A FAST TWO-DIMENSIONAL ENTROPIC THRESHOLDING ALGORITHM†

WEN-TSUEH CHEN,** CHIA-HSIEN WEN,§ and CHIN-WEN YANG§

**Department of Computer Science, National Tsing Hua University, Hsinchu 30043, Taiwan, R.O.C.
§Computer Center, Taichung Veterans General Hospital, Taichung 40705, Taiwan, R.O.C.

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Abstract—Two-dimensional (2D) entropic thresholding is one of the important thresholding techniques for image segmentation. The selection of the global threshold vector is usually through a “maximin” optimization procedure. A fast two-phase 2D entropic thresholding algorithm is proposed. In order to reduce the computation time, first $9L^{2/3}$ candidate threshold vectors are estimated from a quantized image of the original. The global threshold vector is then obtained by checking candidates only. The optimal computation complexity is $O(L^{S/3})$ by quantizing the gray level into $L^{2/3}$ levels. Experimental results show that the processing time of each image is reduced from more than 2 h to about 2 min. The required memory space is also greatly reduced.

Thresholding Entropy Segmentation Algorithm Image

1. INTRODUCTION

Thresholding is one of the important techniques for image segmentation based on the similarity of brightness of image objects. By choosing one or more thresholds of gray level, the object regions can be separated from their background for further processing. Most of the proposed methods select thresholds which depend solely on the first-order gray level histogram of the image. However, more information contained in the image can be utilized to obtain better segmentation. Haralick et al. and Kirby and Rosenfeld advocated their two-dimensional (2D) thresholding approaches which employed point pixel information as well as the local average gray level of the neighborhood pixels.

The entropic thresholding methods find the optimal threshold gray level by applying information theory. One-dimensional (1D) entropic thresholding was first introduced by Pun. His basic idea was that the optimal threshold partitions all pixels of the image into the object class and the background class, and maximizes the a posteriori entropy that is the sum of the entropies of the two classes. Kapur et al. modified Pun’s method by deriving the two entropies directly from the original gray level distribution of the image. Combining the technique of Kapur et al. and the method of Kirby et al., Abutaleb and Pal and Pal proposed their 2D entropic thresholding methods. Brink then refined Abutaleb’s method by maximizing the smaller entropy of the background class and the object class instead of maximizing the sum of the two entropies.

The computation complexity of Brink’s method, using a “maximin” optimization procedure, is bounded by $O(L^4)$, where $L$ is the number of gray levels. We propose a fast two-phase algorithm for Brink’s method. In phase one, a set of candidate threshold vectors is obtained by applying Brink’s method to a quantized image of the original. In phase two, the exact threshold vector is found by checking the candidates only. We show that the computation complexity can be reduced to $O(L^{S/3})$.

In Section 2 the 2D entropy is defined and the method proposed by Brink is depicted. In Section 3 our fast 2D entropic thresholding algorithm is described. Section 4 contains the experimental results and conclusions.

2. TWO-DIMENSIONAL ENTROPIC THRESHOLDING

An image is defined as a 2D light-intensity function, $f(x, y)$, which contains $N \times N$ pixels each with a value of brightness, i.e. gray level, from 0 to $L - 1$. Gray level 0 is the darkest and gray level $L - 1$ is the lightest. Every 1D thresholding method partitions all pixels of the image into two classes at gray level $T$, and assigns two specific gray level values to all pixels of each class respectively for segmentation. Thus, it can be defined as a function $f_T(x, y)$, such that

$$f_T(x, y) = \begin{cases} b_0 & \text{if } f(x, y) < T \\ b_1 & \text{if } f(x, y) \geq T \end{cases} \quad \text{where } 0 \leq b_0, T, b_1 < L.$$  

(1)

A 2D thresholding method employs both the gray level
of a pixel and its local neighborhood information, usually the local average gray level, as the criteria for partition. Let \( g(x, y) \) and \( f_{r,s}(x, y) \) be the function of the local average gray level and the function of the 2D thresholding method, respectively. If the image is partitioned at gray level \( T \) and local average gray level \( S \), then

\[
\begin{align*}
  g(x, y) &= \frac{1}{n^2} \sum_{j=-n/2}^{n/2} \sum_{i=-n/2}^{n/2} f(x + i, y + j) \\
  \text{where } n \leq N \text{ and } n < M (2)
\end{align*}
\]

\[
\begin{align*}
  f_{r,s}(x, y) &= \begin{cases} 
    b_0 & \text{if } f(x, y) < T \vee g(x, y) < S \\
    b_1 & \text{if } f(x, y) \geq T \wedge g(x, y) \geq S 
  \end{cases} \\
  \text{where } 0 \leq b_0, T, S, b_1 < L. (3)
\end{align*}
\]

In a 2D (gray level, local average) histogram depicted in Fig. 1, the origin is defined as the upper left-hand corner, the gray level increases from left to right, and the local average increases from top to bottom. Since the number of gray levels is \( L \), there are \( L^2 \) elements in the histogram. Each element represents the occurrence, \( r_{ij} \), of the corresponding (gray level, local average) pair and \( 0 \leq r_{ij} \leq NM \). If the image is segmented at thresholds \( T \) and \( S \), the histogram will be divided into four quadrants. Since pixels interior to the objects or background should contribute mainly to the near-diagonal elements because of homogeneity, quadrants 0 and 1 contain the distributions of object and background classes. The off-diagonal quadrants, 2 and 3, contain the distributions of pixels near edges and noise.

The idea of 2D entropic thresholding is to select the threshold vector \((T, S)\) which can maximize both the a posteriori entropy of the object class and that of the background class. Since the distributions of the two classes, contained in quadrants 0 and 1, are independent, the probability of each (gray level, local average) pair in the two classes is thus defined as

\[
\begin{align*}
  p_{0ij}(T, S) &= \frac{r_{ij}}{\sum_{i=0}^{L-1} \sum_{k=0}^{L-1} r_{ik}} \\
  p_{1ij}(T, S) &= \frac{r_{ij}}{\sum_{i=S+1}^{L-1} \sum_{k=T+1}^{L-1} r_{ik}}.
\end{align*}
\]

The two class entropies, \( H_0(T, S) \) and \( H_1(T, S) \), are then given by

\[
\begin{align*}
  H_0(T, S) &= -\sum_{j=0}^{S} \sum_{i=0}^{L} p_{0ij}(T, S) \log p_{0ij}(T, S) \\
  H_1(T, S) &= -\sum_{j=S+1}^{L-1} \sum_{i=T+1}^{L} p_{1ij}(T, S) \log p_{1ij}(T, S)
\end{align*}
\]

and the threshold vector \((T, S)\), selected to perform the segmentation, has to satisfy the following criterion function:

\[
H(T, S) = \max \left( \min_{t=0, \ldots, L-1} \{ H_0(t, s), H_1(t, s) \} \right). \quad (8)
\]

In order to find the global maximum, the above optimization procedure invokes the exhaustive searching technique to obtain the threshold vector. Obviously, the calculation of entropies \( H_0(t, s) \) and \( H_1(t, s) \), which includes operations of logarithm, dominates the efficiency of the above method. For each pair of \((t, s)\), it takes \( O(L^2) \) computation time to obtain the two entropies. Since there are \( L^2 \) pairs of \((t, s)\), the computation time for thresholding is \( O(L^4) \).

3. THE FAST 2D ENTROPIC THRESHOLDING ALGORITHM

The 2D entropic thresholding method is time consuming for calculating entropies for the \( L^2 \) pairs of \((t, s)\). If we can estimate a set of possible threshold vectors and then apply the above method only to this set, the computation complexity will be greatly reduced. The estimation is achieved by applying the same method to a quantized image of the original. On the basis of this idea, a fast two-phase 2D entropic thresholding algorithm is implemented as follows:

Let \( f(x, y) \) be the original image with \( N \times M \) pixels and \( L \) the number of gray levels.

**Phase 1: [Estimation]**

Step 1. Transform the original image, \( f(x, y) \), into the quantized image

\[
f'(x, y) = \left[ \frac{f(x, y)}{L^{1/3}} \right] \quad \text{where } 0 \leq x < N, 0 \leq y < M.
\]

Step 2. Calculate the quantized local average, \( g'(x, y) \), over an \( n \times n \) squared window as shown in equation (2).

Step 3. Draw the 2D histogram, \( F' \), of the quantized image such that
A fast two-dimensional entropic thresholding algorithm

\[ F' = \{ r_{ij} \mid 0 \leq i < L^{2/3}, 0 \leq j < L^{2/3} \} \]
where \( r_{ij} \) represents the occurrence of a (gray level, local average) pair of the quantized image.

**Step 4.** For each \((t, s), 0 \leq t < L^{2/3}, 0 \leq s < L^{2/3}\), calculate the a posteriori probabilities

\[
p_{0ij}(t, s) = \frac{r_{ij}}{\sum_{i=0}^{L-1} \sum_{k=0}^{L-1} r_{ik}}, \quad p_{1ij}(t, s) = \frac{r_{ij}}{\sum_{i=1}^{L-1} \sum_{k=1}^{L-1} r_{ik}}
\]

and the quantized entropies

\[
H_0'(t, s) = -\sum_{j=0}^{L^{2/3}-1} \sum_{i=0}^{L^{2/3}-1} p_{0ij}(t, s) \log p_{0ij}(t, s)
\]
\[
H_1'(t, s) = -\sum_{j=1}^{L^{2/3}-1} \sum_{i=1}^{L^{2/3}-1} p_{1ij}(t, s) \log p_{1ij}(t, s).
\]

**Step 5.** The quantized threshold vector \((T', S')\) is the one which satisfies

\[
H'(T', S') = \max \left( \min_{(t, s) \in C} \{ H_0'(t, s), H_1'(t, s) \} \right)
\]

The set of candidate threshold vectors is

\[
C = \{(T, S) \mid (T'_1 - 1)L^{1/3} \leq T < (T'_1 + 2)L^{1/3};
\]
\[
(S'_1 - 1)L^{1/3} \leq S < (S'_1 + 2)L^{1/3}\}.
\]

**Phase 2: [Thresholding]**

**Step 1.** Calculate the local average, \(g(x, y)\), of the original image over an \(n \times n\) squared window as shown in equation (2).

**Step 2.** Draw the 2D histogram, \(F\), of the original image such that \(F = \{ r_{ij} \mid 0 \leq i < L, 0 \leq j < L \} \) where \( r_{ij} \) represents the occurrence of a (gray level, local average) pair.

**Step 3.** For each \((t, s) \in C\), calculate the a posteriori probabilities

\[
p_{0ij}(t, s) = \frac{r_{ij}}{\sum_{i=0}^{L-1} \sum_{k=0}^{L-1} r_{ik}}, \quad p_{1ij}(t, s) = \frac{r_{ij}}{\sum_{i=1}^{L-1} \sum_{k=1}^{L-1} r_{ik}}
\]

and the entropies

\[
H_0(t, s) = -\sum_{j=0}^{L-1} \sum_{i=0}^{L-1} p_{0ij}(t, s) \log p_{0ij}(t, s)
\]
\[
H_1(t, s) = -\sum_{j=1}^{L-1} \sum_{i=1}^{L-1} p_{1ij}(t, s) \log p_{1ij}(t, s).
\]

**Step 4.** The global threshold vector \((T, S)\) is the one which satisfies

\[
H(T, S) = \max \left( \min_{(t, s) \in C} \{ H_0(t, s), H_1(t, s) \} \right)
\]

In phase one, a set of candidate threshold vectors is estimated by applying Brink’s method to a quantized image of the original. Figure 2 shows the employed uniform quantizer which equally divides the range of original gray levels into \(L^{2/3}\) segments. Since each gray level of the quantized image represents a set of \(L^{1/3}\) continuous gray levels of the original image, the quantized threshold vector \((T', S')\), obtained in step 5, represents a region of \(L^{2/3}\) pairs in the original histogram. As shown in Fig. 3, the candidate threshold vectors are chosen as those \((t, s)\) pairs which locate in the nine regions centered by the region that \((T', S')\) represents. In phase two, the global threshold vector is obtained by applying Brink’s method only to the \(9L^{2/3}\) candidates.

Some may argue the validity of the above technique to select candidates. Actually, it is difficult to prove using the mathematical model because of the use of the “maximin” optimization procedure in the method. However, according to our experiments, the global threshold vector usually locates in the set of candidates. Figure 4 shows the 1D histograms of some images and
Fig. 4. The comparison of 1D histograms of some images (left) with those of the quantized images (right).
their quantized images. Each histogram and its quantized one are almost the same in their shapes. Since the rationale of the entropic thresholding technique is based on the probability distribution of an image, the similarity between the two histograms implies that the quantized threshold vector is approximate to the global one. The 2D histogram of a quantized image is also similar to that of the original. Table 1 shows that the global threshold vector of each image locates in either the original region represented by \((T', S')\) or its neighbors. It is worth discussing what is the best number of quantized levels we have to choose in phase one. Theorem 1 proves that the computation complexity of the algorithm will be bounded by \(O(L^{8/3})\) by quantizing into \(L^{2/3}\) levels.

**Theorem 1.** The optimal computation complexity of the two-phase 2D entropic thresholding algorithm is \(O(L^{8/3})\) by quantizing the gray level into \(L^{2/3}\) levels.

**Proof.** Assume the number of gray levels of the quantized image, which will lead to the optimal computation complexity, is \(L^{2/3}\). There are \(L^{2/3} \times L^{2/3}\) elements in the 2D histogram of the quantized image. Since the exhaustive searching technique is applied to the quantized image in phase one, the computation time of phase one is \(O(L^{2/3})\).

In phase two, the maximum optimization procedure is applied only on candidate threshold vectors of the original image. Since the number of both the gray level and the local average of candidate threshold vectors is \(3L^{2/3} / p^2\), there are \(9L^{2/3} / p\) candidates. The computation time to calculate entropies for each candidate is \(O(L^2)\). The computation complexity of phase two is \(O(L^{2 + 2(p - q)/p})\). Thus, the computation complexity of the algorithm is

\[
T_{2DET} = \min_{n} \left( \max_{n} \left( O(L^{4/3}), O(L^{2 + 2(p - q)/p}) \right) \right). \tag{9}
\]

The optimal value of \(T_{2DET}\) is obtained when the two computation times are equal, i.e.

\[
\frac{4q}{p} = 2 + \frac{2(p - q)}{p}. \tag{10}
\]

Any pair \((p, q)\) of natural numbers which satisfies \(2p = 3q\) will satisfy equation (10). Thus, there are \(L^{2/3}\) gray levels of the quantized image in phase one and the computation complexity of the algorithm is bounded by \(O(L^{8/3})\).

### 4. RESULTS AND CONCLUSIONS

The fast 2D entropic thresholding algorithm has been experimentally applied to some images stored in a SUN SPARC 2 workstation. These images consist of \(256 \times 256\) pixels each with their gray levels from 0 to 255. Since the optimal quantized level \(L^{2/3}\) is not an integer, it is given by the nearest integer power of two, 32, for convenience. Each gray level of the quantized image represents eight original gray levels. The local averages are calculated over a \(3 \times 3\) squared window for both the quantized images and their originals.

The images after thresholding and their originals are shown in Fig. 5 for comparison. In order to exhibit the unclassified pixels in off-diagonal quadrants in Fig. 1, they are assigned as the gray level threshold \(T\) and are represented as the gray areas in the resulting images. The black and white regions in these results correspond to those in quadrants 0 and 1. Table 2 lists the results from the two phases of our algorithm and those of the exhaustive searching method. Each threshold vector obtained from phase two definitely or nearly falls in the original region represented by the quantized threshold vector from phase one. The threshold vectors obtained from our algorithm are the same as those from the exhaustive searching.

In our experiments, the processing time for 2D entropic thresholding is greatly reduced. As shown in Table 3, it takes more than 2 h to obtain the threshold vector by exhaustive searching while our algorithm only takes about 2 min.

Memory space required by our algorithm is also greatly reduced. To calculate the a posteriori entropies, a matrix of the a posteriori probabilities for the object and background classes has to be used. Since the matrix is an \(L \times L\) floating point array, if the number of gray levels is 256, there must be at least \(4 \times 64\) K bytes of memory space to store this array. This makes it difficult to implement the 2D entropic thresholding method in the PC environment for the maximum offset of an array in a turbo C compiler is 64 K. In phase one of our algorithm, the number of elements of the quantized...
Fig. 5. Some experimental results of the fast 2D entropic thresholding algorithm, those on the left are original images and those on the right are images after thresholding: (a) Baboon; (b) Barbara; (c) Boat; (d) Lenna; (e) Peppers; (f) Text.
Fig. 5 (Continued)
Table 2. The comparison between threshold vectors of the fast 2D entropic thresholding algorithm and those of the exhaustive searching algorithm

<table>
<thead>
<tr>
<th>Images</th>
<th>Quantized thresholds</th>
<th>Corresponding original regions</th>
<th>Phase 1</th>
<th>Phase 2</th>
<th>Exhaustive searching</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baboon</td>
<td>(13,13)</td>
<td>(104–111,104–111)</td>
<td>(119,110)</td>
<td>(119,110)</td>
<td></td>
</tr>
<tr>
<td>Barbara</td>
<td>(15,15)</td>
<td>(120–127,120–127)</td>
<td>(126,126)</td>
<td>(126,126)</td>
<td></td>
</tr>
<tr>
<td>Boat</td>
<td>(12,13)</td>
<td>(96–103,104–111)</td>
<td>(107,110)</td>
<td>(107,110)</td>
<td></td>
</tr>
<tr>
<td>Lenna</td>
<td>(13,12)</td>
<td>(104–111,96–103)</td>
<td>(107,94 )</td>
<td>(107,94 )</td>
<td></td>
</tr>
<tr>
<td>Peppers</td>
<td>(13,13)</td>
<td>(104–111,104–111)</td>
<td>(109,110)</td>
<td>(109,110)</td>
<td></td>
</tr>
<tr>
<td>Texts</td>
<td>(5,5)</td>
<td>(40–47,40–47)</td>
<td>(43,43 )</td>
<td>(43,43 )</td>
<td></td>
</tr>
</tbody>
</table>

Table 3. The computation times (in seconds) of the fast 2D entropic thresholding algorithm are far less than those of the exhaustive searching algorithm

<table>
<thead>
<tr>
<th>Images</th>
<th>Computation times of fast algorithm</th>
<th>Computation times of exhaustive searching</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Phase 1</td>
<td>Phase 2</td>
</tr>
<tr>
<td>Baboon</td>
<td>10</td>
<td>111</td>
</tr>
<tr>
<td>Barbara</td>
<td>9</td>
<td>115</td>
</tr>
<tr>
<td>Boat</td>
<td>10</td>
<td>106</td>
</tr>
<tr>
<td>Lenna</td>
<td>10</td>
<td>98</td>
</tr>
<tr>
<td>Peppers</td>
<td>11</td>
<td>99</td>
</tr>
<tr>
<td>Texts</td>
<td>12</td>
<td>93</td>
</tr>
</tbody>
</table>

histogram is reduced to $L^{2/3} \times L^{2/3}$, i.e. $4 \times 4$ K for $L=256$. As Brink's method is applied only to the candidates in phase two, the entropies of areas A and B in Fig. 3 are fixed and are carried from phase one for calculation. Although the number of elements of the histogram is $L \times L$, the space for the floating point array is far less than $L^{2/3} \times L^{2/3}$ in phase two.

According to the experimental results, our algorithm is feasible for most of the images although some may consider that the two classes of an improperly controlled image will get merged owing to invoking the quantization technique to estimate candidate threshold vectors. Both the computation time and the memory space are greatly reduced. The quality of thresholding for image segmentation is also maintained.

REFERENCES

in 1984, received the Outstanding Achievement Award from the Ministry of Defense in 1987, and the Technology Innovation Award from the Sun Yat-Sen Academics and Culture Foundation in 1989, and was a recipient of the Distinguished Research Award from the National Science Council in 1990. From 1984 to 1985, he was elected as an IEEE Distinguished Visitor in region 10. Since 1988, he has been a member of the Science and Technology Advisory Board of the Ministry of Education, Republic of China. He is currently the Secretary General of the Hsinchu Section, the Chinese Institute of Engineers and on the Board of Directors of the IEEE Taipei Section. Dr Chen is a senior member of IEEE and a member of the Association for Computing Machinery, the Chinese Institute of Engineers, and Phi Tau Phi.

About the Author—CHIA-HSIEH WEN was born in Taipei, Taiwan, on 2 August 1954. He received the B.S. degree in computer sciences from TamKang University, Taiwan, Republic of China, in 1976, and the M.S. degree in applied mathematics from National Tsing Hua University, in 1978. He is currently a Ph.D. degree candidate at the National Tsing Hua University. From 1979 to 1982 he worked at Taipei Veterans General Hospital, Taiwan, as a system analyst. Since 1982, he has been a director of Computer Center, Taichung Veterans General Hospital, Taiwan, Republic of China. His research interests include image coding, biomedical image processing, and biomedical expert systems.

About the Author—CHIN-WEN YANG was born in Taiwan, on 30 July 1964. He received the B.S. degree in information engineering from Feng Chia University, Taiwan, Republic of China, in 1987, and the M.S. degree in information engineering from National Cheng–Kung University in 1989. He is currently a Ph.D. degree candidate of National Cheng–Kung University. Since 1989, he has been a system programmer at the Computer Center, Taichung Veterans General Hospital, Taiwan, Republic of China. His research interests include computer networking and biomedical image processing.