Nonlinear sliding mode high-gain observers for fault estimation

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A robust high gain observer for state and unknown inputs/faults estimations for a special class of nonlinear systems is developed in this article. Ensuring the observability of the faults/unknown inputs with respect to the outputs, the faults can be estimated from the sliding surface. Under a Lipschitz condition for the nonlinear part, the high gain observers are designed under some regularity assumptions. In the sliding mode, the convergence of the estimation error dynamics is proven similar to the analysis of high-gain observers.

Keywords: sliding mode observer; high-gain observer; fault estimation; nonlinear observer; robot-arm; fault reconstruction

1. Introduction

Sliding mode control has been an effective approach in handling disturbances and modelling uncertainties through the concepts of sliding surface design and equivalent control (Utkin 1992). Based on the same concept, sliding mode observers (SMO) have been developed to robustly estimate the system states (Utkin 1992; Edwards, Spurgeon, and Patton 2000; Xiong and Saif 2001; Koshkouei and Zinober 2004; Veluvolu, Soh, and Cao 2007; Bandyopadhyay and Thakar 2008; Veluvolu and Soh 2009; Pai 2010; Jafarov 2011). The Lyapunov based approach of Walcott and Zak (1987) considered the problems of state observation in the presence of bounded uncertainties/unknown inputs based on a matching condition. The approach in Edwards and Spurgeon (1994) and Edwards et al. (2000) extended the design of Utkin (1992) to linear systems such that the states affected by the unknown inputs are dealt with by the switching terms. The reconstruction of unknown inputs/faults from equivalent control was also discussed in the same work.

Observer based fault detection schemes rely on estimation of outputs from the measurements with the observer in order to detect the fault (Patton and Chen 1994; Chen, Patton, and Zhang 1996; Patton 1997; Edwards and Spurgeon 1998; Edwards et al. 2000). So development of robust estimation is very important for state monitoring and efficient control of the uncertain nonlinear systems. High-gain observers (HGO) (Deza, Busvelle, Gauthier, and Rakotopara 1992; Gauthier, Hammouri, and Othman 1992; Gauthier and Kupka 1994) have been proposed for a general class of single output systems that is uniformly observable. The approach was generalised to a more general class of nonlinear systems in Deza, Bossanne, Busvelle, Gauthier, and Rakotopara (1993) and Busawon, Farza, and Hammouri (1998). Further, a robust HGO with sliding mode was developed for nonlinear state and unknown input estimations (Veluvolu and Soh 2009). However, most of the HGO designs rely on the nonlinear transformation to obtain the form viable for the design of high-gain observer. The gain design relies on inverse of the Jacobian of the state transformation. In practical applications, computation of Lie derivatives for systems of high-order complicates the observer design and singularities may exist in the inverse of the Jacobian matrix. To overcome these aspects, in Busawon et al. (1998), a constant gain observer is proposed for a special class of nonlinear systems that does not require the nonlinear transformation. Khalil and co-workers (Suengrohk and Khalil 1997; Dabroom and Khalil 1999; Atassi and Khalil 2000; Mahmoud and Khalil 2002) further explored the use of HGO for feedback control, numerical differentiation and sampled-data control.

In this article, we consider a special class of nonlinear systems (Busawon et al. 1998) in the presence of faults/unknown inputs. The design methodology follows the combination of high gain observer and the sliding mode observer (Veluvolu and Soh 2009; Veluvolu and Lee 2011). The considered class of systems does not require any state transformation for the design of the observer. Instead of de-coupling the fault signals, the proposed method deals with the fault
signals directly and so the estimation of the faults becomes feasible parallel with the state estimation. The convergence of the error dynamics of the state estimation is proven similar to the analysis of high-gain observer. The proposed method does not require the matching condition on the faults distribution vector.

The rest of this article is organised as follows: Section 2 presents the system description and background results. Section 3 presents the design of robust nonlinear observer that incorporates a sliding mode observer. In Section 4, the convergence analysis and the design of sliding mode gain are discussed. Section 5 presents the estimation of the faults from the sliding mode. An illustrative example is presented in Section 6. Section 7 concludes this article. Throughout this article, \( \lambda_{\text{max}}(A) \) denotes the maximum eigenvalue of a symmetric matrix \( A \), \( ||A|| \) denotes the 2-norm \( \sqrt{\lambda_{\text{max}}(A^T A)} \) of a matrix \( A \), and \( \sigma(A) \) denotes the condition number \( \sqrt{\lambda_{\text{max}}(A)} / \sqrt{\lambda_{\text{min}}(A)} \) of the matrix.

2. System description and background results

2.1. System description

In this article, the following class of uncertain nonlinear systems are considered for design of the robust sliding mode observer:

\[
\begin{align*}
\dot{x}_1 &= \alpha_1(s, y)x_1 + \gamma_1(s, x, u) + p_1(s, x, u)f_1(t) \\
\dot{x}_2 &= \alpha_2(s, y)x_2 + \gamma_2(s, x, u) + p_2(s, x, u)f_2(t) \\
\cdots \\
\dot{x}_q &= \alpha_q(s, y)x_q + \gamma_q(s, x, u) + p_q(s, x, u)f_q(t) \\
\dot{y}_1 &= C_1x_1 \\
\dot{y}_2 &= C_2x_2 \\
\cdots \\
\dot{y}_q &= C_qx_q
\end{align*}
\]

(1)

where \( \dot{x} \triangleq \begin{bmatrix} x_1 & x_2 & \cdots & x_q \end{bmatrix}^T \in M \), a compact subset \( M \subset \mathbb{R}^n \), \( x_i \triangleq \begin{bmatrix} x_{i1} & x_{i2} & \cdots & x_{in} \end{bmatrix}^T \in \mathbb{R}^{n_i}, i = 1, \ldots, q \), with \( \sum_{i=1}^{q} n_i = n; \quad y \triangleq \begin{bmatrix} y_1 & y_2 & \cdots & y_q \end{bmatrix}^T \). When global properties are considered, notions are simplified by assuming that \( M \) accepts a global coordinate system.

The \((n \times n_1)\)-dimensional matrix \( \alpha(s, y) \) and the \((1 \times n_1)\)-dimensional matrix \( C_i \), respectively of the form:

\[
\begin{bmatrix}
0 & \alpha_1(s, y) & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \alpha_{i,n-1}(s, y) & 0 \\
0 & \cdots & 0 & 0
\end{bmatrix},
\]

\[
\begin{bmatrix}
1 & 0 & \cdots & 0
\end{bmatrix}
\]

\( \alpha(s, y) \), \( \gamma(s, x, u) \), \( p(s, x, u) \) are the known nonlinear functions; \( u \) represents a class of bounded inputs; \( s \) is the known signal, may represent an output injection or part of the inputs that are differentiable; faults or the unknown inputs \( f(t) \) can be a function of \( f(s, x, u) \) is upper bounded with \( f_j \).

The functions \( \gamma_i = (\gamma_i, \ldots, \gamma_{im}) \), \( p_i = (p_i, \ldots, p_{im}) \), \( i = 1, \ldots, q \) are such that for every \( k = 1, \ldots, q \):

\[
\frac{\partial \gamma_i}{\partial x_k}(s, x, u) = 0, \quad \frac{\partial p_i}{\partial x_k}(s, x, u) = 0,
\]

\( i > j; \quad 1 \leq l \leq n_k; \quad j = 1, \ldots, n_l \) (2)

The nonlinearities of each subsystem are coupled in triangular or hierarchical fashion with state variables of the other subsystems.

System (1) can be written in the condensed form as

\[
\begin{bmatrix}
\dot{x} \\
y
\end{bmatrix} = \begin{bmatrix}
\begin{bmatrix}
0 & \alpha_1(s, y) & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \alpha_{i,n-1}(s, y) & 0 \\
0 & \cdots & 0 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_q
\end{bmatrix} + \begin{bmatrix}
\gamma_1(s, x, u) \\
\gamma_2(s, x, u) \\
\ddots \\
\gamma_q(s, x, u)
\end{bmatrix}
\end{bmatrix} + \begin{bmatrix}
p_1(s, x, u) \\
p_2(s, x, u) \\
\ddots \\
p_q(s, x, u)
\end{bmatrix} \begin{bmatrix}
f_1(t) \\
f_2(t) \\
\ddots \\
f_q(t)
\end{bmatrix}
\]

(3)

where \( \alpha(s, y) = \text{diag}[\alpha_1(s, y), \ldots, \alpha_q(s, y)] \), \( p(s, x, u) = \text{diag}[p_1(s, x, u), \ldots, p_q(s, x, u)] \), \( C = \text{diag}[C_1, \ldots, C_q] \) and \( \gamma(s, x, u) = [\gamma_1(s, x, u), \ldots, \gamma_q(s, x, u)]^T, \quad f = [f_1(t), \ldots, f_q(t)]^T. \)

The system satisfies the following assumptions:

Assumption 2.1: There exists a class \( \mathcal{U} \) of bounded admissible controls, a compact set \( M \subset \mathbb{R}^n \) and two positive constants \( \beta_{ij}, \beta_2 \) such that for every \( u \in \mathcal{U} \) and every output \( y(t) \) associated to \( u \) and to an initial state \( x(0) \in M \), we have: \( 0 < \beta_{ij} \leq |\alpha_i(s, y)| \leq \beta_2; \quad i = 1, \ldots, q \) and \( j = 1, \ldots, n_i - 1 \).

Assumption 2.2: \( s(t) \) and its time derivative \( ds/dt \) are bounded.

Assumption 2.3: The functions \( \alpha_i(s, y), \quad i = 1, \ldots, q, \quad j = 1, \ldots, n_i - 1 \) belongs to class \( C^r, r \geq 1 \), w.r.t. their arguments.

Assumption 2.4: The functions \( \gamma_i(s, x, u), \quad i = 1, \ldots, q \) are global Lipschitz functions w.r.t. \( x \) uniformly in \( u \) and \( s \) with \( p_i(s, x, u) = 1 \).

Assumption 2.5: The distribution vector \( p(s, x, u) \) is bounded with respect to its arguments.

Remark 1: Assumptions 2.1 and 2.4 can be conservative, but they characterise a system that is uniformly observable for any bounded input. It has been proven in Gauthier et al. (1992) that Assumption 2.1 is a sufficient condition, but not necessary, to ensure uniform observability for any input. The triangular structure in Assumption 2.1 is necessary for the equivalent control based sliding mode observers (Barbot, Boukhobza, and Djemai 1996; Xiong and Saif 2001) to facilitate successive evaluation of higher-order derivative terms from the measurable functions.
estimation error. Assumption 2.5 is necessary for the development of robust sliding mode observer to deal with the fault estimation.

For ease of analysis, we define the following matrices and deduce the equalities.

- We denote $\Gamma(s, y)$: $i = 1, \ldots, q$ the diagonal matrix is defined by

\[
\Gamma_i(s, y) = \begin{bmatrix}
C_i \\
C_i \alpha_i(s, y) \\
\vdots \\
C_i \alpha_i^{n-1}(s, y)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 \\
\alpha_1(s, y) \\
\alpha_1(s, y)\alpha_2(s, y) \\
\vdots \\
\prod_{j=1}^{n-1} \alpha_j(s, y)
\end{bmatrix}
\]

- We denote by $S_{\theta}$ the unique solution of the Lyapunov equation

\[
\theta S_{\theta} + A_i^T S_{\theta} + S_{\theta} A_i - C_i^T C_i = 0
\]

where

\[
A_i = \begin{bmatrix}
0 & I_{(n-1) \times (n-1)} \\
0 & I_{n}
\end{bmatrix}
\]

and $\theta > 0$ is a positive parameter. The explicit solution of (4) can be obtained as

\[
S_{\theta}(m, j) = \frac{(-1)^{m+j} C_{m+j-2}}{\theta^{m+j-1}}, \quad 1 < m, j < n
\]

where

\[
C_n = \frac{n!}{(n-r)!r!}
\]

Furthermore, $S_{\theta}$ is symmetric positive definite (SPD) for every $\theta > 0$ (Gauthier et al. 1992).

- Finally, we form the $\Gamma(s, y)$ and $S_{\theta}$ as block diagonal matrices given by

\[
\Gamma(s, y) = \text{diag}[\Gamma_1(s, y), \ldots, \Gamma_q(s, y)]
\]

\[
S_{\theta} = \text{diag}[S_{\theta 1}, \ldots, S_{\theta q}]
\]

- We define $\Delta_{\theta}$ as a diagonal matrix and deduce the following equalities:

\[
\Delta_{\theta} = \text{diag}\left(1, \frac{1}{\theta}, \ldots, \frac{1}{\theta^{n-1}}\right); \quad i = 1, \ldots, q,
\]

\[
\Delta_{\theta} = \text{diag}(\Delta_{\theta 1}, \Delta_{\theta 2}, \ldots, \Delta_{\theta q})
\]

3. High-gain observer with sliding mode observer

For system (3) satisfying Assumptions 2.1–2.5, the robust nonlinear estimator with a sliding mode observer is of the form

\[
\dot{x}_i = \alpha_i(s, y)\dot{x}_i + \gamma_i(s, \dot{x}, u) + L_i(y_i - C_i \dot{x}_i) + p_i(s, \dot{x}, u)\nu_i
\]

In the above equation,

\[
L_i = \begin{bmatrix}
l_{i1} \\
l_{i2} \\
\vdots \\
l_{im}
\end{bmatrix}^T = \Gamma_i^{-1}(s, y)S_{\theta 1}^{-1} C_i^T
\]

is the feedback estimation gain based on high-gain observer, and the design of the scalar-valued robust term $\nu_i$ is based on sliding mode theory, and is given as

\[
\nu_i = \rho_i \text{sign}(y_i - C_i \dot{x}_i)
\]

for $i = 1, \ldots, q$. The sliding mode estimation gain $\rho_i$ will be discussed in Lemma 4.2.

For simplicity, the observer can be written in the condensed form as

\[
\dot{x} = \alpha(s, y)\dot{x} + \gamma(s, \dot{x}, u) + \Gamma^{-1}(s, y)S_{\theta}^{-1} C^T(y - C\dot{x})
\]

\[
+ p(s, \dot{x}, u)\nu
\]

where

\[
\nu = \begin{bmatrix}
\nu_1 \\
\nu_2 \\
\vdots \\
\nu_q
\end{bmatrix}^T
\]

It is clear that the proposed robust observer copies the form of the nonlinear system with appended feedback correction term and switching terms (robust terms). The feedback gain ensures the exponential convergence where as the robust terms deals with the faults/unknown inputs. The switching terms serves as ‘tracking elements’ for the faults occurring which can be reconstructed from the sliding mode. The convergence analysis and the design of feedback gain and sliding mode gain will be discussed later.

4. Convergence analysis

The convergence of the proposed estimator is discussed in this section. For ease of analysis and organisation, the analysis is divided into three separate sections. The boundedness of the estimation error is first proved
in Lemma 4.1. With the established boundedness of error, the design of sliding mode gain \( p \) and the error convergence to the sliding surface are then addressed in Lemma 4.2. With the sliding mode established, the asymptotic stability of the estimation error in the sliding mode is then proved in Theorem 4.3.

### 4.1. Boundedness of error dynamics

For the error dynamics

\[ e = \begin{bmatrix} e_1 & e_2 & \cdots & e_q \end{bmatrix}^T \triangleq \hat{x} - x \]

with \( e_i = [e_{i1} \ e_{i2} \ \cdots \ e_{i\omega}]^T \triangleq \hat{x}_i - x_i \) for \( i = 1, \ldots, q \), it can be obtained from (1) and (9) that

\[ \hat{e} = (\alpha(s, y) - \Gamma^{-1}(s, y)S_0^{-1}C^TC) e + \gamma(\hat{x}, u, s) - \gamma(x, u, s) + p(s, \hat{x}, u)v_i - p(s, x, u)f_i \]

(14)

For the \( i \)th system we have

\[ \dot{e}_i = (\alpha_i(s, y) - \Gamma_i^{-1}(s, y)S_0^{-1}C_i^TC_i) e_i + \gamma_i(\hat{x}, u, s) - \gamma_i(x, u, s) + p_i(s, \hat{x}, u)v_i - p_i(s, x, u)f_i \]

(15)

#### Lemma 4.1: Consider system (1) satisfying Assumptions 2.1–2.5. For the estimator (9), there exists \( \theta_0 > 0 \) such that \( \forall \theta > \theta_0 \ \forall u \in U, \forall x(0) \in K \). The error dynamics \( e \) remains bounded such that

\[ \|e\| \leq \frac{2q \delta_1 \lambda_{\max}(S) b_p}{\theta - \theta_0} \]

(16)

where \( \delta_1 = \sup_{\omega \geq 0, x \in K} \{\|\gamma(s(t), Cx)\|\} \), \( b_p = \sup_i b_{p_i} \times (\rho_i + f_i) \) and \( b_p \leq \|p(s, x, u)\| \).

**Proof:** Set \( \xi_i = \Gamma_i(s, y)\Delta_i e, \xi = \Gamma(s, y)\Delta e \). Since \( \Delta_i \) and \( \Gamma_i(s, y) \) are diagonal matrices, they commute with each other and their inverses. Using the equalities in (8), it can be obtained that

\[ \dot{\xi}_i = \theta(A_i - S_0^{-1}C_i^TC_i)\xi_i + \Gamma_i(s, y)\Delta_i \]

\[ \times \left[ \gamma_i(s, \hat{x}, u) - \gamma_i(s, x, u) \right] + \Gamma_i(s, y)\Delta_i \]

\[ \times \left[ p_i(s, \hat{x}, u)v_i - p_i(s, x, u)f_i(t) \right] + \dot{\Gamma}_i(s, y)\Gamma_i^{-1}(s, y)\xi_i \]

(17)

Using the results in (8) and with the Lyapunov function \( V_i = \xi_i^T S_0 \xi_i \), it can be evaluated that

\[ V_i \leq -\theta \xi_i^T S_0 \xi_i - \|C_i\xi_i\|^2 + 2\xi_i^T S_0 \Gamma_i(s, y)\Delta_i \]

\[ \times \left[ \gamma_i(s, \hat{x}, u) - \gamma_i(s, x, u) \right] + 2\xi_i^T S_0 \Gamma_i(s, y)\Delta_i \]

\[ \times \left[ p_i(s, \hat{x}, u)v_i - p_i(s, x, u)f_i(t) \right] + 2\xi_i^T S_0 \dot{\Gamma}_i(s, y)\Gamma_i^{-1}(s, y)\xi_i \]

(18)

Together with the Lipschitz assumption in Assumption 2.4 and triangular structure on \( \gamma_i \), and assuming \( \theta \geq 1 \), it can be evaluated that

\[ \|\Delta_i \left[ \gamma(s, \hat{x}, u) - \gamma(s, x, u) \right] \| \]

\[ \leq \|\Delta_i \left[ \gamma(s, \hat{x}, u) - \gamma(s, x, u) \right] \| \]

\[ \leq \|D_i \| \sum_{i=1}^n \|\Delta_i e\| \]

(20)

where \( D_i = \sup_{\omega \geq 0, x \in K} \|\gamma^{-1}(s(t), Cx)\| \). Under Assumptions 2.4 and 2.5, \( p(s, x, u) \) is a Lipschitz function and bounded for some upper bound \( b_p \). Similar to (21), due to the triangular structure of \( p(s, x, u) \), it can be obtained that

\[ \|\Delta_i \left[ p_i(s, \hat{x}, u) - p_i(s, x, u) \right] \| \leq nD_p \|\xi_i\| \]

(22)

Using the above inequalities, the derivative of Lyapunov function can be evaluated as

\[ \dot{V}_i \leq -\theta V_i + 2n\delta_1 \rho_i \|S_i \xi_i\| \|\xi_i\| + 2n\delta_2 \rho_i \|S_i \xi_i\| \|\rho_i\| \rho_i \]

\[ + 2\delta_1 \|S_i \xi_i\| b_p \|\rho_i + \tilde{f}_i\| + 2\delta_1 \|S_i \xi_i\| b_p \|\rho_i + \tilde{f}_i\| \]

(24)

where \( \delta_1 = \sup_{\omega \geq 0, x \in K} \{\|\gamma(s(t), Cx)\|\} \) and \( \delta_2 = \sup_{\omega \geq 0, x \in K} \{\|\dot{\Gamma}_i(s(t), Cx)\|\} \). Then, we can obtain

\[ V = \sum_{i=1}^q \dot{V}_i \leq -\theta V + 2q\rho_1 \|S\| b_p \|\xi\| + 2q\delta_2 \rho_1 \|S\| \rho_1 \]

\[ + 2q\delta_1 \lambda_{\max}(S)b_p \|\xi\| + 2q\delta_2 \rho_1 \|S\| \rho_1 \]

\[ - \left[ \theta - 2q\rho_1 \|S\| b_p \rho_1 - 2q\delta_2 \rho_1 \|S\| \rho_1 \right] V + 2q\delta_1 \lambda_{\max}(S)b_p \|\xi\| \]

(25)

\[ = -c_1 \|\xi\|^2 + c_2 \|\xi\| \]

(26)
where \( \sigma(S_1) = \sup \sigma(S_1), \) \( \rho = \sup \rho \), \( b_p = \sup b_p \), and

\[
e_i = 0
\]

(30)

for \( i = 1, \ldots, q \), the aim is to design the sliding mode estimation as (11) to reach and maintain the sliding modes.

(b) To ensure that the term \([v_i - f_i(t)]\) of the estimation error dynamics (15) in the sliding mode \( e_i = 0 \), i.e., a zero dynamics, can be substituted by an increment of Lipschitzian function through an equivalent control signal, so that the asymptotic convergence of \( e \) can be proved.

Lemma 4.2 and Theorem 4.3 are devoted to the above mentioned points (a) and (b), respectively.

**Lemma 4.2:** For system (1) satisfying Assumptions 2.1–2.5 and the estimator (9), the sliding mode estimation (11) ensures that the sliding surface \( e_i = 0 \) can be reached and maintained provided there exists \( \hat{\theta}_i > 0 \) such that \( \theta > \hat{\theta}_i \) and the sliding mode gain satisfies

\[
\rho_i > \eta_i + \beta_2 e_i 2 \max + f_i
\]

(31)

where \( \|e_2\| \leq e_2 \max \), \( |\alpha_i(s, y)| \leq \beta_2 \), \( |f_i(t)| \leq f_i \) and \( \eta_i > 0 \) is a small positive quantity.

**Proof:** The first dynamics \( e_i \) from (15) can be obtained as follows:

\[
\dot{e}_i = \begin{align*}
\dot{e}_i &= \alpha_i(s, y)e_i + l_i e_i + [\gamma_i(s, \dot{x}, u) - \gamma_i(s, x, u)] \\
&+ v_i - f_i(t)
\end{align*}
\]

For the Lyapunov function \( V_i = \frac{1}{2} e_i^2 \), using the above and the sliding mode estimation (11) it can be evaluated that

\[
\dot{V}_i = e_i \dot{e}_i
\]

\[
= -l_i e_i^2 + e_i [\gamma_i(s, \dot{x}, u) - \gamma_i(s, x, u)] \\
+ [\alpha_i(s, y)e_i + f_i(t)] - \eta_i |e_i|
\]

Under the condition in Lemma 4.1 that ensures boundedness of \( e_i \), and together with the boundedness of \( \alpha_i(s, y) \) in Assumption 2.4, there exists a finite constant gain satisfying (31) such that, if \( e_i \neq 0 \), one has

\[
\dot{V}_i < -l_i e_i^2 + e_i [\gamma_i(s, \dot{x}, u) - \gamma_i(s, x, u)] - \eta_i e_i
\]

Under the Lipschitzian condition in Assumption 2.4 and the boundedness of the input, it can be obtained that

\[
\dot{V}_i < -l_i e_i^2 + e_i [\gamma_i(s, \dot{x}, u) - \gamma_i(s, x, u)] - \eta_i e_i
\]

where \( L_{n_i} \) is the Lipschitz constant of \( \gamma_i(t) \). Due to triangular structure in (2), the subsystems converge sequentially in the sliding modes of \( e_1, \ldots, e_{q(i-1)} \), and eventually we have \( e \equiv [0 \cdots 0, e_1]^T \). Therefore

\[
\dot{V}_i < -l_i e_i^2 + L_{n_i} |e_i| - \eta_i e_i
\]

From the design of high gain, \( l_i \) of \( L_i \) from (10) can be evaluated as \( l_i = n_i \theta \) where \( n_i \) is the order of the system. By selecting \( \theta \) such that

\[
\theta > \frac{l_{n_i}}{n_i} = \hat{\theta}_i
\]

ensures that \( l_i > l_{n_i} \). Without loss of generality, we have \( l_i = \xi_i + l_{n_i} \) where \( \xi_i > 0 \) is a small positive quantity. Hence, it can be shown that

\[
\dot{V}_i < -2 \xi_i V_i - \sqrt{2} \eta_i \sqrt{V_i}
\]

Since \( \xi_i V_i > 0 \), it implies that

\[
\dot{V}_i < -\eta_i \sqrt{2} V_i
\]

Integrating (32) implies that the time taken to reach the sliding surface \( e_1 = 0 \) denoted by \( t_s \) satisfies

\[
t_s \leq -\sqrt{2} n_i^{-1} \sqrt{V_i(e_1(0))}
\]

(33)

where \( e_1(0) \) represents the initial value of \( e_1(t) \) at \( t = 0 \). Hence, the robust term (11) using the gain (31) ensures that the sliding surface \( e_1 = 0 \) can be reached in a finite time and maintained thereafter.

**Remark 1:** The boundedness of \( e \) is established in Lemma 4.1 and is dependent on \( \theta \). As \( \theta \) is an
independent parameter, one can choose arbitrarily to reduce the bound of $e$ in (29). This bound can be used for the calculation of sliding mode gain $\rho$.

4.3. Error dynamics in the sliding mode

Since the estimator design (9) using the robust term (11) ensures the sliding mode, it is only required to examine the convergence of the dynamics of $e$ during the sliding mode. In the sliding mode when $e_i = 0$ and $\dot{e}_i = 0$, the equivalent control of $v_i$ can be obtained from (32) as in Utkin (1992):

$$ (v_i)_{eq} = f(t) - \alpha_i(s,y)e_i^2 $$

Substituting the above equivalent control (34) into (15), the estimation error dynamics in the sliding mode of $e_i = 0$ can be obtained as

$$ \dot{e}_i = (\alpha_i(s,y) - \Gamma_i^{-1}(s,y)S_{\theta_i}(C_i)\Gamma_i + \gamma_i(\dot{x},u,s) - \gamma_i(x,u,s) + (p_i(s,\dot{x},u) - p_i(s,\dot{x},u))(v_i)_{eq} + p_i(s,\dot{x},u)(v_i)_{eq} = f(t) - \alpha_i(s,y)e_i^2 $$

$$ = \left(\alpha_i(s,y) - \Gamma_i^{-1}(s,y)S_{\theta_i}(C_i)\Gamma_i + \gamma_i(\dot{x},u,s) - \gamma_i(x,u,s) + (p_i(s,\dot{x},u) - p_i(s,\dot{x},u))(v_i)_{eq} + p_i(s,\dot{x},u)\alpha_i(s,y)e_i^2 \right) $$

The equivalent control in the sliding mode clearly cancels the fault dynamics effect in the state estimation. The following theorem proves the asymptotic stability of the estimation error.

**Theorem 4.3:** Assumptions that system (1) satisfies Assumptions 2.1–2.5. For the estimator (9) with the robust term (11) and the sliding mode gain (31), there exists $\theta_2 > 0$ such that $\forall \theta > \theta_2$, the estimation error is asymptotically stable in the sliding mode of $e_i = 0$.

**Proof:** The proof follows a similar pattern as in Lemma 4.1. Since $e_{2i}$, $f(t)$ and $\alpha_i(s,y)$ are bounded, then according to (34), $(v_i)_{eq} \leq \bar{v}$ for some upper bound $\bar{v}$. Similar to (22), we have $\|\Delta p_i(s,x,u)\alpha_i(s,y)\| \leq b_{\beta_i} \beta_2 \|\xi\| \leq b_{\beta_i} \beta_2 \|\xi\|$. With Lyapunov function $V_i = \xi^T_i S_i \xi_i$, it can be evaluated with (36) similar to (24):

$$ \dot{V}_i \leq -\theta V_i + 2n\delta l_i \delta_1 \|S_{\theta_i}\| \|\xi\| + 2n\delta l_i \delta_1 \|S_{\theta_i}\| \|\xi\| \bar{v}_i + 2\delta_1 \|S_{\theta_i}\| \|\xi\| b_{\beta_i} \beta_2 \delta_1 + 2\delta_2 \|S_{\theta_i}\| \|\xi\| $$

Then, we can obtain

$$ \dot{\bar{V}} = \sum_{i=1}^{q} \dot{V}_i \leq -\left[2\theta - 2q\sigma(S_{\theta_i}) \delta l_i \delta_1 - 2q\delta l_i \delta_1 \sigma(S_{\theta_i}) \bar{v} \right. $$

$$ - 2q\delta l_i \delta_2 \beta_1 \sigma(S_{\theta_i}) - 2q\delta_2 \sigma(S_{\theta_i}) \right]\bar{V} $$

where $\bar{v} = \sup_i(\bar{v}_i)$, $\beta_2 = \sup_i(\beta_2)$ and $b_{\beta_i} = \sup_i(b_{\beta_i})$. Defining

$$ \theta_2 = 2q\sigma(S_{\theta_i}) \delta l_i \delta_1 + 2q\delta l_i \delta_1 \sigma(S_{\theta_i}) \bar{v} + 2q\delta l_i \delta_2 \beta_1 \sigma(S_{\theta_i}) + 2q\delta_2 \sigma(S_{\theta_i}) $$

and selecting $\theta > \theta_2$, it can be shown that $\dot{V} < 0$. Hence, the robust estimator guarantees asymptotic convergence of the estimation error to zero.

**Remark 2:** To avoid the switching caused by the sign($\cdot$) function, the sign function $\text{sign}(\cdot)$ is approximated by a saturation function

$$ \text{sat}(\cdot, \varepsilon) \triangleq \begin{cases} \frac{\varepsilon}{\varepsilon} & \text{if } |\cdot| \leq \varepsilon \\ \text{sign}() & \text{if } |\cdot| > \varepsilon \end{cases} $$

where $\varepsilon > 0$ is the threshold. By replacing the $\text{sign}(\cdot)$ function with $\text{sat}(\cdot)$, the error dynamics finally settle to a bound inside the boundary layer rather than converging to zero in the sliding plane. Decreasing the boundary layer thickness $\varepsilon$ in $\text{sat}(\cdot)$ increases the accuracy in estimation.

5. Robust fault estimation from sliding mode

Once all trajectories reaches their respective sliding modes, all the states converge to the true states. Therefore

$$ e_{2i} \approx 0 $$

The equivalent control $(v_i)_{eq}$ information can then be used to reconstruct the fault. From (34), the equivalent control can be approximated as

$$ (v_i)_{eq} \approx f(t) $$

The use of a low-pass filter for recovering the equivalent control signal was given by Utkin (1992). Continuous approximation of equivalent injection signal by using a small positive scalar $\delta_{\varepsilon}$ was also implemented in Edwards et al. (2000). Similar to the analysis of Edwards et al. (2000), in the proposed approach, faults can be estimated from equivalent control as follows:

$$ \dot{\hat{F}}(t) \approx (\rho, \text{sign}(e_{2i}))_{eq} \approx \rho \frac{e_{2i}}{|e_{2i}| + \delta_{\varepsilon}} $$

The fault estimation only depends on the measurement error $e_{2i}$ and hence can be performed online with state estimation as given below:

$$ \dot{x}_i = \alpha_i(s,y) \hat{x}_i + \gamma_i(s,\dot{x},u) + \Gamma_i^{-1}(s,y)S_{\theta_i}C_i^T(y_i - C\hat{x}_i) + p_i(s,\dot{x},u)v_i $$

(22)
with the robust term \( v_i = \rho_i \text{sign}(y_i - C_i \hat{x}_i) \) and the fault estimation

\[
\hat{f}_i(t) \approx \rho_i \frac{y_i - C_i \hat{x}_i}{(|y_i - C_i \hat{x}_i| + \delta_c)} \tag{43}
\]

for \( i = 1, \ldots, q \). The accuracy of the disturbance estimation will depend on \( \delta_c \). \( \varepsilon \) discussed in Remark 2 is employed in state observer to reduce the chattering effect and is not related to the constant \( \delta_c \) involved in fault reconstruction. The state observer error finally settles to a bound in the boundary layer \( \varepsilon \). As the error lies inside the boundary layer, this allows for estimation of the fault signal from the output error dynamics with \( \delta_c \).

6. Application to flexible joint robot arm

We consider the laboratory model of a single-link flexible joint robot (Raghavan and Hedrick 1994; Rajamani and Cho 1998; Koshkouei and Zinober 2004) shown in Figure 1 which is defined by the following nonlinear system equations:

\[
\begin{align*}
\dot{\theta}_m &= \omega_m \\
\dot{\omega}_m &= \frac{k}{J_m} (\theta_l - \theta_m) - \frac{S}{J_m} \omega_m + \frac{K_r}{J_m} u \\
\dot{\theta}_l &= \omega_l \\
\dot{\omega}_l &= -k \frac{J_l}{J_l} (\theta_l - \theta_m) - \frac{mgb}{J_l} \sin(\theta_l)
\end{align*}
\]  

\tag{44}

where \( \theta_m \) and \( \theta_l \) are the angular rotations of the motor and the link respectively, with \( \omega_m \) and \( \omega_l \) being their angular velocities, \( J_m \) represents the inertia of the dc motor, \( J_l \) the inertia of the controlled link, \( k \) the elastic constant, \( m \) the link mass, \( u \) the input motor torque, \( g \) the acceleration due to gravity, \( h \) the centre of mass, \( K_r \) the amplifier gain and \( S \) the viscous friction coefficient.

In order to evaluate the effectiveness of the proposed method, we consider the following perturbations with the nominal model rewritten in the following form with \( \mathbf{x} = [x_{11} \ x_{12} \ x_{21} \ x_{22}]^T = [\theta_m \ \omega_m \ \theta_l \ \omega_l]^T \):

\[
\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/J_m & -B/J_m & k/J_m & 0 \\ 0 & 0 & 0 & 1 \\ k/J_l & 0 & -k/J_l & 0 \end{bmatrix} \mathbf{x} \\
+ \begin{bmatrix} 0 \\ \frac{K_r u}{J_m} \\ 0 \\ -\frac{mgb}{J_l} \sin(x_{21}) \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0.3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}
\]  

\tag{45}

where \( f_1 \) and \( f_2 \) denote the faults. The angular rotations of motor and link, \( x_{11} \) and \( x_{21} \) are measurable as outputs. The faults \( f_1, f_2 \) and their distribution matrix are selected for illustration purpose and to highlight the effectiveness of the proposed method in reconstruction. \( f_1 \) can be considered as a fault in the input channel, whereas \( f_2 \) can be treated as an external disturbance.

Consequently, system (45) is of the form (1) (here \( n_1 = n_2 = q = 2 \)), where

\[
\begin{align*}
\mathbf{x}_1 &= \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}, \ & \mathbf{x}_2 = \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}, \\
\gamma_1(s, \mathbf{x}, \mathbf{u}) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
\mathbf{y}_1(s, \mathbf{x}, \mathbf{u}) &= \begin{bmatrix} 0 \end{bmatrix}, \\
\gamma_2(s, \mathbf{x}, \mathbf{u}) &= \frac{k/J_m x_{11} - k/J_m x_{21} - \frac{mgb}{J_l} \sin(x_{21})}{J_m} \\
\mathbf{p}_1 &= \begin{bmatrix} 1 \\ 0.3 \end{bmatrix}, \ & \mathbf{p}_2 = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}, \\
\mathbf{y} &= \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}, \ & \mathbf{C}_1 = \mathbf{C}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\
\mathbf{p}(s, \hat{x}, \mathbf{u}) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\mathbf{C}_1 \mathbf{p}(s, \hat{x}, \mathbf{u}) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \Gamma_1(s, \mathbf{y})
\end{align*}
\]

It can be easily verified that the system satisfies all the assumptions. Hence, an observer of the form (42)–(43) can be designed for the system (45) and is given by:

\[
\begin{align*}
\dot{\mathbf{x}} &= \mathbf{a}(s, \mathbf{y}) \dot{\mathbf{x}} + \gamma(s, \hat{x}, \mathbf{u}) + \mathbf{p}(s, \hat{x}, \mathbf{u})^T \mathbf{y} \\
&+ \mathbf{p}(s, \hat{x}, \mathbf{u})^T \mathbf{y} \\
&+ \mathbf{p}(s, \hat{x}, \mathbf{u})^T \mathbf{y} \\
&+ \mathbf{p}(s, \hat{x}, \mathbf{u})^T \mathbf{y} \tag{46}
\end{align*}
\]
The sliding mode gain is designed according to (11) as
\[
\Gamma(s, y) = \begin{bmatrix} I_2 & 0 \\ 0 & I_2 \end{bmatrix}, \quad S_{10} = S_{20} = \begin{bmatrix} 1 & -1/s^2 \\ -1/s^2 & 2/s^3 \end{bmatrix}.
\]
\[
S_{10}^{-1} = \begin{bmatrix} 2\theta & \theta^2 \\ \theta^2 & \theta^3 \end{bmatrix} \text{ (Since } S_{ij}(m, j) = \frac{(-1)^{m+j} C_{m+j-2}}{\theta^{m+j-1}}, \text{ we have } S_{ii}(1, 1) = \frac{(-1)^2 C_0^0}{\theta^3} = \frac{1}{\theta})
\]

The sliding mode gain is designed according to (11) as
\[
v_1 = \rho_1 \text{sign}(y_1 - C_1 \hat{x}_1), \quad v_2 = \rho_2 \text{sign}(y_2 - C_2 \hat{x}_2)
\]

The fault signals can be estimated from the robust terms similar to (41) as follows:
\[
\hat{f}_1(t) \approx \rho_1 \frac{e_{11}}{|e_{11}| + \delta_e}, \\
\hat{f}_2(t) \approx \rho_1 \frac{e_{21}}{|e_{21}| + \delta_e}
\]

where \(\hat{f}_1\) and \(\hat{f}_2\) are the estimates of unknown inputs obtained through multiple sliding modes.

### 6.1 Simulation results

For the simulation results, we considered the following values (Rajamani and Cho 1998): \(J_m = 0.0037 \text{ kg m}^2\), \(J_f = 0.0093 \text{ kg m}^2\), \(m = 0.21 \text{ kg}\), \(2b = 0.3 \text{ m}\), \(k = 0.18 \text{ Nm/rad}\), \(B = 0.0083 \text{ Nm/V}\), \(K_1 = 0.08 \text{ Nm/V}\), and \(g = 9.81 \text{ m/s}^2\). A constant input torque \(u = 0.01 \text{ Nm}\) is applied to the system.

The sliding mode gains are chosen to be \(\rho_1 = \rho_2 = 12\). The initial conditions for plant and estimator are set as \(x(0) = [1 \ 3 \ 1 \ 2]\) and \(\hat{x}(0) = [0 \ 0 \ 0 \ 0]\). For the high-gain observer, we choose \(\theta = 12\). The fault signals for the system are selected as \(f_1(t) = 9 \times \text{square}(0.3\pi t)\) and \(f_2(t) = -9 \times \text{square}(0.2\pi t)\). The boundary layer thickness for implementation of saturation function to reduce the chattering effect in state observer is selected to be \(\epsilon = 0.01\). For approximation of fault signals, we choose \(\delta_e = 0.03\).
Figure 2 shows the tracking performance in the presence of fault signal in the channel $f_1(t)$. States which are measured quickly track the actual trajectories, states $\hat{x}_{11}$, $\hat{x}_{21}$ converge early in the sliding modes of $e_{11} = 0$ and $e_{22} = 0$ as shown in Figure 2(a) and (c). The robust estimates $\hat{x}_{12}$, $\hat{x}_{22}$ converge to the actual states after both $\hat{x}_{11}$ and $\hat{x}_{21}$ have converged are shown in Figure 2(b) and (d). As the trajectories track the true states, the robust term track the fault signal accurately. The fault signal $f_1(t)$ is reconstructed from the sliding mode is shown in Figure 3(a). For the fault signal $f_2(t)$ in the second channel, the reconstruction is shown in Figure 3(b). Higher magnitude of faults are selected in order to highlight the effectiveness of the proposed method.

7. Conclusions
A robust sliding mode high-gain observer design is developed for a special class of nonlinear systems. The sliding surface uses only the measurable outputs estimation error, and the faults are estimated from the measurable sliding surfaces. The gain design is based on HGO that guarantees exponential convergence of the estimation error in the presence of the faults.

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Note
1. diag[ ] represents the block-diagonal of the matrix.

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