathbf{R}^k \to \mathbf{R}^m$ by

$$g(x) = c_\delta \int_{\|u\| \leq \delta} f(x + u) \eta\left(\frac{\|u\|}{\delta}\right) du \text{ where } c_\delta^{-1} = \int_{\|u\| \leq \delta} \eta\left(\frac{\|u\|}{\delta}\right) du$$

$$h(x) = (1 - \eta(2\rho \|x\|)) f + \eta(2\rho \|x\|) g.$$

Set $a = \sup \left\|(\eta(2\rho\|x\|))'\right\|$ and $b = \sup \left\|(\eta(2\rho\|x\|))''\right\|$.

The proofs of (i-iv) are elementary and will be left to the reader. We will only give a proof of (v). Let $\lambda(t) = f(x+tv)\cdot w$. Then $\lambda'(t) = f_v(x+tv)\cdot w$ which is lipschitz and hence absolutely continuous, and $\lambda''(t) = f_{vv}(x+tv)\cdot w$ which is defined almost everywhere.

$$\|(f_v(x+tv) - f_v(x))\cdot w\| = \|\lambda'(t) - \lambda'(0)\| = \left\|\int_0^t \lambda''(u)du\right\| \le Ct, \forall x$$

$$\begin{aligned}
\|(g_v(x+tv) - g_v(x))\cdot w\| &\le c_\delta \int_{\|u\|\le\delta} \|(f_v(x+tv+u) - f_v(x+u))\cdot w\|\,\eta\left(\frac{\|u\|}{\delta}\right)du \\
&\le Ct, \forall x \\
\|g_{vv}(x)\cdot w\| &\le C, \forall x \text{ a.e.} \\
\|(g-f)(x)\cdot w\| &\le c_\delta \int_{\|u\|\le\delta} \|(f(x+u) - f(x))\cdot w\|\,\eta\left(\frac{\|u\|}{\delta}\right)du \\
&\le A\delta, \forall x, \\
\|(g_v - f_v)(x)\cdot w\| &\le c_\delta \int_{\|u\|\le\delta} \|(f_v(x+u) - f_v(x))\cdot w\|\,\eta\left(\frac{\|u\|}{\delta}\right)du \\
&\le B\delta, \forall x \\
\forall x \text{ a.e.}, \|h_{vv}(x)\cdot w\| &\le \|(1-\eta(2\rho\|x\|))f_{vv}\cdot w + \eta(2\rho\|x\|)g_{vv}\cdot w\| + \\
&\quad 2\|\eta(2\rho\|x\|)_v(g-f)_v\cdot w\| + \|\eta(2\rho\|x\|)_{vv}(g-f)\cdot w\| \\
&\le C + 2aB\delta + bA\delta
\end{aligned}$$

□

Proposition 10. *i. Let K be a compact $C^{1,1}$ submanifold of $\mathbf{R}^n$ such that $F_g(K) \ge R_1$. Then $\forall R_2 < R_1, \forall\sigma > 0$, there exists a smooth approximation K^δ of K in M such that $d_{C^1}(K, K^\delta) < \sigma$ and $F_g(K^\delta) \ge R_2$.*

ii. Hence, $\forall\sigma > 0$, $\mathcal{A}(k,\varepsilon,D;\mathbf{R}^n) \subset \overline{\mathcal{A}^\infty(k,\varepsilon,D_\sigma;\mathbf{R}^n)}$ with respect to C^1 topology, where D_σ is the closure of the $\sigma-$neighborhood of D.

Proof. i. Let $p \in K$ be any point. Rotate and translate K in $\mathbf{R}^k \times \mathbf{R}^{n-k}$ so that $p = 0$ and $TK_p = \mathbf{R}^k \times \{\mathbf{0}\}$. $\exists r_p > 0, \exists f_p : B(0,3r,\mathbf{R}^k) \to \mathbf{R}^{n-k}$ such that $U = \{(x, f_p(x)) : \|x\| < 3r_p\}$ is an open neighborhood of p in K, and $\left\|f_p'(x)\right\| \le 1$ for $\|x\| < 3r$. Set $V(p) = \{(x, f_p(x)) : \|x\| < r_p\}$. By the compactness of K, there are finitely many $V(p_j)$ covering K. Let r_j, f_j, U_j, V_j be defined as above associated to p_j. Choose k_j and A_j so that

$$\frac{1}{R_1} = k_1 < k_2 < ... < k_{j_0+1} = \frac{1}{R_2} \text{ and } 1 = A_1 < A_2 < ... < A_{j_0+1} = 2.$$

Rotate K so that $p_1 = 0$ and $TK_{p_1} = \mathbf{R}^k \times \{\mathbf{0}\}$. $F_g(U_1) \ge R_1$ by the hypothesis. By Lemma 3:

$$\|f_{1,vv}(x)\cdot w\| \le k_1 I(f_1,x,v,w), \forall x \in B(0,3r_1), \forall v \in U\mathbf{R}^k, w \in U\mathbf{R}^{n-k}, a.e.$$

By Lemma 4, for a given $\delta > 0$, there exists a $C^{1,1}$ approximation h_1^δ of f_1 which is C^∞ for $\|x\| < r_1$ and coincides with f_1 for $\|x\| \ge 2r_1$. Let K_1^δ be the submanifold of $\mathbf{R}^n$ obtained from K by replacing U_1 with the graph U_1^δ of h_1^δ.

$$\left\|f_1 - h_1^\delta\right\| \leq A_1\delta$$
$$\left\|(f_1)' - \left(h_1^\delta\right)'\right\| \leq \delta \cdot c_3(A_1, R_1, \|f_1'\|)$$
$$\text{and } \forall x \in B(0, 3r_1), \forall v \in U\mathbf{R}^k, w \in U\mathbf{R}^{n-k}, a.e.,$$
$$\left\|h_{1,vv}^\delta(x) \cdot w\right\| \leq k_1 I(f_1, x, v, w) + \delta \cdot c_4(A_1, R_1, \|f_1'\|)$$

Choose $\delta_1 > 0$ small enough so that $\forall\delta$ with $0 < \delta \leq \delta_1$,

a. $(1+\delta)A_1 \leq A_2$, and

b. $\forall x \in B(0, 3r_1), \forall v \in U\mathbf{R}^k, w \in U\mathbf{R}^{n-k}, a.e.,$

$$\frac{\left\|h_{1,vv}^\delta(x) \cdot w\right\|}{I(h_1^\delta, x, v, w)} \leq \frac{k_1 I(f_1, x, v, w) + \delta \cdot c_4(A_1, R_1, \|f_1'\|)}{I(h_1^\delta, x, v, w)} \leq k_2, \text{ and}$$

c. the adjustment of f_1 by h_1^δ does not change f_j being graphs and keeps $\left\|f_j'\right\| \leq A_2$, for $j \geq 2$.

Consequently, $F_g(K_1^\delta) \geq \frac{1}{k_2}$ and K_1^δ is smooth on U_1^δ. One proceeds inductively to obtain a C^∞ approximation K^δ of K, for all $0 < \delta \leq \delta_{j_0}$ for some $\delta_{j_0} > 0$. Then,

a. $F_g(K^\delta) \geq R_2, \forall 0 < \delta \leq \delta_{j_0}$, and

b. $\lim_{\delta\to 0} d_{C^1}(K, K^\delta) = 0$, that is $\forall\sigma > 0, \exists K^\delta$ such that $d_{C^1}(K, K^\delta) < \sigma$.

ii. Let $\sigma > 0$ and $K \in \mathcal{A}(k, \varepsilon, D; \mathbf{R}^n)$ be given, that is $F_g(K) \geq i(K, \mathbf{R}^n) = R_O(K, \mathbf{R}^n) > \varepsilon$. Then, $\forall m \in \mathbf{N}^+$ with $m > \frac{1}{\sigma}, \exists K_m$, a smooth approximation of K in $\mathbf{R}^n$ such that $F_g(K) - \frac{1}{m} \leq F_g(K_m)$ and $d_{C^1}(K, K_m) \leq \frac{1}{m}$.

Claim 1. $\liminf_m MDC(K_m) > 0$. Suppose that $\liminf_m MDC(K_m) = 0$ and follow the proof of Proposition 3, to obtain $p_0 = q_0$. If $\eta_{p_m q_m}$ denotes a minimal geodesic of K_m between p_m and q_m, then $\eta_{p_m q_m}$ is normal to the segment $\gamma_{p_m q_m}$ at p_m and q_m. Since $\|p_m - q_m\| \to 0$, the maximum of the ambient curvature of $\eta_{p_m q_m}$ in $\mathbf{R}^n$ becomes arbitrarily large as $m \to \infty$. But, the sectional curvatures of K_m are bounded by $\frac{2}{\varepsilon}$ for large m by Proposition 8(i). Thus, Claim 1 holds.

By Propositions 2 and 3:

$$\limsup_{m\to\infty} R_O(K_m, \mathbf{R}^n) \leq R_O(K, \mathbf{R}^n) \leq \frac{1}{2} MDC(K, \mathbf{R}^n) \leq \liminf_{m\to\infty} \frac{1}{2} MDC(K_m, \mathbf{R}^n).$$

Claim 2. $\limsup_{m\to\infty} R_O(K_m, \mathbf{R}^n) = R_O(K, \mathbf{R}^n)$.

Suppose that $\limsup_{m\to\infty} R_O(K_m, \mathbf{R}^n) < R_O(K, \mathbf{R}^n)$. Then for sufficiently large m,

$$R_O(K_m, \mathbf{R}^n) < \frac{1}{2} MDC(K_m, \mathbf{R}^n), \text{i.e. } R_O(K_m, \mathbf{R}^n) = F_g(K_m, \mathbf{R}^n).$$

However, this brings all to a contradiction:

$$R_O(K, \mathbf{R}^n) \leq F_g(K, \mathbf{R}^n) \leq \limsup_{m\to\infty} F_g(K_m, \mathbf{R}^n) = \limsup_{m\to\infty} R_O(K_m, \mathbf{R}^n) < R_O(K, \mathbf{R}^n)$$

Hence, $\limsup_{m\to\infty} R_O(K_m, \mathbf{R}^n) = R_O(K, \mathbf{R}^n) > \varepsilon$, where the smooth submanifolds $K_m \subset D_\sigma$ and $K_m \to K$ in C^1 sense. In other words, $K \in \overline{\mathcal{A}^\infty(k, \varepsilon, D_\sigma; \mathbf{R}^n)}$. □

Proposition 11. *i. For any given $\varepsilon > 0$, a complete Riemannian manifold M and a compact subset $D \subset M$, there exists $\varepsilon'(\varepsilon, D_\varepsilon, M) > 0$ with $\varepsilon' < \varepsilon$ satisfying that*

"$\forall \sigma > 0$ and for any given compact $C^{1,1}$ submanifold K of M with $K \subset D$ and $F_g(K) > \varepsilon$, there exists a smooth approximation K' of K in M with $d_{C^1}(K, K') < \sigma$ and $F_g(K') > \varepsilon'$".

ii. Hence, $\forall \sigma > 0$, $\mathcal{A}(k, \varepsilon, D; M) \subset \overline{\mathcal{A}^\infty(k, \varepsilon', D_\sigma; M)}$ with respect to C^1 topology, where D_σ is the closure of the $\sigma-$neighborhood of D.

Proof. i. Let $D' = \overline{B}(D, \varepsilon)$ and $r_0 = \frac{1}{4} i(D') > 0$. Choose a finite collection of points p_α such that $\{B(p_\alpha, r_0; M) : \alpha = 1, ..., \alpha_0\}$ covers D' and let $\varphi_\alpha := \left(\exp^M_{p_\alpha} | B(0, 3r_0; TM_{p_\alpha})\right)^{-1}$.

Define $\varepsilon_0 = \min(\varepsilon, \frac{1}{2} i(D'))$ and $S(p, \alpha, \varepsilon_0) = \varphi_\alpha(B(p_\alpha, 2r_0; M) \cap \partial B(p, \varepsilon_0; M))$ which are smooth since $\varepsilon_0 \leq \frac{1}{2} i(D')$. In all of the second fundamental form assertions below, $\partial B(p, \varepsilon_0; M)$ are codimension 1 smooth submanifolds of (M, g_0), and $S(p, \alpha, \varepsilon_0)$ are codimension 1 smooth submanifolds of $\mathbf{R}^n$ with the flat metric.

$$\exists c_5, \forall p \in D', \left\| II^{\partial B(p, \varepsilon_0; M)} \right\| \leq c_5$$
$$\exists c_6, \forall p \in D', \forall \alpha, \left\| II^{S(p, \alpha, \varepsilon_0)} \right\| \leq c_6 \text{ whenever } S(p, \alpha, \varepsilon_0) \neq \emptyset$$

The first assertion follows the smoothness of the metric g_0 of M, $\varepsilon_0 \leq \frac{1}{2} i(D')$, and compactness of D'. The second assertion follows the facts that there are finitely many α, and $(\varphi_\alpha)_* g_0$ are uniformly C^i-bounded for $i = 0, 1, 2$, as well as quasi-isometric to the Euclidean metric: $\infty > a \geq \frac{\|(\varphi_\alpha)_* g_0(v)\|}{\|v\|} \geq b > 0$, uniformly. $\forall \tau > 0$, define

$$\min_{p,v,w} \left\| II_w^{\partial B(p, \tau; M)}(v) \right\| = \lambda(\tau, D') \geq 0$$
$$\min_{p,\alpha,v',w'} \left\| II_w^{S(p, \alpha, \tau)}(v) \right\| = \mu(\tau, D') \geq 0$$

where $p \in D', q \in \partial B(p, \tau; M)$, $v \in UT\partial B(p, \tau; M)_q$, $w \in UN\partial B(p, \tau; M)_q$, $\alpha = 1, ..., \alpha_0$, $q' \in S(p, \alpha, \tau) \neq \emptyset$, $v' \in UT\partial B(p, \tau; M)_{q'}$, and $w' \in UN\partial B(p, \tau; M)_{q'}$. Then $\lim_{\tau \to 0^+} \lambda(\tau, D') = \infty$, since D' is compact. By the reasons stated above, also $\mu(\tau, D') \to \infty$, as $\tau \to 0^+$.

Choose any $c_7 > c_6$, $\varepsilon_1 > 0$ and $\varepsilon'(\varepsilon, D', M) > 0$ such that $\mu(\varepsilon_1, D') > c_7$ and $\varepsilon_1 > \varepsilon'$.

Let K be any given compact $C^{1,1}$ submanifold of M with $K \subset D$ and $F_g(K) > \varepsilon$. Then, K avoids tangential balls $B(p, \varepsilon_0, M)$ in an open neighborhood W of the point of tangency $q \in \partial B(p, \varepsilon_0, M)$. Then any nonempty $\varphi_\alpha(K)$ avoids the open set $\varphi_\alpha(B(p, \varepsilon_0, M))$ in an open neighborhood W' of the point of tangency $\varphi_\alpha(q) \in S(p, \alpha, \varepsilon_0) \subset \partial \varphi_\alpha(B(p, \varepsilon_0, M))$. Let n be the unit normal to $S(p, \alpha, \varepsilon_0)$ at $\varphi_\alpha(q)$ towards $\varphi_\alpha(B(p, \varepsilon_0, M))$. Then for any $r < \frac{1}{c_6}$, $W' \cap B(\varphi_\alpha(q), r, \mathbf{R}^n) \subset W' \cap \varphi_\alpha(B(p, \varepsilon_0, M))$ and $W' \cap \varphi_\alpha(K) \cap B(\varphi_\alpha(q), r, \mathbf{R}^n) = \emptyset$. Hence,

$$F_g(\varphi_\alpha(K \cap B(p_\alpha, 2r_0; M)), \mathbf{R}^n) \geq \frac{1}{c_6}, \forall \alpha.$$

By applying the method of Proposition 10 to $\varphi_\alpha(K \cap \overline{B(p_\alpha, 2r_0; M)})$, for any c with $c_6 < c < c_7$, there exists a $C^{1,1}$ approximation K_α of K such that $\varphi_\alpha(K_\alpha \cap B(p_\alpha, r_0; M))$ is a smooth submanifold of $\mathbf{R}^n$, $F_g(\varphi_\alpha(K_\alpha \cap B(p_\alpha, 2r_0; M)), \mathbf{R}^n) \geq \frac{1}{c}$,

K coincides with K_α outside $B(p_\alpha, 3r_0; M)$ and $d_{C^1}(K, K_\alpha) < \frac{\sigma}{\alpha_0}$. One proceeds inductively on finitely many α to obtain a C^∞ submanifold K' of M satisfying

$$d_{C^1}(K, K') < \sigma,$$
$$F_g(\varphi_\alpha(K' \cap B(p_\alpha, 2r_0; M)), \mathbf{R}^n) > \frac{1}{c_7}, \forall \alpha, \text{ and}$$
$$\left\| II^{\varphi_\alpha(K' \cap B(p_\alpha, 2r_0; M))} \right\| < c_7$$

$\varphi_\alpha(K' \cap B(p_\alpha, 2r_0; M))$ avoids tangential submanifolds $S(p, \alpha, \varepsilon_1)$ in a deleted open neighborhood of $\varphi_\alpha(q)$ since $\left\| II_w^{S(p,\alpha,\varepsilon_1)}(v) \right\| > c_7$ for all possible choices of p, v and w. Hence, $F_g(K', M) \geq \varepsilon_1 > \varepsilon'$.

ii. Let $\sigma > 0$ and $K \in \mathcal{A}(k, \varepsilon, D; M)$ be given: $F_g(K) \geq i(K, M) = R_O(K, M) > \varepsilon$. Then, $\forall m \in \mathbf{N}^+$ with $m > \frac{1}{\sigma}, \exists K_m$, a smooth approximation of K in M such that $\varepsilon_1 \leq F_g(K_m, M)$ and $d_{C^1}(K, K_m) \leq \frac{1}{m} < \sigma$.

As in Proposition 10, $\liminf_m MDC(K_m) > 0$.

$$\limsup_{m\to\infty} R_O(K_m, M) \leq R_O(K, M) \leq \frac{1}{2} MDC(K, M) \leq \liminf_{m\to\infty} \frac{1}{2} MDC(K_m, M)$$

If $\limsup_{m\to\infty} R_O(K_m, M) = R_O(K, M) > \varepsilon$, where the smooth submanifolds $K_m \subset D_\sigma$ and $K_m \to K$ in C^1 sense, then,

$$K \in \overline{\mathcal{A}^\infty(k, \varepsilon, D_\sigma; M)} \subset \overline{\mathcal{A}^\infty(k, \varepsilon', D_\sigma; M)}.$$

If $\limsup_{m\to\infty} R_O(K_m, M) < R_O(K, M)$, then for sufficiently large m of the last subsequence,

$$R_O(K_m, M) < \frac{1}{2} MDC(K_m, M)$$
$$R_O(K_m, M) = F_g(K_m, M) \geq \varepsilon_1 > \varepsilon'$$
$$K \in \overline{\mathcal{A}^\infty(k, \varepsilon', D_\sigma; M)}.$$

□

Remark 2. *Different versions of the following lemma have been used by Whitney [W], Cheeger and Gromov [CG], Gromov [Gr], Pugh [P] and others. Especially, the proof of "Hausdorff convergence implies Lipschitz convergence" is along the same lines. All versions known to the author are done in* $\mathbf{R}^N$ *to find a diffeomorphism between two Whitney embeddings. We include the following version since it is in Riemannian manifolds with a uniform choice of radius and an isotopy conclusion.*

Lemma 5. *i. There exists* $\rho(k, \varepsilon, D_\varepsilon, M) > 0$ *such that for all* $K, L \in \mathcal{A}^\infty(k, \varepsilon, D; M)$ *satisfying* $L \subset B(K, \rho; M)$ *there exists a smooth isotopy between* K *and* L *in* $B(K, \rho; M)$.

ii. There exists $\rho'(k, \varepsilon, D_\varepsilon, M) > 0$ *such that for all* $K, L \in \mathcal{A}(k, \varepsilon, D; M)$ *satisfying* $L \subset B(K, \rho'; M)$ *there exists a continuous isotopy between* K *and* L *in* $B(K, \rho'; M)$ *through* $C^{1,1}$ *embeddings of* L.

Proof. i. Let $D' = \overline{B}(D, \varepsilon)$ and $\varepsilon_0 = \min(\varepsilon, \frac{1}{2} i(D'))$. By Proposition 8(i), $\forall K \in \mathcal{A}^\infty(k, \varepsilon, D; M)$, $\left\| II^K \right\| \leq C_0$ and $|Sect(K)| \leq C_1$. By Lemma 2, $\exists d_2 = \min(d_1, \varepsilon_0) > 0$ such that and any C^2 curve $\alpha : [0, d_2] \to D'$ with $\|\alpha'(s)\| = 1$ and $\|\nabla_{\alpha'} \alpha'\| \leq C_0$

must satisfy $d_M(\gamma(s),\alpha(s)) \leq \frac{s}{4}, \forall s \in [0,d_2]$ where $\gamma(s) = \exp_{\alpha(0)} s\alpha'(0)$. Given $p \in D'$ and $v \in TM_p$, one can naturally identify $T(TM_p)_v \cong TM_p$. The vector in $UT(TM_p)_v$ corresponding to $u \in UTM_p$ under this identification will be denoted by u', and let $u'' = (d\exp_p)_v(u')$.

Claim 1: $\exists d_3 > 0$ such that $d_3 \leq d_2$ and $\forall p \in D', \forall u \in UTM_p, \forall v \in TM_p, \|v\| \leq d_3$, one must have $d(\exp_p d_2 u, \exp_q \frac{d_2 u''}{\|u''\|}) \leq \frac{d_2}{4}$ where $q = \exp_p v$, and u', u'' are defined as above. Suppose that such d_3 does not exist, then by using compactness, extract a subsequence $p_m \to p_0$, $\|v_m\| \to 0$, $q_m \to q_0$, $u_m \to u_0$, $u''_m \to u''_0$, with $d(\exp_{p_m} d_2 u_m, \exp_{q_m} \frac{d_2 u''_m}{\|u''_m\|}) > \frac{d_2}{4}$. By continuity and $d(\exp_p)_0 = Id$, one obtains $p_0 = q_0$ and $u''_0 = u_0$. Then $d\left(\exp_{p_m} d_2 u_m, \exp_{q_m} \frac{d_2 u''_m}{\|u''_m\|}\right) \to 0$ which leads to a contradiction. Thus, Claim 1 holds.

Set $\rho = \min\left(\frac{d_2}{3}, \frac{i_0}{4}, d_3\right)$ where $0 < i_0 \leq i(K), \forall K \in \mathcal{A}^\infty(k,\varepsilon,D;M)$, by Proposition 8(iv). Obviously, $\rho < \varepsilon_0 < i(K,M)$.

Let K and L be given as in the hypothesis. Define $E_p = \exp_p^N(B(0,\rho;NK_p)), \forall p \in K$, which are C^∞ $(n-k)$ dimensional submanifolds of M. K^k is obviously transversal to all E_p.

Claim 2. If $E_p \cap L \neq \emptyset$, then E_p intersects L transversally at finitely many points. Suppose not: $\exists u'' \in T(E_p)_q \cap TL_q - \{0\}$ for some $p \in K$ and $q \in E_p \cap L$, since $\dim L + \dim E_p = \dim M$. $\exists v \in NK_p \subset TM_p$ such that $\|v\| < \rho \leq d_3$ and $\exp_p v = q$. Let $u' = (d(\exp_p)_v)^{-1}(u'')$ and adjust the length of u'' so that $\|u'\| = 1$. Find $u \in UTM_p \cong UT(TM_p)_v$ corresponding to u'. Then one has $u \in UNK_p$, since $u'' \in T(E_p)_q$ and $E_p \subset \exp_p(UNK_p)$. Set $a_0 = \exp_p u d_2$.

$$\begin{aligned} d_2 &\leq \varepsilon_0 < i(K,M) \\ B(a_0,d_2;M) \cap K &= \emptyset \\ d_M(a_0,K) &= d_2 \\ d_M\left(a_0, \exp_q \frac{d_2 u''}{\|u''\|}\right) &\leq \frac{d_2}{4} \text{ by choice of } d_3 \geq \|v\| = d(p,q) \\ d_M\left(\exp_q \frac{d_2 u''}{\|u''\|}, \exp_q^{\mathbf{L}} \frac{d_2 u''}{\|u''\|}\right) &\leq \frac{d_2}{4} \text{ by choice of } d_2\text{, Lemma 2 and } \left\|II^L\right\| \leq C_0 \\ d_M\left(\exp_q^{\mathbf{L}} \frac{d_2 u''}{\|u''\|}, K\right) &\geq \frac{d_2}{2} > \rho \end{aligned}$$

The last assertion contradicts with $L \subset B(K,\rho;M)$. Hence, $\forall q \in E_p \cap L$, $T(E_p)_q \cap TL_q = \{0\}$ and Claim 2 holds.

Since K is smooth and $\rho < i(K,M)$, $\Psi = \left(exp^N | B(0,\rho,NK)\right)$ is a diffeomorphism of $B(0,\rho,NK)$ onto $B(K,\rho,M)$. Define $\Pi : B(K,\rho,M) \to K$ by $\Pi^{-1}(p) = E_p$. Π is a smooth submersion onto K. By Claim 2, $\Pi|L : L \to K$ is a maximal rank map. Since L is compact, K is connected, and $\dim L = \dim K$, $\Pi|L$ must be onto and a covering map. Let $q_1, q_2 \in L$ be such that $\Pi(q_1) = \Pi(q_2) = p$.

$$\begin{aligned} p &\in B(q_1,\rho,M) \cap B(q_2,\rho,M) \\ q_1 &\in L \cap B(q_2,2\rho,M) \subset B(q_2,3\rho,L) \end{aligned}$$

by $3\rho \leq \min(d_2, i_0)$, Lemma 2(i) and $\left\| II^L \right\| \leq C_0$. Let γ be a normal minimal geodesic of L from q_1 and q_2. The loop $\Pi(\gamma)$ is contractible in K, since

$$d_M(\Pi(\gamma(t)), p) \leq d_M(\Pi(\gamma(t)), \gamma(t)) + d_M(\gamma(t), q_j) + d_M(p, q_j) \leq \frac{7}{2}\rho < i_0 \leq i(K)$$
$$\text{where } j = 1 \text{ for } t \leq \frac{3\rho}{2}, \text{ and } j = 2 \text{ otherwise.}$$

By the homotopy lifting property, $q_1 = q_2$. Consequently, $\Pi|L$ is a diffeomorphism of L onto K, and $\forall p \in K, E_p \cap L$ consists only one point. Hence, $\Psi^{-1}(L)$ is a smooth section of the normal bundle $B(0, \rho, NK)$ transverse to the fibers NK_p.The same is true for $t_0 \cdot \Psi^{-1}(L)$, $\forall t_0 \in [0, 1]$. Define $\Omega : L \times [0, 1] \to M$ by $\Omega(q, t) = \Psi(t \cdot \Psi^{-1}(q))$. Obviously, Ω is a smooth map, $\Omega(q_0, t)$ is the minimal geodesic between $\Pi(q_0)$ and q_0, and $\Omega(., t_0)$ is a smooth embedding of L into M, for all $t_0 \in [0, 1]$.

ii. Choose ε' with $\mathcal{A}(k, \varepsilon, D; M) \subset \overline{\mathcal{A}^\infty(k, \varepsilon', D_\sigma; M)}$ and $\rho'(k, \varepsilon, D_\varepsilon, M) = \rho(k, \varepsilon', D_\varepsilon, M)$. Consider any $K, L \in \mathcal{A}(k, \varepsilon, D; M)$ satisfying $L \subset B(K, \rho'; M)$. By using Proposition 11, find smooth approximations K' and L' with $d_{C^1}(K, K') < \sigma$ and $d_{C^1}(L, L') < \sigma$ for sufficiently small σ to secure $L' \subset B(K', \rho'; M)$. Recall that we constructed the smooth approximations K' by using mollifiers locally in coordinate systems. One can construct the obvious "vertical" isotopies between the graphs: $(1-t)f(x) + th^\delta(x)$ for each local smoothing and then push them forward into M by the coordinate maps. By applying these finitely many isotopies successively, one can construct an isotopy between K and K'. Similarly, one constructs an isotopy between L and L', and combining all one obtains an isotopy between K and L. Other than the times of attachment of successive isotopies constructed by using different local graphs or part (i), the isotopy is $C^{1,1}$, and at any fixed time t_0 the embedding of L is $C^{1,1}$. □

4.1. **Proof of Theorem 2.**

Proof. We will take subsequences for several times, to simplify the notation all subsequences will be denoted by the same index m, and $\forall m$ means within the last chosen subsequence. The letter "i" appearing as a subindex such as in g_{ij} never means injectivity radius as in $i(K, M)$, $i(D')$ or i_0.

i. By Proposition 11, $\exists \delta > 0$ such that $\mathcal{D}(k, \varepsilon, D; M) \subset \overline{\mathcal{A}^\infty(k, \delta, D_\varepsilon; M)}$ in C^1 topology. By Proposition 9, $\mathcal{A}^\infty(k, \delta, D_\varepsilon; M)$ has finitely many diffeomorphism types, and hence, the same is true for $\mathcal{D}(k, \varepsilon, D; M)$. The finiteness of isotopy classes will follow (ii) and Lemma 5(ii).

ii. Let a sequence $\{(K_m, M)\}_{m=1}^\infty$ in $\mathcal{D}(k, \varepsilon, D; M)$ be given. By Proposition 11, $\forall m \in \mathbf{N}^+$, choose smooth submanifolds $(L_m, M) \in \mathcal{A}^\infty(k, \delta, D_\varepsilon; M)$ such that $d_{C^1}(K_m, L_m) < \frac{1}{m}$. Choose a subsequence so that all L_m are diffeomorphic to a fixed C^∞ manifold L, by the finiteness of diffeomorphism types. Hence, $\forall m \in \mathbf{N}^+$, there are C^∞ embeddings $e_m : L \to (M, g_0)$ such that $L_m = e_m(L)$ and Riemannian metrics $g_m = e_m^* g_0$ are C^∞ on L. By the intrinsic form of Gromov's Compactness Theorem, as it was stated in [Pe, Thm. 4.4], there exists a subsequence $g_m \to g_\infty$ in C^1 sense on L with respect to harmonic coordinates, where g_∞ is a $C^{1,\alpha}$ Riemannian metric on L.

We will show below that there exists an isometric embedding $e_\infty : (L, g_\infty) \to (M, g_0)$ such that $e_m \to e_\infty$ in C^1 sense. Let $D' = \overline{B}(D, \varepsilon)$ and $\delta_0 = \min(\delta, \frac{1}{2} i(D'))$. Choose a finite collection of points p_α such that $\{B(p_\alpha, \delta_0; M) : \alpha = 1, ..., \alpha_0\}$ covers

D' and define $\varphi_\alpha := \left(\exp^M_{p_\alpha} | \overline{B(0,\delta_0;TM_{p_\alpha})}\right)^{-1}$. There exists a Lebesgue number $c_9 > 0$ for the covering $\{B(p_\alpha,\delta_0;M) : \alpha = 1,...,\alpha_0\}$, that is: $\forall q \in D', \exists \alpha(q)$ such that $B(q,c_9;D') \subset B(p_{\alpha(q)},\delta_0;M)$. Let $r_1 = \min(\frac{i_0}{4},\frac{c_9}{2})$ where $i(L,g_m) \geq i_0 > 0, \forall m$, by Proposition 8(iv).

By following [Pe, Thm. 4.4], for sufficiently large $m \in \mathbf{N}^+ \cup \{\infty\}$, choose a finite open cover of L by balls $\{B(q_s,r;(L,g_m)) : s = 1,...,s_0(n,d_0,v_0,C_1)\}$ such that the harmonic coordinates of [JK] exist on $B(q_s,2r;(L,g_m))$ for some $r = r(n,d_0,v_0,C_1)) \in (0,r_1]$. By [JH] and [Pe], the components of the metrics in the harmonic coordinates satisfy

$$(g_m)_{ij} \to (g_\infty)_{ij} \text{ in } C^1 \text{ sense as } m \to \infty. \qquad (2.1)$$

$\forall s = 1,...,s_0$, the sequence $\{e_m(q_s)\}_{m=1}^\infty$ is in compact D'. By taking a subsequence, assume that $\forall s = 1,...,s_0$, $e_m(q_s) \to z_s$, as $m \to \infty$ for some $z_s \in D'$.

Fix $s = 1$. Set $\psi_m : B(q_1,2r;(L,g_m)) \to \mathbf{R}^k$ to be the harmonic coordinates and $V = \overline{B(q_s,r;(L,g_\infty))}$. By the construction of harmonic coordinates [JH], (2.1), and e_m being isometric embeddings, there exists a compact $W \subset \mathbf{R}^k$ such that for sufficiently large m, all of the following holds.

$$\begin{aligned} V &\subset B(q_1,2r;(L,g_m)) \\ \psi_m(V) &\subset int(W) \subset W \subset \psi_m(B(q_1,2r;(L,g_m))) \\ e_m(B(q_1,2r;(L,g_m))) &\subset B(z_1,2r;D') \subset B(p_{\alpha(1)},\delta_0;M) \end{aligned}$$

Define the following C^∞ functions and vector fields for the last subsequence:

$$h_m(y_1,y_2,..,y_k) = \varphi_{\alpha(1)} \circ e_m \circ \psi_m^{-1} : W \subset \mathbf{R}^k \to TM_{p_{\alpha(1)}} \cong \mathbf{R}^n$$

$$Y_i^m = \left(\psi_m^{-1}\right)_* \left(\frac{\partial}{\partial y_i}\right) \text{ and } Z_i^m = (e_m)_* Y_i^m$$

All of the estimates below are uniformly on W, or the corresponding domains by ψ_m^{-1} and e_m, for all present indices i,j,l, and m of the last chosen subsequence when limit is not taken.

$$\left\langle Y_i^m, Y_j^m \right\rangle_{g_m} = (g_m)_{ij} \to (g_\infty)_{ij} \text{ as } m \to \infty \qquad (2.2)$$

Since g_∞ is a non-degenerate metric and e_m are isometric embeddings, there exists constants c_{10}, $c_{11}$and c_{12} such that

$$0 < c_{10} \leq \left\|Z_j^m\right\|_{g_0} = \left\|Y_j^m\right\|_{g_m} \leq c_{11} < \infty \qquad (2.3)$$

$$Z_l^m \left\langle Z_i^m, Z_j^m \right\rangle_{g_0} = Y_l^m \left\langle Y_i^m, Y_j^m \right\rangle_{g_m} = \frac{\partial (g_m)_{ij}}{\partial y_l} \to \frac{\partial (g_\infty)_{ij}}{\partial y_l} \text{ as } m \to \infty \qquad (2.4)$$

$$\left\|Z_l^m \left\langle Z_i^m, Z_j^m \right\rangle_{g_0}\right\| = \left\|Y_l^m \left\langle Y_i^m, Y_j^m \right\rangle_{g_m}\right\| \leq c_{12} < \infty \qquad (2.5)$$

$$\begin{aligned} \left\|\left\langle \nabla^m_{Y_l^m} Y_i^m, Y_j^m \right\rangle_{g_m}\right\| &= \frac{1}{2}\left\|Y_l^m \left\langle Y_i^m, Y_j^m \right\rangle_{g_m} - Y_i^m \left\langle Y_l^m, Y_j^m \right\rangle_{g_m} + Y_j^m \left\langle Y_i^m, Y_l^m \right\rangle_{g_m}\right\| \\ &\leq \frac{3}{2} c_{12} \qquad (2.6) \end{aligned}$$

where ∇^m denotes the connection of (K,g_m). Let ∇^M denote the connection of (M,g_0). Since e_m are isometric embeddings, L_m are C^∞ submanifolds and

$R_O(L_m, M) \geq \delta, \exists c_{13}, c_{14}$ such that

$$\left\| \left\langle \nabla^M_{Z_l^m} Z_i^m, Z_j^m \right\rangle_{g_o} \right\| \leq \frac{3}{2} c_{12} \qquad (2.6')$$

$$\left\| \left\langle \nabla^M_{Z_j^m} Z_j^m, \overrightarrow{n} \right\rangle_{g_o} \right\| \leq C_0(\delta) \left\| Z_j^m \right\|^2_{g_0} \leq c_{13},\ \forall \overrightarrow{n} \in UNL_m \qquad (2.7)$$

$$\left\| \left\langle \nabla^M_{Z_l^m} Z_j^m, \overrightarrow{n} \right\rangle_{g_o} \right\| \leq \frac{3}{2} c_{13}, \forall \overrightarrow{n} \in UNL_m \qquad (2.8)$$

$$\left\| \nabla^M_{Z_l^m} Z_j^m \right\|_{g_o} \leq c_{14}, \forall \overrightarrow{n} \in UNL_m \qquad (2.9)$$

There exists c_{15}, c_{16} depending on M, $p_{\alpha(1)}$, and $\exp^M_{p_{\alpha(1)}}$ such that

$$\forall v \in UTM|\overline{B(p_{\alpha(1)}, \delta_0; M)},\ 0 < c_{15} \leq \left\| \left(\varphi_{\alpha(1)}\right)_* (v) \right\|_{\mathbf{R}^n} \leq c_{16} < \infty.$$

$$\left\| \frac{\partial h_m}{\partial y_j} \right\|_{\mathbf{R}^n} = \left\| (h_m)_* \left(\frac{\partial}{\partial y_j} \right) \right\|_{\mathbf{R}^n} = \left\| \left(\varphi_{\alpha(1)}\right)_* (Z_j^m) \right\|_{\mathbf{R}^n} \leq c_{16} \cdot c_{11} \qquad (2.10)$$

$$\left(\varphi_{\alpha(1)}\right)_* \left(\nabla^M_{Z_l^m} Z_j^m \right) = \sum_\gamma \left(\sum_{\beta,\eta} \left(\frac{\partial h_m}{\partial y_l} \right)_\beta \left(\frac{\partial h_m}{\partial y_j} \right)_\eta \Gamma^\gamma_{\beta\eta} + \left(\frac{\partial^2 h_m}{\partial y_l \partial y_j} \right)_\gamma \right) \frac{\partial}{\partial x_\gamma}$$

in the local coordinates $\varphi_{\alpha(1)}$, where $()_\beta$ denotes the βth component in $\mathbf{R}^n$. Hence, $\exists c_{17}(k, n, c_{11}, c_{16}, g_0, \varphi_{\alpha(1)}) < \infty$ such that

$$\left\| \frac{\partial^2 h_m}{\partial y_l \partial y_j} \right\|_{\mathbf{R}^n} \leq c_{17}. \qquad (2.11)$$

Since W is compact, $e_m(q_1) \to z_1$ as $m \to \infty$, (2.10) and (2.11), there exists a subsequence of h_m converging in C^1 topology to a $C^{1,1}$ function h_∞ over W, by Arzela-Ascoli Theorem. Hence, there exists a subsequence of e_m converging to a $C^{1,1}$ function over $B(q_1, r; (L, g_\infty))$. By applying this process on all finitely many $B(q_s, r; (L, g_\infty))$, there exists a subsequence $e_m \to e_\infty \in C^{1,1}$ on L in C^1 topology.

e_∞ is an immersion since h_∞ is non-singular: $\forall v \in \mathbf{R}^n - \{0\}$, and sufficiently large m,

$$\begin{aligned} \|(h_m)_* (v)\|_{\mathbf{R}^n} &= \left\| \left(\varphi_{\alpha(1)}\right)_* \circ (e_m)_* \circ \left(\psi_m^{-1}\right)_* (v) \right\|_{\mathbf{R}^n} \\ &\geq c_{15} \left\| \left(\psi_m^{-1}\right)_* (v) \right\|_{g_m} \geq \frac{c_{15}}{2} \left\| \left(\psi_m^{-1}\right)_* (v) \right\|_{g_\infty} > 0. \end{aligned}$$

Suppose that e_∞ is not one-to-one, $e_\infty(a) = e_\infty(b)$ for some $a, b \in L$. Let $A = \frac{1}{5} \min(d_2, d(a, b; (L, g_\infty)))$ with d_2 of Proposition 8(iii). For sufficiently large m, $d(a, b; (L, g_m)) \leq \frac{5}{4} d(a, b; (L, g_\infty))$. As in the proof of Proposition 8(iii),

$$B(e_m(a), A; M) \cap B(e_m(b), A; M) = \emptyset.$$

This contradicts with $e_m(a) \to e_\infty(a)$ and $e_m(b) \to e_\infty(b) = e_\infty(a)$. Hence, e_∞ is one-to-one.

$g_m = e_m^* g_0 \to e_\infty^* g_0$ in C^0 topology, since $e_m \to e_\infty$ in C^1 topology. However, $g_m \to g_\infty$ in C^1 topology on L in harmonic coordinates. Consequently, $e_\infty^* g_0 = g_\infty$, i.e., $e_\infty : (L, g_\infty) \to (M, g_0)$ is an isometric embedding.

Since we have chosen $d_{C^1}(K_m, L_m) < \frac{1}{m}$ in the beginning of the proof, the initial sequence $\{(K_m, M)\}_{m=1}^\infty$ has a C^1-convergent subsequence $\{(K_{m_j}, M)\}_{j=1}^\infty$ whose limit is $(e_\infty(L), M) := (K_0, M)$.

(K_0, M) belongs to $\mathcal{D}(k, \varepsilon, D; M)$ by $K_{m_j} \subset D$ and Propositions 1 and 2:

$$\varepsilon \leq \limsup_{j\to\infty} \ i(K_{m_j}, M) \leq i(K_0, M).$$

iii. Given For every $(K, M) \in \mathcal{D}(k, \varepsilon, D; M)$ with the given embedding $e : K \to (M, g_0)$, find smooth approximations (L_m, M) of (K, M) in $\mathcal{A}^\infty(k, \delta, D_\varepsilon; M)$ such that $d_{C^1}(K, L_m) < \frac{1}{m}$, repeat (ii) to show that $e(K) = e_\infty(L)$ by uniqueness of limits. Of course, g_∞ is $C^{1,\alpha}$ $(\alpha < 1)$ by [Pe] in harmonic coordinates [JK] and it is a limit of C^∞ Riemannian metrics of bounded curvature and injectivity radius with respect to Lipschitz distance, [Ni], [Pe], [GW]. (K, g_∞) is a $C^{1,\alpha}$ Alexandrov space [Ni] with a well defined exponential map, [D3], [Pu]. □

Corollary 2. *By Proposition 2 and Theorem 2, there exists thickest submanifolds in every nonempty diffeomorphism or isotopy class $\mathcal{E} \subset \mathcal{D}(k, \varepsilon, D; M)$:*

$$\exists (K_0, M) \in \mathcal{E} \text{ such that } \forall (K, M) \in \mathcal{E},\ i(K_0, M) \geq i(K, M)$$

The same conclusion holds for any nonempty closed subset $\mathcal{E}$ of $\mathcal{D}(k, \varepsilon, D; M)$. For example, subsets with fixed volume or diameter.

4.2. C^1-**Compactness for K with many components.** Define $\mathcal{D}^*(k, \varepsilon, D; M) = \{(K, M) : K \in C^{1,1},\ \dim K = k,\ K \subset D, \text{ and } i(K, M) \geq \varepsilon\}$ where K is not necessarily connected.

Corollary 3.

i. The number of components of K in $\mathcal{D}^*(k, \varepsilon, D; M)$ are uniformly bounded.

ii. $\mathcal{D}^*(k, \varepsilon, D; M)$ is sequentially compact in C^1-topology, and it has finitely many isotopy and diffeomorphism types.

iii. There exists a thickest submanifold in each nonempty isotopy class of $\mathcal{D}^*(k, \varepsilon, D; M)$.

Proof. i. Let $\varepsilon_0 = \min(\varepsilon, \frac{1}{2} i(D_\varepsilon))$. For each component K^α of K, choose $p_\alpha \in K^\alpha$. $B(K^\alpha, \varepsilon; M) \cap B(K^\beta, \varepsilon; M) = \emptyset$ for $\alpha \neq \beta$, since $i(K, M) \geq \varepsilon$. Hence, $B(p_\alpha, \varepsilon_0; M) \cap B(p_\beta, \varepsilon_0; M) = \emptyset$. By Croke [Cr, Prop. 14], $\exists v_1(n, \varepsilon_0) > 0$ such that $v_1 \leq vol_n(B(p_\alpha, \varepsilon_0; M)), \forall \alpha$. Hence, K has at most $vol_n(D_\varepsilon)/v_1$ components.

ii. Given any sequence $\{(K_j, M)\}_{j=1}^\infty$ in $\mathcal{D}^*(k, \varepsilon, D; M)$, choose a subsequence (by using same index j) where the number of components of K_j is constant, and enumerate the components K_j^α. $\forall \alpha, \varepsilon \leq i(K_j, M) \leq i(K_j^\alpha, M)$. By Theorem 2, choose a subsequence where K_j^1 converges in C^1 topology, and choose its subsequence where K_j^2 converges in C^1 topology and so on. Hence, $\mathcal{D}^*(k, \varepsilon, D; M)$ is sequentially compact in C^1-topology and a subsequence $(K_j, M) \to (K_\infty, M) \in \mathcal{D}^*(k, \varepsilon, D; M)$, by Proposition 2. $\exists j_0 \forall j \geq j_0 \forall \alpha$, $K_j^\alpha \subset B(K_\infty^\alpha, \rho'; M)$, for ρ' of Lemma 5 and hence K_j^α is isotopic to K_∞^α in $B(K_\infty^\alpha, \varepsilon; M)$. These isotopies can be combined to give an isotopy of K_j to K_∞ without any self intersections $\forall j \geq j_0$, since $B(K_\infty^\alpha, \varepsilon; M)$ are mutually disjoint.

iii. This is an immediate consequence of (ii) and Proposition 2. □

4.3. **Normal curvatures and Thickness Formula in Euclidean Spaces.** $\exp_p : T(K, g_\infty)_p \to (K, g_\infty)$ is of class $C^{0,1}$, see [D3] and [Pu]. Even though the geodesics $\exp_p sv$ of (K, g_∞) are C^2 in limit harmonic coordinates, [D3, 5.10.3], the corresponding geodesics $e_\infty(\exp_p sv)$ in (K, M) are $C^{1,1}$. Hence $\nabla_{\gamma'}^M \gamma'$ is defined almost everywhere in s.

Definition 10. *We define the supremum of the "absolute normal curvatures"* $\sup \kappa_N(K)$ *for a* $C^{1,1}$ *submanifold* K *to be*

$\sup \left\{ \left\| \nabla^M_{\gamma'} \gamma'(s) \right\| : \gamma : \mathbf{R} \to K \text{ is a geodesic of } K \text{ with } \|\gamma'\| = 1 \text{ and } \nabla^M_{\gamma'}\gamma'(s) \text{ exists} \right\}.$

Proposition 12. *For a* $C^{1,1}$ *submanifold* K^k *of* $\mathbf{R}^n$, $F_g(K, \mathbf{R}^n) = \frac{1}{\sup \kappa_N(K)}$. *Hence,*

$$i(K, M) = \min\{\frac{1}{\sup \kappa_N(K)}, \frac{1}{2} MDC(K)\}.$$

Proof. The following is a basic result in $\mathbf{R}^n$, we refer to [D6, Proposition 2] for an elementary proof.

Let $\gamma : I = (-\frac{\pi}{2\kappa}, \frac{\pi}{2\kappa}) \to \mathbf{R}^n$ be with $\|\gamma'\| \equiv 1$, $\|\gamma''\| \le \kappa \neq 0$ a.e. Then,

i. $\gamma \cap O_{\gamma(0)}(\gamma'(0), \frac{1}{\kappa}; \mathbf{R}^n) = \emptyset$, and

ii. if $\gamma''(0)$ exists and $\|\gamma''(0)\| = \kappa$, then $\forall R > \frac{1}{\kappa}, \exists \delta > 0$ such that $\gamma((0, \delta)) \subset B(\gamma(0) + R\frac{\gamma''(0)}{\|\gamma''(0)\|}, R)$.

Let $\kappa_0 = \sup \kappa_N(K)$ and $\varepsilon = \frac{\pi}{2\kappa_0}$. By (i), for any $p \in K$, $v \in UTK_p$,

$$\exp_p^K((-\varepsilon, \varepsilon)v) \cap O_p(v, \frac{1}{\kappa_0}, \mathbf{R}^n) = \emptyset$$

$$B(p, \varepsilon; K) \cap O_p(\frac{1}{\kappa_0}, K; \mathbf{R}^n) = \emptyset$$

$$F_g(p, K; \mathbf{R}^n) \ge \frac{1}{\kappa_0}$$

$$F_g(K; \mathbf{R}^n) \ge \frac{1}{\kappa_0} = \inf_{\gamma \text{ geodesic}} \frac{1}{\|\gamma''\|}$$

Suppose that $F_g(K; \mathbf{R}^n) > \inf_{\gamma} \frac{1}{\|\gamma''\|}$. Then, there exists a geodesic γ and R such that $\gamma''(0)$ exists and $F_g(K; \mathbf{R}^n) > R > \frac{1}{\|\gamma''(0)\|}$. Then by (ii) above,

$\gamma((0, \delta)) \subset B(\gamma(0) + R\frac{\gamma''(0)}{\|\gamma''(0)\|}, R)$ which implies that $R \ge F_g(\gamma(0), K; \mathbf{R}^n) \ge F_g(K; \mathbf{R}^n)$ by the definition of F_g. Hence, one obtains a contradiction. Consequently, $F_g(K, \mathbf{R}^n) = \frac{1}{\sup \kappa_N(K)}$. The rest follows Theorem 1. ☐

5. Estimates on the number of Isotopy and Diffeomorphism types

The number $\#(k, \varepsilon, D; M)$ of the different diffeomorphism classes and isotopy classes of $C^{1,1}$ manifolds of $\mathcal{D}(k, \varepsilon, D; M)$ is bounded above by a constructible constant in terms $n, k, \delta(\varepsilon)$ and D_ε where $\mathcal{D}(k, \varepsilon, D; M) \subset \overline{\mathcal{A}^\infty(k, \delta, D_\varepsilon; M)}$ in C^1 topology. It is clear from the proofs of Propositions 8 and 11, and Lemmas 2 and 5 that $\rho = \rho(n, k, \delta(\varepsilon), C_0, |Sect(M)|, i(D_\varepsilon))$. The dependence of δ on ε relies on a finite number of fixed coordinate charts of M, in fact on their derivatives up to second order. Using normal coordinates may bring in $\|\nabla R\|$, but in harmonic coordinates, one can control them with only $|Sect(M)|$ and $i(D_\varepsilon)$.

The number of different diffeomorphism classes and isotopy classes in $\mathcal{D}(k, \varepsilon, D; M)$ can be bounded in terms of $\rho(\delta(\varepsilon))$ as follows. Take a minimal cover $\mathcal{B} = \{B(p_\alpha, \rho/2) : \alpha = 1, ...\Lambda_0\}$ of D_ε by open discs: $B(p_\alpha, \rho/4) \cap B(p_\beta, \rho/4) = \emptyset$ if $\alpha \neq \beta$. Then, $\Lambda_0 \le vol(D_\varepsilon)/\min(vol(B(p_\alpha, \rho/4))) \le c(n)vol(D_\varepsilon)\rho^{-n}$ by volume estimates of [Cr]. Define $\Phi(K) = \{\alpha : K \cap B(p_\alpha, \rho/2) \neq \emptyset\}$. Any $K, L \in \mathcal{D}(k, \varepsilon, D; M)$ with $\Phi(K) = \Phi(L)$ must satisfy $L \subset B(K, \rho; M)$, and hence, K and L are isotopic and diffeomorphic. Consequently, there are at most $2^{c(n)vol(D_\varepsilon)\rho^{-n}}$ distinct diffeomorphism classes and isotopy classes of $C^{1,1}$ manifolds in $\mathcal{D}(k, \varepsilon, D; M)$.

We calculate these estimates in $\mathbf{R}^n$ below. Let $D = \overline{B(0,r,\mathbf{R}^n)}$ and $i(K,\mathbf{R}^n) \geq \varepsilon$. Rescale the metric so that $R = \frac{r}{\varepsilon}, D = \overline{B(0,R,\mathbf{R}^n)}$ and $i(K,\mathbf{R}^n) \geq 1$. Let $\alpha_n = vol(S^n(1))$.

$$i(\mathbf{R}^n) = \infty,\ Sect(\mathbf{R}^n) = 0,\ \text{and } C_0 = C_1 = 1$$

$$1 = \varepsilon \geq \min(\varepsilon', \delta, \delta_0) \approx 1$$

$$d_1 = d_2 = \frac{1}{2} \text{ and } d_3 = \frac{1}{8}$$

$$d_0 \leq \frac{1}{2}(8R)^n$$

$$v_0 \geq \frac{(n-1)\alpha_n}{n\alpha_{n-k-1}} \cdot e^{1-n}$$

$$i_0 \geq \min(\pi, \frac{\pi v_0}{\alpha_n} \sinh^{1-n} d_0) \text{ by [HK]}$$

$$\geq e^{-\frac{n}{2}(8R)^n},\ \text{for } k \geq 2,$$

$$i_0 \geq \pi,\ \text{for } k = 1$$

$$\rho = \min\left(\frac{d_2}{3}, \frac{i_0}{4}, d_3\right) = \frac{i_0}{4} \text{ for } k \geq 2$$

$$\rho = \min\left(\frac{d_2}{3}, \frac{i_0}{4}, d_3\right) = d_3 = \frac{1}{8},\ \text{for } k = 1$$

$$\Lambda_0 \leq \left(\frac{4R}{\rho}\right)^n = \left(\frac{16R}{i_0}\right)^n$$

$$\#(k,\varepsilon,D;M) \leq 2^{\Lambda_0}$$

Almost all of the estimates are reasonable, except for i_0 and v_0 for $k \geq 2$.

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