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FREE FIELD APPROACH TO D-BRANES IN GEPNER MODELS.

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Abstract

We represent free field construction of boundary states in Gepner models basing on the free field realization of N=2 superconformal minimal models. Using this construction we consider the open string spectrum between the boundary states and show that it can be described as chiral de Rham complex of the LG orbifold. It allows to establish direct relation of the open string spectrum for boundary states in Gepner models to the open string spectrum for fractional branes in LG orbifolds.

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0. Introduction

Dirichlet branes give realization of solitonic states in string theory which have Ramond-Ramond charges and play important role in non-perturbative aspects of the theory [1]. The description of D-branes on curverd string backgrounds in Calabi-Yau models of superstring compactification is one of the important problems of string theory.

At large volume the D branes can be described by classical geometric techniques of bundles on submanifolds of Calabi-Yau manifold. The extrapolation into the stringy regime usually requires boundary conformal field theory (CFT) methods. In this approach D-brane configurations are given by conformally invariant boundary states or boundary conditions in Gepner models of CFT and have been considered first by Recknagel and Schomerus [2].

In contrast to the large volume limit the quantum string generalization of geometric objects such as a manifold, or a submanifold for example, are developed in much less extent and most of the investigations devoted to the boundary state approach to D-branes are related to the problem of the string generalization of the geometry. The considerable progress in the understanding of the quantum geometry of D-branes at small volume of the Calabi-Yau manifold has been achieved mainly due to the works [2]- [14]. The idea developed in these papers is to relate the intersection index of boundary states with the bilinear form of the K-theory classes of bundles on the large volume Calabi-Yau manifold and use this relation to associate the K-theory classes to the boundary states establishing thereby the correspondence betwenn th boundary states and bundles on Calabi-Yau manifold. As a result very important relations between the boundary states in Gepner models, fractional branes on the LG orbifolds, the bundles on the singular manifolds and derived categories has been found.

In this paper we consider another aspect of boundary states in Gepner models coming from their $N = 2$ minimal models construction [15], [16] to find more direct relation of the geometric properties of D-branes with algebraic structures of BCFT description of D-branes.

$N = 2$ superconformal minimal models represent a subclass of rational CFT where the construction of the boundary states leaving a whole chiral symmetry algebra unbroken can be given in principle and the interaction of these states with closed strings can be calculated exactly. But in practice the calculation of closed string amplitudes in general CFT backgrounds is available only if the corresponding free field realization of the model is known. Therefore, it is important to extend free field approach to the case of rational models of CFT with a boundary. This problem has been treated recently in [17]- [19], where free field representations of $sl(2)$ Kac-Moody and $N = 2$ Virasoro algebra [20]- [25] has been used to boundary states construction and boundary correlation function calculation.

In this paper we extend the free field construction [18] of boundary states in $N = 2$ minimal models to the case of boundary states in Gepner models. In section 1 we review free field construction of irreducible representations in $N = 2$ minimal models developed by Feigin and Semikhatov. In section 2 the free field realization of Gepner models is briefly discussed. Section 3 is devoted to the explicit free field construction of Ishibashi states. In section 4 free field representation of A and B-type Recknagel Shomerus boundary states in Gepner models is given and the open string spectrum of states is calculated using free field realization of Ishibashi states. In section 5 we investigate the open string spectrum between the boundary states using the ideas of vertex operator algebra approach to the string theory on Calabi-Yau manifolds recently developed in [26], [28], [27], [29]. The detailed calculation of the open string spectrum is carried out in the simplest example of 1^3 Gepner model which appears to be quite representative to illustrate the idea how to to establish by the free fields the direct relation of the Gepner models boundary states to the fractional branes in LG orbifolds and toric geometry of Calabi-Yau manifolds. More detailed investigation of the geometry of boundary states and the relation with the results of [2]- [13] will be represented in the future publication.

1. Free-field realization of $N = 2$ minimal models irreducible representations.

In this section we briefly discuss free-field construction of Feigin and Semikhatov [25] of the irreducible modules in $N = 2$ superconformal minimal models. Free field approach to $N = 2$ minimal models considered also in [30]- [32].

1.1. Free-field representations of $N = 2$ super-Virasoro algebra.

We introduce (in the left-moving sector) the free bosonic fields $X(z), X^*(z)$ and free fermionic fields $\psi(z), \psi^*(z)$, so that its OPE's are given by

$$X^*(z_1)X(z_2) = \ln(z_{12}) + reg.,$$
$$\psi^*(z_1)\psi(z_2) = z_{12}^{-1} + reg, \tag{1}$$

where $z_{12} = z_1 - z_2$. Then for an arbitrary number μ the currents of $N = 2$ super-Virasoro algebra are given by

$$
\begin{aligned}
G^{+}(z)=\psi^{*}(z)\partial X(z)-\frac{1}{\mu}\partial\psi^{*}(z),\ G^{-}(z)=\psi(z)\partial X^{*}(z)-\partial\psi(z),\\
J(z)=\psi^{*}(z)\psi(z)+\frac{1}{\mu}\partial X^{*}(z)-\partial X(z),\\
T(z)=\partial X(z)\partial X^{*}(z)+\frac{1}{2}(\partial\psi^{*}(z)\psi(z)-\psi^{*}(z)\partial\psi(z))-\\
\frac{1}{2}(\partial^{2}X(z)+\frac{1}{\mu}\partial^{2}X^{*}(z)),
\end{aligned} \tag{2}
$$

and the central charge is

$$c=3(1-\frac{2}{\mu}). \tag{3}$$

As usual, the fermions in NS sector are expanded into half-integer modes:

$$
\begin{aligned}
\psi(z)=\sum_{r\in 1/2+Z}\psi[r]z^{-\frac{1}{2}-r},\ \psi^{*}(z)=\sum_{r\in 1/2+Z}\psi^{*}[r]z^{-\frac{1}{2}-r},\\
G^{\pm}(z)=\sum_{r\in 1/2+Z}G^{\pm}[r]z^{-\frac{3}{2}-r},
\end{aligned} \tag{4}
$$

and they are expanded into integer modes in R sector:

$$
\begin{aligned}
\psi(z)=\sum_{r\in Z}\psi[r]z^{-\frac{1}{2}-r},\ \psi^{*}(z)=\sum_{r\in Z}\psi^{*}[r]z^{-\frac{1}{2}-r},\\
G^{\pm}(z)=\sum_{r\in Z}G^{\pm}[r]z^{-\frac{3}{2}-r}.
\end{aligned} \tag{5}
$$

The bosons $X(z), X^{*}(z), J(z), T(z)$ are expanded in both sectors into integer modes:

$$
\begin{aligned}
\partial X(z)=\sum_{n\in Z}X[n]z^{-1-n},\ \partial X^{*}(z)=\sum_{n\in Z}X^{*}[n]z^{-1-n},\\
J(z)=\sum_{n\in Z}J[n]z^{-1-n},\ T(z)=\sum_{n\in Z}L[n]z^{-2-n}.
\end{aligned} \tag{6}
$$

In NS sector $N=2$ Virasoro superalgebra is acting naturaly in Fock module $F_{p,p^{*}}$ generated by the fermionic operators $\psi^{*}[r]$, $\psi[r]$, $r<\frac{1}{2}$, and bosonic operators $X^{*}[n]$, $X[n]$, $n<0$ from the vacuum state $|p,p^{*}>$ such that

$$
\begin{aligned}
\psi[r]|p,p^{*}>=\psi^{*}[r]|p,p^{*}>=0, r\geq\frac{1}{2},\\
X[n]|p,p^{*}>=X^{*}[n]|p,p^{*}>=0, n\geq 1,\\
X[0]|p,p^{*}>=p|p,p^{*}>,\ X^{*}[0]|p,p^{*}>=p^{*}|p,p^{*}>.
\end{aligned} \tag{7}
$$

It is a primary state with respect to the $N=2$ Virasoro algebra

$$
\begin{aligned}
G^{\pm}[r]|p,p^{*}>=0, r>0,\\
J[n]|p,p^{*}>=L[n]|p,p^{*}>=0, n>0,\\
J[0]|p,p^{*}>=\frac{j}{\mu}|p,p^{*}>=0,\\
L[0]|p,p^{*}>=\frac{h(h+2)-j^{2}}{4\mu}|p,p^{*}>=0,
\end{aligned} \tag{8}
$$

where $j = p^* - \mu p$, $h = p^* + \mu p$. The vacuum state $|p, p^* >$ corresponds to the vertex operator $V_{(p,p^*)}(z) \equiv \exp(pX^*(z) + p^*X(z))$ placed at $z = 0$.

It is easy to calculate the character $f_{p,p^*}(q, u)$ of the Fock module F_{p,p^*}. By the definition

$$f_{p,p^*}(q, u) = Tr_{F_{p,p^*}}(q^{L[0]-\frac{c}{24}} u^{J[0]}). \tag{9}$$

Thus we obtain

$$f_{p,p^*}(q, u) = q^{\frac{h(h+2)-j^2}{4\mu} - \frac{c}{24}} u^{\frac{j}{\mu}} \frac{\Theta(q, u)}{\eta(q)^3}, \tag{10}$$

where we have used the Jacobi theta-function

$$\Theta(q, u) = q^{\frac{1}{8}} \sum_{m \in Z} q^{\frac{1}{2}m^2} u^{-m} \tag{11}$$

and the Dedekind eta-function

$$\eta(q) = q^{\frac{1}{24}} \prod_{m=1} (1 - q^m). \tag{12}$$

The $N = 2$ Virasoro algebra has the following set of automorphisms which is known as spectral flow [33]

$$\begin{aligned} G^{\pm}[r] &\rightarrow G^{\pm}_t[r] \equiv G^{\pm}[r \pm t], \\ L[n] \rightarrow L_t[n] \equiv L[n] + tJ[n] + t^2\frac{c}{6}\delta_{n,0},\ J[n] &\rightarrow J_t[n] \equiv J[n] + t\frac{c}{3}\delta_{n,0}, \end{aligned} \tag{13}$$

where $t \in Z$. Note that spectral flow is intrinsic property of $N = 2$ super-Virasoro algebra and hence, it does not depend on a particular realization. Allowing in (13) t to be half-integer, we obtain the isomorphism between the NS and R sectors.

The spectral flow action on the free fields can be easily described if we bosonize fermions ψ^*, ψ

$$\psi(z) = \exp(-y(z)),\ \psi^*(z) = \exp(+y(z)). \tag{14}$$

and introduce spectral flow vertex operator

$$U^t(z) = \exp(-t(y + \frac{1}{\mu}X^* - X)(z)). \tag{15}$$

The following OPE's

$$\begin{aligned} \psi(z_1)U^t(z_2) &= z_{12}^t : \psi(z_1)U^t(z_2) :, \\ \psi^*(z_1)U^t(z_2) &= z_{12}^{-t} : \psi^*(z_1)U^t(z_2) :, \\ \partial X^*(z_1)U^t(z_2) &= z_{12}^{-1} tU^t(z_2) + r., \\ \partial X(z_1)U^t(z_2) &= -z_{12}^{-1}\frac{t}{\mu}U^t(z_2) + r. \end{aligned} \tag{16}$$

give the action of spectral flow on the modes of the free fields

$$\begin{aligned} \psi[r] \rightarrow \psi[r - t],&\ \psi^*[r] \rightarrow \psi^*[r + t], \\ X^*[n] \rightarrow X^*[n] + t\delta_{n,0},&\ X[n] \rightarrow X[n] - \frac{t}{\mu}\delta_{n,0}. \end{aligned} \tag{17}$$

The action of the spectral flow on the vertex operator $V_{(p,p^*)}(z)$ is given by the normal ordered product of the vertex $U^t(z)$ and $V_{p,p^*}(z)$. It follows from (17) that spectral flow generates twisted sectors.

1.2. Irreducible $N=2$ super-Virasoro representations and butterfly resolution.

The $N=2$ minimal models are characterized by the condition that μ is integer and $\mu \geq 2$. In NS sector the irreducible highest-weight modules, constituting the (left-moving) space of states of the minimal model, are unitary and labeled by two integers h, j, where $h = 0, ..., \mu-2$ and $j = -h, -h+2, ..., h$. The highest-weight vector $|h, j>$ of the module satisfies the conditions (which are similar to (8))

$$\begin{aligned} G^{\pm}[r]|h,j> &= 0, r > 0, \\ J[n]|h,j> = L[n]|h,j> &= 0, n > 0, \\ J[0]|h,j> &= \frac{j}{\mu}|h,j>, \\ L[0]|h,j> &= \frac{h(h+2)-j^2}{4\mu}|h,j>. \end{aligned} \tag{18}$$

If in addition to the conditions (18) the relation

$$G^{+}[-1/2]|h,j> = 0 \tag{19}$$

is satisfied we call the vector $|h,j>$ and the module $M_{h,j}$ chiral highest-weight vector (chiral primary state) and chiral module, correspondingly. In this case we have $h = j$. Analogously, anti-chiral highest-weight vector (anti-chiral primary state) and anti-chiral module can be defined if instead of (19)

$$G^{-}[-1/2]|h,j> = 0 \tag{20}$$

The Fock modules are highly reducible representations of $N=2$ Virasoro algebra. To see this we introduce following to [25] two fermionic screening currents $S^{\pm}(z)$ and the charges $Q^{\pm}$ of the currents

$$\begin{aligned} S^{+}(z) = \psi^{*}\exp(X^{*})(z),\ S^{-}(z) &= \psi\exp(\mu X)(z), \\ Q^{\pm} &= \oint dz S^{\pm}(z) \end{aligned} \tag{21}$$

These charges commute with the generators of $N=2$ super-Virasoro algebra (2). Moreover they are nilpotent and mutually anticommute

$$(Q^{+})^2 = (Q^{-})^2 = \{Q^{+}, Q^{-}\} = 0. \tag{22}$$

But they do not act within each Fock module. Instead they relate the different Fock modules. The space where the screening charges are acting can be constructed as follows. One has to introduce the two-dimensional lattice of the momentums:

$$\pi = \{(p,p^*)|p = \frac{n}{\mu}, p^* = m, n, m \in Z\} \tag{23}$$

and associate to this lattice the space

$$F_{\pi} = \oplus_{(p,p^*)\in\pi} F_{p,p^*}. \tag{24}$$

Due to the properties (22 one can combine the charges $Q^{\pm}$ into BRST operator acting in F_{π} and build a BRST complex consisting of Fock modules $F_{p,p^*} \in F_{\pi}$ such that its cohomology is given by NS sector $N = 2$ minimal model irreducible module $M_{h,j}$. This complex has been constructed in [25].

Let us consider first free field construction for the chiral module $M_{h,h}$. In this case the complex (which is known due to Feigin and Semikhatov as butterfly resolution) can be represented by the following diagram

$$
\begin{array}{cccccccc}
 & & \vdots & & \vdots & & & \\
 & & \uparrow & & \uparrow & & & \\
\ldots & \leftarrow & F_{1,h+\mu} & \leftarrow & F_{0,h+\mu} & & & \\
 & & \uparrow & & \uparrow & & & \\
\ldots & \leftarrow & F_{1,h} & \leftarrow & F_{0,h} & & & \\
 & & & & & \nwarrow & & \\
 & & & & & & F_{-1,h-\mu} \quad \leftarrow \quad F_{-2,h-\mu} & \leftarrow \quad \ldots \\
 & & & & & & \uparrow \qquad\qquad\qquad \uparrow & \\
 & & & & & & F_{-1,h-2\mu} \quad \leftarrow \quad F_{-2,h-2\mu} & \leftarrow \quad \ldots \\
 & & & & & & \uparrow \qquad\qquad\qquad \uparrow & \\
 & & & & & & \vdots \qquad\qquad\qquad \vdots &
\end{array}
\tag{25}
$$

We shall denote this resolution by C_h and denote by Γ the set where the momentums of the Fock spaces of the resolution take values. The horizontal arrows in this diagram are given by the action of Q^+ and vertical arrows are given by the action of Q^-. The diagonal arrow at the middle of butterfly resolution is given by the action of Q^+Q^- (which equals $-Q^-Q^+$ due to (22)). Ghost number operator g of this complex is defined for an arbitrary vector $|v_{n,m}>\in F_{n,m\mu+h}$ by

$$
\begin{aligned}
g|v_{n,m}> &= (n+m)|v_{n,m}>, \; if \; n,m \geq 0, \\
g|v_{n,m}> &= (n+m+1)|v_{n,m}>, \; if \; n,m < 0.
\end{aligned}
\tag{26}
$$

For an arbitrary vector of the complex $|v_N>$ with the ghost number N the differential d_N is defined by

$$
\begin{aligned}
d_N|v_N> &= (Q^+ + Q^-)|v_N>, \; if \; N \neq -1, \\
d_N|v_N> &= Q^+Q^-|v_N>, \; if \; N = -1.
\end{aligned}
\tag{27}
$$

and rises the ghost number by 1. Note that when $N = -1$ one has to take $d_N = Q^-Q^+ = -Q^+Q^-$ which does not affect the cohomology.

The main statement of [25] is that the complex (25) is exact except at the $F_{0,h}$ module, where the cohomology is given by the chiral module $M_{h,j=h}$.

The butterfly resolution allows to write the character $\chi_h(q,u) \equiv Tr_{M_{h,h}}(q^{L[0]-\frac{c}{24}}u^{J[0]})$ of the module $M_{h,h}$ as the Euler charcteristic of the complex:

$$
\begin{aligned}
\chi_h(q,u) &= \chi_h^{(l)}(q,u) - \chi_h^{(r)}(q,u), \\
\chi_h^{(l)}(q,u) &= \sum_{n,m\geq 0} (-1)^{n+m} f_{n,h+m\mu}(q,u), \\
\chi_h^{(r)}(q,u) &= \sum_{n,m>0} (-1)^{n+m} f_{-n,h-m\mu}(q,u),
\end{aligned}
\tag{28}
$$

where $\chi_h^{(l)}(q,u)$ and $\chi_h^{(r)}(q,u)$ are the characters of the left and right wings of the resolution.

To get the resolutions for other (anti-chiral and non-chiral) modules one can use the observation [25] that all irreducible modules can be obtained from the chiral modules $M_{h,j=h}$, $h=0,...,\mu-2$ by the spectral flow action $U^{-t}, t=h,h-1,...1$. Equivalently one can restrict the set of chiral modules by the range $h=0,...,[\frac{\mu}{2}]-1$ and extend the spectral flow action by $t=\mu-1,...,1$ (when μ is even and $h=[\frac{\mu}{2}]-1$ the spectral flow orbit becomes shorter: $t=[\frac{\mu}{2}]-1,...,1$) [34]. Thus the set of irreducible modules can be labeled also by the set $\{(h,t)|h=0,...,[\frac{\mu}{2}]-1,\ t=\mu-1,...,0\}$, except the case when μ is even and the spectral flow orbit becomes shorter. It turns out that one can get all the resolutions by the spectral flow action also. Indeed, the charges $Q^{\pm}$ commute with spectral flow operator U^t as it is easy to see from (15) and the corresponding OPE's, moreover, U^t is not BRST-exact ($U(z)$ corresponds to the anti-chiral primary field) hence, the resolutions in NS sector are generated from (25) by the operators U^{-t}, where $t=h,h-1,...,1$ or $t=\mu-1,...,1$ (or $t=[\frac{\mu}{2}]-1,...,1$ if μ is even and $h=[\frac{\mu}{2}]-1$).

Due to this discussion it is more convenient to change the notation for irreducible modules. In what follows we shall denote the irreducible modules as $M_{h,t}$, indicating by t spectral flow parameter.

As well as the modules and resolutions one can get the characters by the spectral flow action:

$$\chi_{h,t}(q,u)=q^{\frac{c}{6}t^2}u^{\frac{c}{3}t}\chi_h(q,uq^t). \tag{29}$$

There are following important automorphism properties of irreducible modules and characters [25], [34]. When μ is odd

$$M_{h,t}\equiv M_{h,t+\mu},\ \chi_{h,t+\mu}(q,u)=\chi_{h,t}(q,u). \tag{30}$$

When μ is even

$$\begin{aligned} M_{h,t}\equiv M_{h,t+\mu},\ \chi_{h,t+\mu}(q,u)=\chi_{h,t}(q,u),\ h\neq[\frac{\mu}{2}]-1,\\ M_{h,t}\equiv M_{h,t+[\frac{\mu}{2}]},\ \chi_{h,t+[\frac{\mu}{2}]}(q,u)=\chi_{h,t}(q,u),\ h=[\frac{\mu}{2}]-1. \end{aligned} \tag{31}$$

$$M_{h,t}\equiv M_{\mu-h-2,t-h-1},\ \chi_{h,t}(q,u)=\chi_{\mu-h-2,t-h-1}(q,u) \tag{32}$$

Note that the butterfly resolution is not periodic under the spectral flow as opposed to the characters. It is also not invariant with respect to the automorphysm (32). Instead, the periodicity and invariance are recovered on the level of chomology. Thus, $U^{\pm\mu}$ spectral flow and automorphysm (32) are the quasi-isomorphims of complexes.

The modules, resolutions and characters in R sector are generated from the modules, resolutions and characters in NS sector by the spectral flow operator $U^{-\frac{1}{2}}$.

2. Free field realization of Gepner model.

2.1.Free field realization of the product of minimal models.

It is easy to generalize the free field representation of the Sect.1. to the case of tensor product of r $N=2$ minimal models which can be characterized by r dimensional vector $\boldsymbol{\mu}=(\mu_1,...,\mu_r)$, where $\mu_i\geq 2$ and integer.

Let E be a real r dimensional vector space and let E^* be the dual space to E. Let us denote by $<,>$ the natural scalar product in the direct sum $E\oplus E^*$. In the subspaces E and E^* we fix the sets of basic vectors $\Re$ and $\Re^*$

$$\begin{aligned}\Re&=\{\mathbf{s}_i, i=1,...,r\},\\ \Re^*&=\{\mu_i\mathbf{s}^*_i, i=1,...,r\}\end{aligned}\tag{33}$$

which constitute an othonormal basis in $E\oplus E^*$.

According to the $\Re$ and $\Re^*$ we introduce (in the left-moving sector) the free bosonic fields $X_i(z), X^*_i(z)$ and free fermionic fields $\psi_i(z), \psi^*_i(z)$, $i=1,...,r$ so that its singular OPE's are given by (1) as well as the following fermionic screening currents and their charges

$$\begin{aligned}S^+_i(z)&=\mathbf{s}_i\psi^*\exp(\mathbf{s}_iX^*)(z),\\ S^-_i(z)&=\mathbf{s}^*_i\psi\exp(\mu_i\mathbf{s}^*_iX)(z),\\ Q^{\pm}_i&=\oint dzS^{\pm}_i(z).\end{aligned}\tag{34}$$

To each $i=1,...,r$ the pair of screenings (S^+_i, S^-_i) defines by the formulas (2) N=2 $c_i=3(1-\frac{2}{\mu_i})$ Virasoro superalgebra

$$\begin{aligned}G^+_i&=\mathbf{s}_i\psi^*\mathbf{s}^*_i\partial X-\frac{1}{\mu_i}\mathbf{s}_i\partial\psi^*,\ G^-_i=\mathbf{s}^*_i\psi\mathbf{s}_i\partial X^*-\mathbf{s}^*_i\partial\psi,\\ J_i&=(\mathbf{s}_i\psi^*\mathbf{s}^*_i\psi+\frac{1}{\mu_i}\mathbf{s}_i\partial X^*-\mathbf{s}^*_i\partial X),\\ T_i(z)&=(\frac{1}{2}(\mathbf{s}_i\partial\psi^*\mathbf{s}^*_i\psi-\mathbf{s}_i\psi^*\mathbf{s}^*_i\partial\psi)+\mathbf{s}_i\partial X^*\mathbf{s}^*_i\partial X-\\ &\frac{1}{2}(\mathbf{s}^*_i\partial^2X+\frac{1}{\mu_i}\mathbf{s}_i\partial^2X^*))\end{aligned}\tag{35}$$

as well as the vertex operators

$$V_{(p_i,p^*_i)}=\exp(p^*_i\mathbf{s}^*_iX+p_i\mathbf{s}_iX^*)),\tag{36}$$

which are the conformal fields:

$$\begin{aligned}G^+_i(z_1)V_{(p_i,p^*_i)}(z_2)&=z^{-1}_{12}p_i\mathbf{s}_i\psi^*V_{(p_i,p^*_i)}+r.,\ G^-_i(z_1)V_{(p_i,p^*_i)}(z_2)=z^{-1}_{12}p^*_i\mathbf{s}^*_i\psi V_{(p_i,p^*_i)}+r.,\\ J_i(z_1)V_{(p_i,p^*_i)}(z_2)&=z^{-1}_{12}\frac{1}{\mu_i}j_iV_{(p_i,p^*_i)}(z_2)+r,\\ T_i(z_1)V_{(p_i,p^*_i)}(z_2)&=z^{-2}_{12}\frac{1}{4\mu_i}(h_i(h_i+2)-j^2_i)V_{(p_i,p^*_i)}(z_2)+\\ &z^{-1}_{12}\partial V_{(p_i,p^*_i)}(z_2)+r.,\end{aligned}\tag{37}$$

where

$$h_i=p^*_i+\mu_ip_i,\ j_i=p^*_i-\mu_ip_i.\tag{38}$$

These vertex operators are naturally associated to the lattice $\Pi=P\oplus P^*\in E\oplus E^*$, where $P\in E, P^*\in E^*$ such that P is generated by $\mathbf{e}_i=\frac{1}{\mu_i}\mathbf{s}_i$ and P^* is generated by the basis $\mathbf{s}^*_i=\mathbf{e}^*_i$, $i=1,...,r$. For an arbitrary vector $(\mathbf{p},\mathbf{p}^*)\in\Pi$, we introduce in NS sector Fock vacuum state

$|\mathbf{p},\mathbf{p}^*>$ by the formulas similar to (7) and denote by $F_{\mathbf{p},\mathbf{p}^*}$ the Fock module generated from $|\mathbf{p},\mathbf{p}^*>$ by the creation operators of the fields $X_i(z), X_i^*(z), \psi_i(z), \psi_i^*(z)$.

Let F_Π be the direct sum of Fock modules associated to the lattice Π. As an obvious generalization of the results from Sec.1. we form for each vector $\mathbf{h}=\sum_i h_i\mathbf{s}_i^* \in P^*$, where $h_i=0,1,...,\mu_i-2$ butterfly resolution $C^\star_{\mathbf{h}}$ as the product $\otimes_{i=1}^3 C^\star_{h_i}$ of butterfly resolutions of minimal models. The corresponding ghost number operator g is given by the sum $\sum_{i=1}^3 g_i$ of ghost number operators of each of the resolutions and the differential ∂ acting on ghost number N subspace $C^N_{\mathbf{h}}$ of the resolution is given by the sum of differentials of each of the complexes $C^\star_{h_i}$ and maps this space in $C^{N+1}_{\mathbf{h}}$. It is obvious that the complex $C^\star_{\mathbf{h}}$ is exact except at the $F_{0,\mathbf{h}}$ module, where the cohomology is given by the product $M_{\mathbf{h},0}=\otimes_{i=1}^r M_{h_i,0}$ of the chiral modules of each minimal model. Hence one can represent the character

$$\chi_{\mathbf{h},0}(q,u)\equiv Tr_{M_{\mathbf{h},0}}(q^{L[0]-\frac{c}{24}})u^{J[0]}) \tag{39}$$

of $M_{\mathbf{h},0}$ as the product of characters $\chi_{\mathbf{h},0}(q,u)=\prod_i \chi_{h_i,0}(q,u)$.

According to the discussion at the end of Sec.1. we can obtain the resolution and character for the product of arbitrary irreducible modules of minimal models acting on $C^\star_{\mathbf{h}}$ by the spectral flow operators $U^{-\mathbf{t}}=\prod_i U_i^{-t_i}$ of the minimal models. Hence one can label the resolutions, modules and characters by the pairs of vectors $(\mathbf{h},\mathbf{t})$, from the set $\Delta'=\{(\mathbf{h},\mathbf{t})|h_i=0,...,[\frac{\mu_i}{2}]-1,\ t_i=0,...,\mu_i-1,\ i=1,...,r\}$. On the equal footing ane can use the set $\Delta=\{(\mathbf{h},\mathbf{t})|h_i=0,...,\mu_i-2,\ t_i=0,...,h_i,\ i=1,...,r\}$. Though Δ and Δ' give the equivalent set of irreducible modules (with respect to the direct sum of $N=2$ minimal models Virasoro algebras) their free field realization in these two sets is different due to different sets of spectral flow parameters.

It is also clear that R sector resolutions, modules and characters are generated from NS sector by the spectral flow $U^{-\mathbf{v}/2}=\prod_{i=1}^r U_i^{-1/2}$, where $\mathbf{v}=(1,...,1)$ is r-dimensional vector.

The same free field realization can be used in the right-moving sector. Thus the sets of screening vectors $\bar{\mathfrak{R}}$ and $\bar{\mathfrak{R}}^*$ have to be fixed in the right-moving sector. It can be done in many ways, the only restriction is that the corresponding cohomology group has to be isomorphic to the space of states of the product of minimal models in the right-moving sector. Therefore $\bar{\mathfrak{R}}$ and $\bar{\mathfrak{R}}^*$ is determined modulo $O(r,r)$ transformations which left unchanged the matrix of scalar products $<\mathbf{s}_i,\mathbf{s}_j^*>$. In what follows we put

$$\bar{\mathfrak{R}}=\mathfrak{R},\ \bar{\mathfrak{R}}^*=\mathfrak{R}^*. \tag{40}$$

Hence, one can use the same complex to describe the irreducible modules in the right-moving sector.

2.2.Free field realization and Calabi-Yau extension.

It is well known that product of minimal models can not be applied straightforward to describe the string theory on $2D$-dimensional Calabi-Yau manifold. First, one has to demand that $\sum_i c_i=3D$. Second, the so called simple current orbifold [35] of the product of minimal models has to be constructed. These orbifold, which is known as Calabi-Yau extension [16], [36], gives the space of states of $N=2$ superconformal sigma model on Calabi-Yau model. We denote this model by $CY_{\boldsymbol{\mu}}$. The currents of $N=2$ Virasoro superalgebra of this model are given by the sum of currents of each minimal model

$$G^\pm(z)=\sum_i G_i^\pm,$$
$$J(z)=\sum_i J_i,\ T(z)=\sum_i T_i. \tag{41}$$

The left-moving (as well as the right-moving) sector of the $CY_{\boldsymbol{\mu}}$ is given by projection of the space of states on the subspace of integer $J[0]$-charges and organizing the projected space into the orbits $[\mathbf{h},\mathbf{t}]$ under the spectral flow operator $U^{\mathbf{v}} = \prod_{i=1}^{r} U_i$, $\mathbf{v} = (1,1,...,1)$ is r-dimensional vector [36].

The partition function in NS sector of $CY_{\boldsymbol{\mu}}$ sigma model is diagonal modular invariant of the spectral flow orbits characters restricted to the subset of integer $J[0]$ charges. From $N=2$ Virasoro superalgebra representations there is no difference what of the sets Δ or Δ' we use to parametrize the orbit characters (though their free field realizations are different). In what follows we combine these to sets into one extended set $\tilde{\Delta} = \{(\mathbf{h},\mathbf{t})|h_i = 0,...,\mu_i - 2,\ t_i = 0,...,\mu_i - 1,\ i = 1,...,r\}$ which is 2^r times biger than Δ or Δ' and take into account this extention by corresponding multiplyer ("field identification") [15].

The orbit characters (with the restriction on integer charges subspace) can be written in explicit form so that the structure of simple current extension becomes clear [16], [36]:

$$
\begin{aligned}
ch_{\mathbf{h},\mathbf{t}}(q,u) = \frac{1}{\kappa^2}\sum_{n,m=0}^{\kappa-1} Tr_{M_{\mathbf{h},\mathbf{t}}}(U^{n\mathbf{v}} q^{(L[0]-\frac{c}{24})} u^{J[0]} \exp{(\imath 2\pi m J[0])} U^{-n\mathbf{v}}) = \\
\frac{1}{\kappa^2}\sum_{n,m=0}^{\kappa-1} \exp{(\imath 2\pi(\frac{cn^2}{6}\tau + \frac{cn}{3}\upsilon))}\chi_{\mathbf{h},\mathbf{t}}(\tau,\upsilon+n\tau+m) = \\
\frac{1}{\kappa^2}\sum_{n,m=0}^{\kappa-1}\chi_{\mathbf{h},\mathbf{t}+n\mathbf{v}}(\tau,\upsilon+m),
\end{aligned}
\tag{42}
$$

where $q = \exp{(\imath 2\pi\tau)}$, $u = \exp{(\imath 2\pi\upsilon)}$ and $\kappa = lcm\{\mu_i\}$. The partition function of $CY_{\boldsymbol{\mu}}$ sigma model is given by

$$
Z_{CY}(q,\bar{q}) = \frac{1}{2^r}\sum_{[\mathbf{h},\mathbf{t}]\in\Delta_{CY}} \kappa |ch_{[\mathbf{h},\mathbf{t}]}(q)|^2 = \sum_{(\mathbf{h},\mathbf{t})\in\Delta_{CY}}\sum_{n=0}^{\kappa-1}\chi_{\mathbf{h},\mathbf{t}}(q)\chi_{\mathbf{h},\mathbf{t}+n\mathbf{v}}(\bar{q}),
\tag{43}
$$

where Δ_{CY} denotes the subset of $\tilde{\Delta}$ restricted to the space of integer $J[0]$ charges. $[\mathbf{h},\mathbf{t}]$ denotes the spectral flow orbit of the point $(\mathbf{h},\mathbf{t})$. Factor $\frac{1}{2^r}$ corresponds to the extended set $\tilde{\Delta}$ of irreducible modules and κ is the length of the orbit $[\mathbf{h},\mathbf{t}]$. In general case the orbits with different lengths could appear but we will not consider these cases to escape the problem of fixed point resolution [35], [36], [10]. Modular invariance of the partition function is due to the following behaviour under modular transformation [36]

$$
ch_{[\mathbf{t},\mathbf{h}]}(-\frac{1}{\tau}) = \kappa \sum_{[\mathbf{h}',\mathbf{t}]\in\Delta_{CY}} S_{[\mathbf{h},\mathbf{t}],[\mathbf{h}',\mathbf{t}']} ch_{[\mathbf{t}',\mathbf{h}']}(\tau),
\tag{44}
$$

where the matrix $S_{[\mathbf{h},\mathbf{t}],[\mathbf{h}',\mathbf{t}']}$ is given by the product of modular S matrices of minimal models factors.

2.1.Free field realization of Gepner models.

The Gepner models [15] of Calabi-Yau superstring compactification are given by (generalized) GSO projection [15], [16] which is carrying out on the product of the space of states of $CY_{\boldsymbol{\mu}}$ σ-model and space of states of external fermions and bosons describing space-time degrees of freedom of the string. In the framework of simple current extension formalism the Gepner's construction has been farther developed in [36], [37], [35].

Let us introduce so called supersymmetrized (Green-Schwartz) characters [15], [16]

$$Ch_{[\mathbf{h},\mathbf{t}]}(q,u)=\frac{1}{4\kappa^2}\sum_{n,m=0}^{2\kappa-1}Tr_{(M_{\mathbf{h},\mathbf{t}}\otimes\Phi)}(U_{tot}^{\frac{m}{2}}\exp{(\imath\pi nJ_{tot}[0])}q^{(L_{tot}-\frac{c_{tot}}{24})}u^{J_{tot}[0]}U_{tot}^{\frac{-m}{2}}), \quad (45)$$

where the trace is calculated in the product of $M_{\mathbf{h},\mathbf{t}}$ and Fock module Φ generated by the external (space-time) fermions and bosons in NS sector, $J_{tot}[0]$ and $L_{tot}[0]$ are zero modes of the total $U(1)$ current and stress-energy tensor which includes the contributions from space-time degrees of freedom, $c_{tot}=c+\frac{3}{2}(8-2D)=12$ is a total central charge and U_{tot} is a total spectral flow operator acting in the product $M_{\mathbf{h},\mathbf{t}}\otimes\Phi$.

The modular invariant Gepner model partition function is given by [15], [16], [36]

$$Z_{Gep}(q,\bar{q})=\frac{1}{2^r}(Im\tau)^{-(4-N/2)}\sum_{[\mathbf{h},\mathbf{t}]\in\Delta_{CY}}\kappa|Ch_{[\mathbf{h},\mathbf{t}]}(q)|^2. \quad (46)$$

3. Free field represention for Ishibashi states in Gepner models.

3.1. Linear Ishibashi states in Fock modules.

The boundary states we are going to construct can be considered as a bilinear forms on the space of states of the model. Thus, it will be implied in what follows that the right-moving sector of the model is realized by the free-fields $\bar{X}_i(\bar{z}),\bar{X}^*_i(\bar{z}),\bar{\psi}_i(\bar{z}),\bar{\psi}^*_i(\bar{z}),\ i=1,...,r$ and the right-moving $N=2$ super-Virasoro algebra is given by the formulas similar to (2)

There are two types of boundary states preserving $N=2$ super-Virasoro algebra [38], usually called B-type

$$\begin{aligned}(L[n]-\bar{L}[-n])|B>>=(J[n]+\bar{J}[-n])|B>>=0,\\ (G^+[r]+\imath\eta\bar{G}^+[-r])|B>>=(G^-[r]+\imath\eta\bar{G}^-[-r])|B>>=0\end{aligned} \quad (47)$$

and A-type states

$$\begin{aligned}(L[n]-\bar{L}[-n])|A>>=(J[n]-\bar{J}[-n])|A>>=0,\\ (G^+[r]+\imath\eta\bar{G}^-[-r])|A>>=(G^-[r]+\imath\eta\bar{G}^+[-r])|A>>=0\end{aligned} \quad (48)$$

where $\eta=\pm1$.

In the tensor product of the left-moving Fock module $F_{\mathbf{p},\mathbf{p}^*}$ and right-moving Fock module $\bar{F}_{\bar{\mathbf{p}},\bar{\mathbf{p}}^*}$ we construct the most simple states fulfilling the solutions (47) and (48). We shall call these states as linear Ishibashi [39] states and denote by $|\mathbf{p},\mathbf{p}^*,\bar{\mathbf{p}},\bar{\mathbf{p}}^*,\eta,B(A)>>$.

We consider most simple B-type linear Ishibashi states in NS sector. They can be easily obtained from the following ansatz for fermions

$$\begin{aligned}(\psi^*_i[r]-\imath\eta\bar{\psi}^*_i[-r])|\mathbf{p},\mathbf{p}^*,\bar{\mathbf{p}},\bar{\mathbf{p}}^*,\eta,B>>=0,\\ (\psi_i[r]-\imath\eta\bar{\psi}_i[-r])|\mathbf{p},\mathbf{p}^*,\bar{\mathbf{p}},\bar{\mathbf{p}}^*,\eta,B>>=0.\end{aligned} \quad (49)$$

Substituting these relations into (47) and using (35), (41) we find

$$\begin{aligned}(X_k[n]+\bar{X}_k[-n]+d_k\delta_{n,0})|\mathbf{p},\mathbf{p}^*,\bar{\mathbf{p}},\bar{\mathbf{p}}^*,\eta,B>>=0,\\ (X^*_k[n]+\bar{X}^*_k[-n]+d^*_k\delta_{n,0})|\mathbf{p},\mathbf{p}^*,\bar{\mathbf{p}},\bar{\mathbf{p}}^*,\eta,B>>=0.\end{aligned} \quad (50)$$

It follows from these relation that we have B-type boundary conditions for each minimal model

$$(L_i[n]-\bar{L}_i[-n])|B>>=(J_i[n]+\bar{J}_i[-n])|B>>=0,$$
$$(G_i^{+}[r]+\imath\eta\bar{G}_i^{+}[-r])|B>>=(G_i^{-}[r]+\imath\eta\bar{G}_i^{-}[-r])|B>>=0. \quad (51)$$

The linear B-type Ishibashi state in NS sector is given by the standard expression [40], [41]

$$|\mathbf{p},\mathbf{p}^{*},\eta,B>>=\prod_{n=1}\exp(-\frac{1}{n}\sum_{i}(X_i^{*}[-n]\bar{X}_i[-n]+X_i[-n]\bar{X}_i^{*}[-n]))$$
$$\prod_{r=1/2}\exp(\imath\eta\sum_{i}(\psi_i^{*}[-r]\bar{\psi}_i[-r]+\psi_i[-r]\bar{\psi}_i^{*}[-r]))|\mathbf{p},\mathbf{p}^{*},-\mathbf{p}-\mathbf{d},-\mathbf{p}^{*}-\mathbf{d}^{*}>. \quad (52)$$

The closed string transition amplitude between such states in NS sector is given by

$$<<\mathbf{p}_2,\mathbf{p}_2^{*},\eta,B|q^{L[0]-c/24}u^{J[0]}|\mathbf{p}_1,\mathbf{p}_1^{*},\eta,B>>=$$
$$\delta(\mathbf{p}_1-\mathbf{p}_2)\delta(\mathbf{p}_1^{*}-\mathbf{p}_2^{*})q^{\frac{1}{2}<\mathbf{p}_1+\mathbf{p}_1^{*},\mathbf{p}_1+\mathbf{p}_1^{*}+\mathbf{d}+\mathbf{d}^{*}>-\frac{c}{24}}u^{<\mathbf{p}_1+\mathbf{p}_1^{*},\mathbf{d}-\mathbf{d}^{*}>}$$
$$\prod_{m=1}(1+uq^{m-\frac{1}{2}})^{r}(1+u^{-1}q^{m-\frac{1}{2}})^{r}(1-q^{m})^{-2r}. \quad (53)$$

Note that the state $<<\mathbf{p},\mathbf{p}^{*},\eta,B|$ is defined in such a way to satisfy conjugate boundary conditions and to take into account the charge asymmetry [42]- [44] of the free-field realization of each minimal model.

The linear A-type Ishibashi states can be found analogously. The linear ansatz for fermions has the form

$$(\psi_i^{*}[r]-\imath\eta\mu_i\bar{\psi}_i[-r])|\mathbf{p},\mathbf{p}^{*},\bar{\mathbf{p}},\bar{\mathbf{p}}^{*},\eta,A>>=0,$$
$$(\psi_i[r]-\frac{\imath\eta}{\mu_i}\bar{\psi}_i^{*}[-r])|\mathbf{p},\mathbf{p}^{*},\bar{\mathbf{p}},\bar{\mathbf{p}}^{*},\eta,A>>=0. \quad (54)$$

Substituting these relations into (48) and using (2) we find

$$(\mu_k X_k[n]+\bar{X}_k^{*}[-n]+d_k^{*}\delta_{n,0})|\mathbf{p},\mathbf{p}^{*},\bar{\mathbf{p}},\bar{\mathbf{p}}^{*},\eta,A>>=0,$$
$$(\frac{1}{\mu_k}X_k^{*}[n]+\bar{X}_k[-n]+d_k\delta_{n,0})|\mathbf{p},\mathbf{p}^{*},\bar{\mathbf{p}},\bar{\mathbf{p}}^{*},\eta,A>>=0. \quad (55)$$

Therefore we have A-type boundary conditions for each minimal model

$$(L_i[n]-\bar{L}_i[-n])|A>>=(J_i[n]-\bar{J}_i[-n])|A>>=0,$$
$$(G_i^{+}[r]+\imath\eta\bar{G}_i^{-}[-r])|A>>=(G_i^{-}[r]+\imath\eta\bar{G}_i^{+}[-r])|A>>=0. \quad (56)$$

The linear A-type Ishibashi state in NS sector is given similar to B-type

$$|\mathbf{p},\mathbf{p}^{*},\eta,A>>=\prod_{n=1}\exp(-\frac{1}{n}\sum_{i}(\mu_i X_i[-n]\bar{X}_i[-n]+\frac{1}{\mu_i}X_i^{*}[-n]\bar{X}_i^{*}[-n]))$$
$$\prod_{r=1/2}\exp(\imath\eta\sum_{i}(\mu_i\psi_i[-r]\bar{\psi}_i[-r]+\frac{1}{\mu_i}\psi_i^{*}[-r]\bar{\psi}_i^{*}[-r]))|\mathbf{p},\mathbf{p}^{*},-\Omega^{-1}\mathbf{p}^{*}-\mathbf{d},-\Omega\mathbf{p}-\mathbf{d}^{*}>, \quad (57)$$

where we have introduced the matrix $\Omega_{ij}=\mu_i\delta_{ij}$.

The closed string transition amplitude between A-type Ishibashi states in NS sector is given by

$$<<\mathbf{p}_2,\mathbf{p}_2^*,\eta,A|q^{L[0]-c/24}u^{J[0]}|\mathbf{p}_1,\mathbf{p}_1^*,\eta,A>>=$$
$$\delta(\mathbf{p}_1-\mathbf{p}_2)\delta(\mathbf{p}_1^*-\mathbf{p}_2^*)q^{\frac{1}{2}<\mathbf{p}_1+\mathbf{p}_1^*,\mathbf{p}_1+\mathbf{p}_1^*+\mathbf{d}+\mathbf{d}^*>-\frac{c}{24}}u^{<\mathbf{p}_1+\mathbf{p}_1^*,\mathbf{d}-\mathbf{d}^*>}$$
$$\prod_{m=1}(1+uq^{m-\frac{1}{2}})^r(1+u^{-1}q^{m-\frac{1}{2}})^r(1-q^m)^{-2r}. \quad (58)$$

Note that linear Fock space Ishibshi states (52), (57) conserve the Virasoro algebra of each minimal model and hence they are related to the tensor product of Ishibashi states of each minimal model. Some comment on the boundary conditions (50), (55) is in order. There is a freedom to interpret the boundary conditions for free fields as a Neumann or Dirichlet because we are completely free to choose the fields $(\bar{X}_i,\bar{X}_i^*)$ or $(-\bar{X}_i,-\bar{X}_i^*)$ to parametrize the right-moving degrees of freedom.

Note also that linear A-type Ishibashi states are related to B-type states by the mirror involution in the right-moving sector

$$\bar{\psi}_i^*[r]\rightarrow\mu_i\bar{\psi}_i[r],\ \bar{\psi}_i[r]\rightarrow\frac{1}{\mu_i}\bar{\psi}_i^*[r],$$
$$\bar{X}_i^*[n]\rightarrow\mu_i\bar{X}_i[n],\ \bar{X}_i[n]\rightarrow\frac{1}{\mu_i}\bar{X}_i^*[n]. \quad (59)$$

3.2. B-type Ishibashi states in the product of minimal models.

Let us consider the following superposition of B-type free field Ishibashi states (52)

$$|I_{\mathbf{h}},\eta,B>>=\sum_{(\mathbf{p},\mathbf{p}^*)\in\Gamma_{\mathbf{h}}}c_{\mathbf{p},\mathbf{p}^*}|\mathbf{p},\mathbf{p}^*,\eta,B>>, \quad (60)$$

where the coefficients $c_{\mathbf{p},\mathbf{p}^*}$ are arbitrary and the summation is performed over the momentums of the butterfly resolution $C^{\star}_{\mathbf{h}}$. It is clear that this state satisfies the relations (47). We define the closed string transition amplitude between the pair of such states by the following expression $<<I_{\mathbf{h}_2},\eta,B|(-1)^g q^{L[0]-\frac{c}{24}}u^{J[0]}|I_{\mathbf{h}_1},\eta,B>>$, where g is ghost number operator of the complex $C^{\star}_{\mathbf{h}}$. It is easy to see that

$$<<I_{\mathbf{h}_2},\eta,B|(-1)^g q^{L[0]-\frac{c}{24}}u^{J[0]}|I_{\mathbf{h}_1},\eta,B>>=$$
$$\delta(\mathbf{h}_1-\mathbf{h}_2)\sum_{(\mathbf{p},\mathbf{p}^*)\in\Gamma_{\mathbf{h}_1}}(-1)^g|c_{\mathbf{p},\mathbf{p}^*}|^2$$
$$q^{\frac{1}{2}<\mathbf{p}^*+\mathbf{p},\mathbf{p}^*+\mathbf{p}+\mathbf{d}+\mathbf{d}^*>-\frac{c}{24}}u^{<\mathbf{p}^*+\mathbf{p},\mathbf{d}-\mathbf{d}^*>}$$
$$\prod_{m=1}(1+uq^{m-\frac{1}{2}})^r(1+u^{-1}q^{m-\frac{1}{2}})^r(1-q^m)^{-2r}. \quad (61)$$

The coefficients $c_{\mathbf{p},\mathbf{p}^*}$ can be fixed partly from the condition that this amplitude is proportional to the character of the module $M_{\mathbf{h}}$:

$$<<I_{\mathbf{h}_2},\eta,B|(-1)^g q^{L[0]-\frac{c}{24}}u^{J[0]}|I_{\mathbf{h}_1},\eta,B>>=\delta(\mathbf{h}_2-\mathbf{h}_1)|c_{0,\mathbf{h}_1}|^2\chi_{\mathbf{h}_1}(q,u). \quad (62)$$

Comparing with (28) we obtain

$$|c_{\mathbf{p},\mathbf{p}^*}|^2=|c_{0,\mathbf{h}}|^2,\ (\mathbf{p},\mathbf{p}^*)\in\Gamma_{\mathbf{h}}. \quad (63)$$

Thus, the state (60) is a good candidate for free-field realization (in NS sector) of B-type Ishibashi state of the module $M_{\mathbf{h}}$. It would be a genuine Ishibashi state if it did not radiate nonphysical closed string states which are present in the free field representation of the model. In other words, the overlap of this state with an arbitrary closed string state which does not belong to the Hilbert space of the $MM_{\boldsymbol{\mu}}$ model should vanish. As we will see this condition can be formulated as a $BRST$ invariance condition of the state (60).

Before GSO projection, closed string states of $MM_{\boldsymbol{\mu}}$ model which can interact with the Ishibashi state (60) come from the product of left-moving and right-moving Fock modules $F_{\mathbf{p},\mathbf{p}^*}\otimes \bar{F}_{-\mathbf{p}-\mathbf{d},-\mathbf{p}^*-\mathbf{d}^*}$, where $(\mathbf{p},\mathbf{p}^*)\in\Gamma_{\mathbf{h}}$. The left-moving modules of the superposition (60) constitute the butterfly resolution $C^{\star}_{\mathbf{h}}$ whose cohomology is given by the module $M_{\mathbf{h}}$. What about the Fock modules from the right-moving sector? To have nontrivial interaction with the states from $MM_{\boldsymbol{\mu}}$ the right-moving Fock modules have to from the product of resolutions of minimal models (25) also. But this contradicts to the relations between left-moving and right-moving momentums from (50). The contradiction may be resolved if we allow that right-moving Fock modules form the product of dual resolutions to (25). The dual resolution $\tilde{C}^{\star}_{h}$ to minimal model resolution (25) is given by the following diagram

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & & & \\
 & & \downarrow & & \downarrow & & & & \\
\ldots & \rightarrow & \bar{F}_{-1-\frac{1}{\mu},-1-h-\mu} & \rightarrow & \bar{F}_{-\frac{1}{\mu},-1-h-\mu} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
\ldots & \rightarrow & \bar{F}_{-1-\frac{1}{\mu},-1-h} & \rightarrow & \bar{F}_{-\frac{1}{\mu},-1-h} & & & & \\
 & & & & & \searrow & & & \\
 & & & & & & \bar{F}_{1-\frac{1}{\mu},-1-h+\mu} \rightarrow & \bar{F}_{2-\frac{1}{\mu},-1-h+\mu} & \rightarrow \ldots \\
 & & & & & & \downarrow & \downarrow & \\
 & & & & & & \bar{F}_{1-\frac{1}{\mu},-1-h+2\mu} \rightarrow & \bar{F}_{2-\frac{1}{\mu},-1-h+2\mu} & \rightarrow \ldots \\
 & & & & & & \downarrow & \downarrow & \\
 & & & & & & \vdots & \vdots &
\end{array}
\tag{64}
$$

The arrows on this diagram are given by the same operators as on the diagram (25).

Similar to the [25] one can proove that complex (64) is exact except at the $\bar{F}_{-\frac{1}{\mu},-1-h}$ module, where the cohomology is given by the anti-chiral module $M_{h,t=2h}$.

Thus the right-moving Fock modules have to form dual resolution $\tilde{C}^{\star}_{\mathbf{h}}=\otimes^{r}_{i=1}\tilde{C}^{\star}_{h_i}$. Now the coefficients $c_{\mathbf{p},\mathbf{p}^*}$ can be fined from the $BRST$ invariance condition which is straightforward generalization of the condition found in [17] for individual $N=2$ minimal model. To formulate this condition one has to describe by the free fields the total space of states of $MM_{\boldsymbol{\mu}}$ model.

To do that we form first the product of complexes $C^{\star}_{\mathbf{h}}\otimes\tilde{C}^{\star}_{\mathbf{h}}$ to build the complex

$$\ldots\rightarrow \mathbf{C}^{-2}_{\mathbf{h}}\rightarrow\mathbf{C}^{-1}_{\mathbf{h}}\rightarrow\mathbf{C}^{0}_{\mathbf{h}}\rightarrow\mathbf{C}^{+1}_{\mathbf{h}}\rightarrow\ldots, \tag{65}$$

which is graded by the sum of the ghost numbers $g+\bar{g}$ and for an arbitrary ghost number I the space $\mathbf{C}^{I}_{\mathbf{h}}$ is given by the sum of products of the Fock modules from the resolution $C^{\star}_{\mathbf{h}}$ and $\tilde{C}^{\star}_{\mathbf{h}}$ such that $g+\bar{g}=I$. The differential δ of the complex (65) is defined by the differentials ∂ and $\bar{\partial}$ of the complexes $C^{\star}_{\mathbf{h}}$ and $\tilde{C}^{\star}_{\mathbf{h}}$

$$\delta_I|v_g\otimes\bar{v}_{\bar{g}}>=|\partial v_g\otimes\bar{v}_{\bar{g}}>+(-1)^{g}|v_g\otimes\bar{\partial}\bar{v}_{\bar{g}}>, \tag{66}$$

where $|v_g>$ is an arbitrary vector from the complex C_h with ghost number g, while $|\bar{v}_{\bar{g}}>$ is an arbitrary vector from the complex $\tilde{C}^{\star}_{h}$ with the ghost number $\bar{g}$ and $g+\bar{g}=I$. The cohomology of the complex (65) is nonzero only at grading 0 and is given by the product of irreducible modules $M_{\mathbf{h}}\otimes\bar{M}_{\mathbf{h},\mathbf{t}=2\mathbf{h}}$, where $\bar{M}_{\mathbf{h},\mathbf{t}=2\mathbf{h}}$ is the product of anti-chiral modules of minimal models.

The Ishibashi state we are looking for can be considered as a linear functional on the Hilbert space of the product of models, then it has to be an element of the homology group. Therefore, the $BRST$ invariance condition for the state can be formulated as follows.

Let us define the action of the differential D on the state $|I_{\mathbf{h}},\eta,B>>$ by the formula

$$<<\delta^{*}I_{\mathbf{h}},\eta,B|v_g\otimes\bar{v}_{\bar{g}}>\equiv<<I_{\mathbf{h}},\eta,B|\delta_{g+\bar{g}}|v_g\otimes\bar{v}_{\bar{g}}>, \tag{67}$$

where $v_g\otimes\bar{v}_{\bar{g}}$ is an arbitrary element from $\mathbf{C}^{g+\bar{g}}_{\mathbf{h}}$. Then, $BRST$ invariance condition means that

$$\delta^{*}|I_{\mathbf{h}},\eta,B>>=0. \tag{68}$$

As a straightforward generalization of Theorem 2 from [17] we find that superposition (60) satisfies $BRST$ invariance condition (68) if the coefficients $c_{\mathbf{p},\mathbf{p}^{*}}$ obey the equations

$$c_{\mathbf{p},\mathbf{p}^{*}}=\sqrt{2}\cos((2g_{\mathbf{p},\mathbf{p}^{*}}+1)\frac{\pi}{4})c_{0,\mathbf{h}}, \tag{69}$$

where $g_{\mathbf{p},\mathbf{p}^{*}}$ is the ghost number. Note that $BRST$ condition doesn't fix the phase of the overall coefficient $c_{0,\mathbf{h}}$.

For an arbitrary module $M_{\mathbf{h},\mathbf{t}}$, $(\mathbf{h},\mathbf{t})\in\tilde{\Delta}$, the Ishibashi state is generated by the action of spectral flow operators on the Ishibashi state $|I_{\mathbf{h}},\eta,B>>$:

$$|I_{\mathbf{h},\mathbf{t}},\eta,B>>=\prod_{i}U_i^{t_i}\bar{U}_i^{-t_i}|I_{\mathbf{h}},\eta,B>>. \tag{70}$$

Indeed, it is easy to see from (17) that this state satisfy the boundary conditions (49), (50), (51). Hence (47) is fulfilled. It is also $BRST$ closed because spectral flow commutes with screening charges and transition amplitude between pair of such states is proportional to the character $\chi_{\mathbf{h},\mathbf{t}}$.

3.3. A-type Ishibashi states in the product of minimal models.

Let us consider free field representation for A-type Ishibashi states. It is obvious that A-type Ishibashi states are given by superpositions like (60), where the coefficients are determined by the relation (63). The $BRST$ condition for A-type states is slightly different from B-type case. The reason is that the application of one of the left-moving $BRST$ charges, say Q^{+}_{i} to A-type state gives according to (54) and (55) the right-moving $BRST$ charge $\bar{Q}^{-}$ multiplied by μ_i as opposed to the B-type case. In fact we are free to rescale arbitrary the right-moving $BRST$ charges because it does not change the cohomolgy of the complex in the right-moving sector and the cohomology of the total complex (65). Hence we define the right-moving $BRST$ charges in such a way to cancel this effect

$$\begin{aligned}\bar{S}^{+}_{i}(\bar{z})&=\frac{\imath\eta}{\mu_i}\mathbf{s}_i\bar{\psi}^{*}\exp(\mathbf{s}_i\bar{X}^{*})(\bar{z}),\\ \bar{S}^{-}_{i}(\bar{z})&=\imath\eta\mu_i\mathbf{s}^{*}_{i}\bar{\psi}\exp(\mu_i\mathbf{s}^{*}_{i}\bar{X})(\bar{z}),\\ \bar{Q}^{\pm}_{i}&=\oint d\bar{z}\bar{S}^{\pm}_{i}(\bar{z}),\end{aligned} \tag{71}$$

As a result $BRST$ invariant A-type Ishibashi state $|I_{\mathbf{h},\mathbf{h}},\eta,A>>$ is given similar to (60), (69) and similar to B-type case the phase of coefficient $c_{0,\mathbf{h}}$ is arbitrary also. For an arbitrary module $M_{\mathbf{h},\mathbf{t}}$ we then obtain

$$|I_{\mathbf{h},\mathbf{t}},\eta,A>>=\prod_{i}U_i^{-t_i}\bar{U}_i^{-t_i}|I_{\mathbf{h}},\eta,A>> . \tag{72}$$

Similar to the B-type case we find that closed string transition amplitude between A-type Ishibashi states is given by the character $\chi_{\mathbf{h},\mathbf{t}}$.

In conclusion of this section the following remark is in order. In case when some of the μ_i coincide one can consider generalization of (49)-(50) or (54)-(55), where left-moving degrees of freedom are related to the right-moving by some matrix. Carring out the similar analysis one can show that this matrix has to be a permutation matrix of identical minimal models. Thus free field realization of permutation branes [45] can be obtained.

4. Free field realization of boundary states in Gepner model.

4.1. A-type boundary states in Calabi-Yau extension.

It is easy to obtain a set of boundary states in the product of minimal models applying to the free-field realized Ishibashi states the formula found by Cardy [46]. As we have seen $BRST$ invariance fixes the Ishibashi states up to the arbitrary constant $c_{\mathbf{h},\mathbf{t}}$ and we put (following to normalization by Cardy [46]) these coefficients to be equal 1.

As it has already been noticed the product of minimal models can not be applied straightforward to describe in the bulk the string theory on Calabi-Yau manifold. Instead, the so called simple current orbifold whose partition function is diagonal modular invariant partition function with respect to orbit characters (42) describes. The extension of this technique to the conformal field theory with a boundary has been developed in [2], [36], [10], [5]. In our approach we follow mainly [5].

A-type boundary states in CY extension are labeled by spectral flow orbits $[\mathbf{\Lambda},\boldsymbol{\lambda}]\in\tilde{\Delta}$. We consider first spectral flow invariant boundary states. Their expansion with respect to the Ishibashi states (72) can be written according to Cardy's formula

$$|[\mathbf{\Lambda},\boldsymbol{\lambda}],\eta,A>>=\frac{\alpha}{\kappa^2}\sum_{(\mathbf{h},\mathbf{t})\in\tilde{\Delta}}W^{\mathbf{h},\mathbf{t}}_{\mathbf{\Lambda},\boldsymbol{\lambda}}\sum_{m,n=0}^{\kappa-1}\exp{(\imath 2\pi nJ[0])}U^{m\mathbf{v}}\bar{U}^{m\mathbf{v}}|I_{\mathbf{h},\mathbf{t}},\eta,A>>, \tag{73}$$

where $W^{\mathbf{h},\mathbf{t}}_{\mathbf{\Lambda},\lambda}$ are Cardy's coefficients

$$W^{\mathbf{h},\mathbf{t}}_{\mathbf{\Lambda},\lambda}=R^{\mathbf{h}}_{\mathbf{\Lambda}}\exp{(\imath\pi\sum_{i=1}\frac{(\Lambda_i-2\lambda_i)(h_i-2t_i)}{\mu_i})},\ R^{\mathbf{h}}_{\mathbf{\Lambda}}=\prod_{i=1}^{r}R^{h_i}_{\Lambda_i},$$
$$R^{h_i}_{\Lambda_i}=\frac{S_{\Lambda_i,h_i}}{\sqrt{S_{0,h_i}}},\ S_{\Lambda_i,h_i}=\frac{\sqrt{2}}{\mu_i}\sin(\frac{\pi(\Lambda_i+1)(h_i+1)}{\mu_i}), \tag{74}$$

and α is the normalization constant. The summation over n makes $J[0]$-projection, while summation over m introduce spectral flow twisted sectors. $J[0]$ projection in closed string sector corresponds to spectral flow action in the open string sector [10], while spectral flow action in the closed string sector corresponds to $J[0]$ projection in the open string sector [5].

This state depends only on the spectral flow orbit class. Moreover, the $J[0]$ integer charge restriction of the orbits $[\mathbf{\Lambda},\boldsymbol{\lambda}]$ is necessary for the selfconsistency of the expression (73). Indeed, on can change the parametrization of the states $(\mathbf{h},\mathbf{t})$ by spectral flow shift $(\mathbf{h},\mathbf{t}'+l\mathbf{v})$ and insert this into (73). Then we obtain that this state is proportional to itself with the coefficient $\exp{(\imath 2\pi l\sum_i \frac{\Lambda_i-2\lambda_i}{\mu_i})}$. Hence

$$\exp{(\imath 2\pi l\sum_i \frac{\Lambda_i-2\lambda_i}{\mu_i})}=1 \tag{75}$$

has to be satisfyed which is nothing else $J[0]$ integer charge restriction. It means obviously that the boundary state (73) is spectral flow invariant.

It is also consistent with "field identifications"

$$|[\mathbf{\Lambda},\boldsymbol{\lambda}],\eta,A>>=|[\Lambda_1,...,\mu_i-\Lambda_i-2,...,\Lambda_r,\lambda_1,...,\lambda_i-\Lambda_i-1,...,\lambda_r],\eta,A>>, \tag{76}$$

which corresponds to automorphysms (32).

The transition amplitude between these boundary states is given by

$$\begin{aligned} Z^A_{[\mathbf{\Lambda}_1,\boldsymbol{\lambda}_1][\mathbf{\Lambda}_2,\boldsymbol{\lambda}_2]}(\tilde{q}) \equiv << [\mathbf{\Lambda}_1,\boldsymbol{\lambda}_1],\eta,A|(-1)^g q^{L[0]-\frac{c}{24}}|[\mathbf{\Lambda}_2,\boldsymbol{\lambda}_2],\eta,A>>= \\ \alpha^2\sum_{\mathbf{h},\mathbf{t}}\prod_i(N^{h_i}_{\Lambda_{1,i},\Lambda_{2,i}}\delta^{(2\mu_i)}(\Lambda_{2,i}-2\lambda_{2,i}-\Lambda_{1,i}+2\lambda_{1,i}+h_i-2t_i)+ \\ N^{\mu_i-h_i-2}_{\Lambda_{1,i},\Lambda_{2,i}}\delta^{(2\mu_i)}(\Lambda_{2,i}-2\lambda_{2,i}-\Lambda_{1,i}+2\lambda_{1,i}+h_i-2t_i-\mu_i))ch_{\mathbf{h},\mathbf{t}}(\tilde{q}) \end{aligned} \tag{77}$$

This expression is obviously invariant with respect to automorphysms (32). It can be rewritten in more compact form [2] where this invariance is not so explicit however.

$$\begin{aligned} Z^A_{[\mathbf{\Lambda}_1,\boldsymbol{\lambda}_1][\mathbf{\Lambda}_2,\boldsymbol{\lambda}_2]}(\tilde{q}) = \\ 2^r\alpha^2\sum_{\mathbf{h},\mathbf{t}}N^{\mathbf{h}}_{\mathbf{\Lambda}_1,\mathbf{\Lambda}_2}\delta^{(2\mu_i)}(\mathbf{\Lambda}_2-2\boldsymbol{\lambda}_2-\mathbf{\Lambda}_1+2\boldsymbol{\lambda}_1+\mathbf{h}-2\mathbf{t})ch_{\mathbf{h},\mathbf{t}}(\tau), \end{aligned} \tag{78}$$

where we have used the automorphism property of characters (32) and the factor 2^r is coused by "field identification" (32). The important feature of the spectral flow invariant boundary states is that the open string spectrum is $J[0]$ projected. Hence in Gepner models the open string spectrum between these states will be BPS.

We fix the constant α

$$\alpha=2^{-r} \tag{79}$$

The internal automorphysm group of Gepner model allows to construct additional boundary states. Namely one can use the operator $\exp{(-\imath 2\pi\sum_i\phi_i J_i[0])}\in U(1)^r$ to generate new boundary states. Let us consider the properties of the state

$$|[\mathbf{\Lambda},\boldsymbol{\lambda}],\boldsymbol{\phi},\eta,A>>\equiv \exp{(-\imath 2\pi\sum_i\phi_i J_i[0])}|[\mathbf{\Lambda},\boldsymbol{\lambda}],\eta,A>> . \tag{80}$$

It satisfies the conditions similar to (48) except the relations for fermionic fields

$$\begin{aligned} (G^{\pm}[r]+\imath\eta\sum_i\exp{(\pm\imath 2\pi\phi_i)}\bar{G}^{\mp}_i[-r])|[\mathbf{\Lambda},\boldsymbol{\lambda}],\boldsymbol{\phi},\eta,A>>=0, \\ (\psi^*_i[r]-\imath\eta\mu_i\exp{(\imath 2\pi\phi_i)}\bar{\psi}_i[-r])|[\mathbf{\Lambda},\boldsymbol{\lambda}],\boldsymbol{\phi},\eta,A>>=0, \\ (\psi_i[r]-\imath\frac{\eta}{\mu_i}\exp{(-\imath 2\pi\phi_i)}\bar{\psi}^*_i[-r])|[\mathbf{\Lambda},\boldsymbol{\lambda}],\boldsymbol{\phi},\eta,A>>=0. \end{aligned} \tag{81}$$

This state does not invariant with respect to diagonal N=2 Virasoro algebra unless

$$\phi_i \in Z,\ \ i = 1, ..., r. \tag{82}$$

Hence the group $U(1)^r$ reduces to Z^r. It is worth to note that one can ignore the case when all ϕ_i are half-integer because it can be canceled by the $\eta \to -\eta$ (brane-anti-brane) redefinition. It is easy to see directly what kind of states we obtain by this way.

$$\begin{aligned}
&|[\mathbf{\Lambda}, \boldsymbol{\lambda}], \boldsymbol{\phi}, \eta, A>>= \\
&\frac{\alpha}{2^r\kappa^2} \sum_{(\mathbf{h},\mathbf{t})\in\tilde{\Delta}} W^{\mathbf{h},\mathbf{t}}_{[\mathbf{\Lambda},\lambda]} \sum_{m,n=0}^{\kappa-1} \exp\left(\imath 2\pi n J[0]\right) \exp\left(-\imath 2\pi m \sum_i \phi_i \frac{c_i}{3}\right) U^{m\mathbf{v}} \bar{U}^{m\mathbf{v}} \\
&\exp\left(-\imath 2\pi \sum_i \phi_i \frac{h_i - 2t_i}{\mu_i}\right) |I_{\mathbf{h},\mathbf{t}}, \eta, A>>= \\
&\frac{\alpha}{2^r\kappa^2} \sum_{(\mathbf{h},\mathbf{t})\in\tilde{\Delta}} R^{\mathbf{h}}_{\mathbf{\Lambda}} \exp\left(\imath\pi \sum_i \frac{(\Lambda_i - 2\lambda_i - 2\phi_i)(h_i - 2t_i)}{\mu_i}\right) \\
&\sum_{m,n=0}^{\kappa-1} \exp\left(\imath 2\pi n J[0]\right) \exp\left(\imath 4\pi m \sum_i \frac{\phi_i}{\mu_i}\right) U^{m\mathbf{v}} \bar{U}^{m\mathbf{v}} |I_{\mathbf{h},\mathbf{t}}, \eta, A>>\ .
\end{aligned} \tag{83}$$

It is one of the states (73) if

$$2\sum_i \frac{\phi_i}{\mu_i} \in Z. \tag{84}$$

In opposite case the new boundary states are generated by this action. It is easy to see that all Recknagel-Schomerus states [2] are generated by this action providing the following parametrization of Recknagel-Shomerus boundary atates [4]

$$|[\mathbf{\Lambda}, \boldsymbol{\lambda}], \eta, A>>= \exp\left(-\imath 2\pi \sum_i \lambda_i J_i[0]\right) |[\mathbf{\Lambda}, 0], \eta, A>>\ . \tag{85}$$

The transition amplitude between these states is given by

$$\begin{aligned}
&Z^{A}_{([\mathbf{\Lambda}_1,\boldsymbol{\lambda}_1],([\mathbf{\Lambda}_2,\boldsymbol{\lambda}_2],)}(\tilde{q}) = \\
&\frac{\alpha^2}{\kappa^2} \sum_{(\mathbf{h},\mathbf{t})\in\tilde{\Delta}} \prod_i \Big(N^{h_i}_{\Lambda_{1,i},\Lambda_{2,i}} \delta^{(2\mu_i)}(\Lambda_{2,i} - 2\lambda_{2,i} - \Lambda_{1,i} + 2\lambda_{1,i} + h_i - 2t_i) + \\
&N^{\mu_i - h_i - 2}_{\Lambda_{1,i},\Lambda_{2,i}} \delta^{(2\mu_i)}(\Lambda_{2,i} - 2\lambda_{2,i} - \Lambda_{1,i} + 2\lambda_{1,i} + h_i - 2t_i - \mu_i)\Big) \\
&\sum_{m,n=0} Tr_{\mathbf{h},\mathbf{t}}\Big(U^{-n\mathbf{v}} \exp\Big(\imath 2\pi m (J[0] + \sum_i \frac{c_i}{3}(\lambda_{2,i} - \lambda_{1,i})\Big) \tilde{q}^{(L[0] - \frac{c}{24})} U^{n\mathbf{v}}\Big)
\end{aligned} \tag{86}$$

We see that $J[0]$ projection is changed. Instead of the projection on the subspace of integers $J[0]$-charges we obtain the projection on the subspace of integers shifted by $\sum_i \frac{c_i}{3}(\lambda_{2,i} - \lambda_{1,i})$. Hence in Gepner model tachyon my appear between such boundary states.

4.2. B-type boundary states in Calabi-Yau extension.

B-type boundary state can couple only to "charge conjugate" parts $M_{\mathbf{h},\mathbf{t}} \otimes \bar{M}_{\mathbf{h},\boldsymbol{\mu}+\mathbf{h}-\mathbf{t}}$of the bulk Hilbert space (t_i are defined modulo μ_i). If all components of vector $\boldsymbol{\mu}$ are odd (short spectral flow orbits do not appear) it forces the restriction on the set of Ishibashi states [2]:

$$\boldsymbol{\mu} + \mathbf{h} - \mathbf{t} = l\mathbf{v} + \mathbf{t}, \tag{87}$$

where we have identified the vector $\mathbf{h}$ with $(h_1, ..., h_r)$ and $l = 0, ..., \kappa - 1$.

Let us denote by $\tilde{\Delta}_B$ the subset of $\tilde{\Delta}$ satisfying (87). Then for an arbitrary pair of vectors $[\boldsymbol{\Lambda}, \boldsymbol{\lambda}] \in \Delta_{CY}$ the ansatz for B-type boundary state is given by

$$|[\boldsymbol{\Lambda}, \boldsymbol{\lambda}], \eta, B >>= \frac{\alpha}{\kappa^2} \sum_{(\mathbf{h},\mathbf{t}) \in \tilde{\Delta}_B} W^{\mathbf{h},\mathbf{t}}_{[\boldsymbol{\Lambda},\lambda]} \sum_{m,n=0}^{\kappa-1} \exp(\imath 2\pi n J[0]) U^{m\mathbf{v}} \bar{U}^{-m\mathbf{v}} |I_{\mathbf{h},\mathbf{t}}, \eta, B >> . \tag{88}$$

One can check that this state depends only on the spectral flow orbit. It is also obvious that $[\boldsymbol{\Lambda}, \boldsymbol{\lambda}]$ has to be restricted to the set of $J[0]$ integer charges by the reasons similar to the A-type case.

The transition amplitude calculation between the pair of B-type states can be carried over similar to [2]

$$Z^B_{[\boldsymbol{\Lambda}_1,\boldsymbol{\lambda}_1][\boldsymbol{\Lambda}_2,\boldsymbol{\lambda}_2]}(\tilde{q}, \tilde{u}) =$$

$$2^r \kappa \alpha^2 \sum_{(\mathbf{h},\mathbf{t}) \in \tilde{\Delta}_{CY}} \delta^{(\kappa)}(\kappa \sum_i \frac{\Lambda_{2,i} - 2\lambda_{2,i} - \Lambda_{1,i} + 2\lambda_{1,i} + h'_i - 2t'_i}{2\mu_i}) N^{\mathbf{h}}_{\boldsymbol{\Lambda}_1,\boldsymbol{\Lambda}_2} ch_{\mathbf{h},\mathbf{t}}(\tau). \tag{89}$$

Thus one has to put

$$\alpha = 2^{-r} \kappa^{-1} \tag{90}$$

to take into account that extended set $\tilde{\Delta}$ has been used.

Similar to the A-type case the open string spectrum between spectral flow invariant states is $J[0]$ projected. Hence they give BPS spectrum in Gepner model.

Similar to the A-type case the discret group of internal automorphisms is acting on the B-type boundary states and all Recknagel-Shomerus states can be recovered by this action

$$|[\boldsymbol{\Lambda}, \boldsymbol{\lambda}], \eta, B >> \equiv \exp(\imath \sum_i 2\pi \lambda_i J_i[0]) |[\boldsymbol{\Lambda}, 0], \eta, B >>, \tag{91}$$

where λ_i are integers restricted to the region $\lambda_i = 0, ..., \mu_i$.

Closed string transition amplitude is given by

$$Z^B_{([\boldsymbol{\Lambda}_1,\boldsymbol{\lambda}_1])([\boldsymbol{\Lambda}_2,\boldsymbol{\lambda}_2])}(\tilde{q}) =$$

$$\alpha^2 \kappa \sum_{(\mathbf{h},\mathbf{t}) \in \tilde{\Delta}} N^{\mathbf{h}}_{\boldsymbol{\Lambda}_1,\boldsymbol{\Lambda}_2} \delta^{(\kappa)}(\kappa \sum_i \frac{(\boldsymbol{\Lambda}_2 - 2\boldsymbol{\lambda}_2 - \boldsymbol{\Lambda}_1 + 2\boldsymbol{\lambda}_1 + \mathbf{h} - 2\mathbf{t})_i}{2\mu_i})$$

$$\sum_{m,n} Tr_{M_{\mathbf{h}'',\mathbf{t}''}} (U^{-n\mathbf{v}} \exp(\imath 2\pi m (J[0] + \sum_i \frac{c_i}{3}(\lambda_{2,i} - \lambda_{1,i})) \tilde{q}^{(L[0] - \frac{c}{24})} U^{n\mathbf{v}}). \tag{92}$$

Similar to the A-type case we see that $J[0]$ projection is shifted by of $\sum_i \frac{c_i}{3}(\lambda_{2,i} - \lambda_{1,i})$. Hence in Gepner model the tachyon is expected in the open string spectrum.

4.4. A-type boundary states in Gepner models.

In this subsection we discuss free field construction of boundary states in Gepner models using the free field realization of Ishibashi states we developed in section 3 and taking into account GSO projection.

In Gepner model partition function the supersymmetrized characters $Ch_{\mathbf{h},\mathbf{t}}$ appear. Hence the natural idea is to combine Cardy's prescrition and supersymmetrization procedure to built the boundary states. In algebraic approach the boundary states in Gepner models has been constructed first in [2]. In the work [5] the relationship between spacetime supersymmetry and spectral flow has been straightforwardly used in BPS boundary state construction. The representation of this paper follows mainly the way of [5].

Let us introduce the notation $|\Im_{\mathbf{h},\mathbf{t}},\eta,A>>$ for the product of internal Ishibashi state $|I_{\mathbf{h},\mathbf{t}},\eta,A>>$ and external Ishibashi state of Fock module Φ. In this notation we omit (temporarily) the indexes labeling external Ishibashi states. Let us consider first the following ansatz for spectral flow invariant A-type boundary states in Gepner model (they are still Ishibashi states in space-time sector)

$$|[\mathbf{\Lambda},\boldsymbol{\lambda}],\eta,A>>=\frac{\alpha}{4\kappa^2}\sum_{(\mathbf{h},\mathbf{t})\in\tilde{\Delta}}\sum_{m,n=0}^{2\kappa-1}(-1)^n\exp{(\imath\pi\frac{m^2}{2})}$$
$$W^{(\mathbf{h},\mathbf{t})}_{(\mathbf{\Lambda},\boldsymbol{\lambda})}\exp{(\imath\pi nJ_{tot}[0])}U_{tot}^{\frac{m}{2}}\bar{U}_{tot}^{\frac{m}{2}}|\Im_{\mathbf{h},\mathbf{t}},\eta,A>>, \qquad (93)$$

where α is given by (79) and $W^{(\mathbf{h},\mathbf{t})}_{(\mathbf{\Lambda},\boldsymbol{\lambda})}$ is given by (74). The summation over n gives the projection on the odd $J_{tot}[0]$ charges. The Ramound sector contribution is included in the summation over m such that it comes with the coefficient $\imath$.

The transition amplitude is given by

$$Z^A_{[\mathbf{\Lambda}_1,\boldsymbol{\lambda}_1][\mathbf{\Lambda}_2,\boldsymbol{\lambda}_2]}(\tilde{q})=<<([\mathbf{\Lambda}_1,\boldsymbol{\lambda}_1],\eta)^*,A|(-1)^g q^{(L_{tot}[0]-\frac{c_{tot}}{24})}|[\mathbf{\Lambda}_2,\boldsymbol{\lambda}_2],\eta,A>>, \qquad (94)$$

where $|([\mathbf{\Lambda}_1,\boldsymbol{\lambda}_1],\eta)^*,A>>$ is CPT conjugated state [2]. Using (93) we obtain

$$Z^A_{[\mathbf{\Lambda}_1,\boldsymbol{\lambda}_1],[\mathbf{\Lambda}_2,\boldsymbol{\lambda}_2]}(\tilde{q},\tilde{u})=$$
$$(-\imath\tilde{\tau})^{D-4}\sum_{\mathbf{h},\mathbf{t}}N^{\mathbf{h}}_{\mathbf{\Lambda}_1,\mathbf{\Lambda}_2}\delta^{(2\mu_i)}(\mathbf{\Lambda}_2-2\boldsymbol{\lambda}_2-\mathbf{\Lambda}_1+2\boldsymbol{\lambda}_1+\mathbf{h}-2\mathbf{t})Ch_{\mathbf{h},\mathbf{t}}(\tilde{q},\tilde{u}), \qquad (95)$$

where the factor $(-\imath\tilde{\tau})^{D-4}$ is coused by modular transformation of spacetime bosons. Because of supersymmetrized characters appears on the righthand side the open string spectrum is BPS.

The other boundary states can be generated similar to (80). For these boundary states it is easy to perform the calculation similar to (95) and see that tachyon decouples from the open string spectrum unless

$$\sum_i\frac{c_i}{3}(\lambda_{2,i}-\lambda_{1,i})\in Z. \qquad (96)$$

4.5. B-type boundary states in Gepner models.

The ansatz for B-type boundary states could be taken in the form similar to A-type case. Obviously the Ishibashi states that contribute to the superposition are restricted by the subsets $\Delta'_B\in\Delta$ which is defined by the relation similar to (87):

$$\boldsymbol{\mu}+\mathbf{h}-(\mathbf{t}-\frac{m}{2}\mathbf{v})=(\mathbf{t}-\frac{m}{2}\mathbf{v})+\frac{n}{2}\mathbf{v}, \qquad (97)$$

where m, n are defined modulo 2κ. The ansatz for B-type spectral flow invariant boundary states is

$$|[\mathbf{\Lambda}, \boldsymbol{\lambda}], \eta, B >>= \frac{\alpha}{4\kappa^2} \sum_{(\mathbf{h},\mathbf{t}) \in \tilde{\Delta}'_B} \sum_{m,n=0}^{2\kappa-1} (-1)^n \exp{(\imath\pi \frac{m^2}{2})}$$

$$W^{(\mathbf{h},\mathbf{t})}_{(\mathbf{\Lambda},\boldsymbol{\lambda})} \exp{(\imath\pi n J_{tot}[0])} U^{\frac{m}{2}}_{tot} \bar{U}^{\frac{m}{2}}_{tot} |\Im_{\mathbf{h}'\mathbf{t}'}, \eta, B >> . \quad (98)$$

The transition amplitude is given by

$$Z^B_{[\mathbf{\Lambda}_1,\boldsymbol{\lambda}_1][\mathbf{\Lambda}_2,\boldsymbol{\lambda}_2]}(\tilde{q}) =$$

$$(-\imath\tilde{\tau})^{D-4} \sum_{(\mathbf{h},\mathbf{t}) \in \tilde{\Delta}} N^{\mathbf{h}}_{\mathbf{\Lambda}_1,\mathbf{\Lambda}_2} \delta^{(\kappa)}(\kappa \sum_i \frac{(\mathbf{\Lambda}_2 - \mathbf{\Lambda}_1 + \mathbf{h} - 2\boldsymbol{\lambda}_2 + 2\boldsymbol{\lambda}_1 - 2\mathbf{t})_i}{2\mu_i})$$

$$Ch_{(\mathbf{h},\mathbf{t})}(\tilde{q}). \quad (99)$$

Thus the spectrum is BPS due to supersymmetrized charcters on the rhs.

The other boundary states can be generated similar to (91). It is easy to perform the calculation similar to (99) and see that tachyon decouples from the open string spectrum unless (96).

5. Open string spectrum and chiral de Rham complex.

In this section we represent an idea how the spectrum of open strings between the boundary states can be investigated using free field representations. Our discussion will be very brief and restricted to the boundary states in $\boldsymbol{\mu} = (3, 3, 3)$ model. More complete investigation will be presented in future publication.

For simplicity we shall ignore the space-time degrees of freedom concentrating on the internal part of the spectrum.

We begin from the open string spectrum between B-type boundary states.

The irreducible representations in $\mu = 3$ minimal model can be labeled by $t = 0, 1, 2$ so that M_0 is vacuum representation with vanishing conformal weight and $J[0]$ charge, while M_1 and M_2 are generated from M_0 by spectral flow operators U^{-1} and U^{-2}. Their highest vectors have conformal weights and $J[0]$ charges $(1/6, 1/3)$ and $(1/6, -1/3)$ correspondingly. Thus in the product of minimal models characterizing by vector $\boldsymbol{\mu} = (3, 3, 3)$ the set Δ' labeling the irreducible modules is $\Delta' = \{(\mathbf{h}, \mathbf{t}) | h_i = 0,\ t_i = 0, ..., 2,\ i = 1, ..., 3\}$.

The set $\tilde{\Delta}_B$ of solutions of (87) is given by

$$\tilde{\Delta}_B = \{(\mathbf{h} = (h, h, h), \mathbf{t} = k\mathbf{v}) | h, = 0, 1,\ k = 0, 1, 2\} \quad (100)$$

The spectral flow invariant boundary states are numbered by the spectral flow orbits $[\boldsymbol{\lambda}] = [\lambda_1, ..., \lambda_3]$, which are given by

$$[\boldsymbol{\lambda}] = \{[0, 0, 0], [1, 2, 0], [2, 1, 0]\}. \quad (101)$$

The B-type boundary states are given by (88). But it is easy to see that they are equivalent to each other because $\sum_i \lambda_i = 0, mod3$. Thus we have only one spectral flow invariant B-type boundary state.

The transition amplitude is given by (89)

$$<< [\boldsymbol{\lambda}], \eta, B|(-1)^g \exp(\imath 2\pi\tau(L[0]-\frac{1}{8}))|[\boldsymbol{\lambda}], \eta, B >>= \frac{1}{3}\sum_{[\mathbf{t}]} ch_{[\mathbf{t}]}(-\frac{1}{\tau}), \quad (102)$$

where

$$[\mathbf{t}] = \{[0,0,0], [1,2,0], [2,1,0]\}. \quad (103)$$

We see that the total set of orbit characters appears in the amplitude. Under the spectral flow $U^{\frac{\mathbf{v}}{2}}$ the NS spectrum (102) goes to R spectrum which can be interprated geometricaly.

Note first that the spectrum of open strings between spectral flow invariant boundary states is given by $J[0]$ projection of $\sum_i Q_i^+ + \sum_i Q_i^-$ cohomology of the free fields space of states which is twisted by spectral flow operators $U^{\mathbf{t}+(n+\frac{1}{2})\mathbf{v}}$. The calculation of cohomolgy can be given by two steps.

At first step we can take for example $\sum_i Q_i^+$ cohomology and then as the second step we calculate $\sum_i Q_i^-$ cohomology. It is well known (see for example [25]) that $\sum_i Q_i^+$ cohomology is generated by the set of $bc\beta\gamma$-system of fields

$$a_i(z) = \exp(\mathbf{s}_i^* X(z)),\ a_i^*(z) = (\mathbf{s}_i \partial X^* - \mathbf{s}_i^* \psi \mathbf{s}_i \psi^*) \exp(-\mathbf{s}_i^* X(z)),$$
$$\alpha_i(z) = \mathbf{s}_i^* \psi \exp(\mathbf{s}_i^* X(z)),\ \alpha_i^*(z) = \mathbf{s}_i \psi^* \exp(-\mathbf{s}_i^* X(z)). \quad (104)$$

The spectral flow operators $U^{\mathbf{t}}$ generate the twisted sectors for these fields due to the relations

$$a_i(z_1)U^{\mathbf{t}}(z_2) = z_{12}^{-\frac{t_i}{\mu_i}} : a_i(z_1)U^{\mathbf{t}}(z_2) :,\ a_i^*(z_1)U^{\mathbf{t}}(z_2) = z_{12}^{\frac{t_i}{\mu_i}} : a_i^*(z_1)U^{\mathbf{t}}(z_2) :,$$
$$\alpha_i(z_1)U^{\mathbf{t}}(z_2) = z_{12}^{t_i-\frac{t_i}{\mu_i}} : \alpha_i(z_1)U^{\mathbf{t}}(z_2) :,\ \alpha_i^*(z_1)U^{\mathbf{t}}(z_2) = z_{12}^{\frac{t_i}{\mu_i}-t_i} : \alpha_i^*(z_1)U^{\mathbf{t}}(z_2) : . \quad (105)$$

Let us consider in more details the space of states generated by the fields (104) from the vacuum vector $|\frac{1}{2}\mathbf{v}>$ which is generated by $U^{\frac{1}{2}\mathbf{v}}$. In this sector all fields are expanded into integer modes and

$$a_i[n]|\frac{1}{2}\mathbf{v}>= 0, n > 0,\ a_i^*[n]|\frac{1}{2}\mathbf{v}>= 0, n > -1,$$
$$\alpha_i[n]|\frac{1}{2}\mathbf{v}>= 0, n > -1,\ \alpha_i^*[n]|\frac{1}{2}\mathbf{v}>= 0, n > 0. \quad (106)$$

In terms of the (104) fields the N=2 Virasoro superalgebra currents (41) are given by

$$G^- = \sum_i \alpha_i a_i^*,\ G^+ = \sum_i (1-\frac{1}{\mu_i})\alpha_i^* \partial a_i - \frac{1}{\mu_i} a_i \partial \alpha_i^*,$$
$$J = \sum_i (1-\frac{1}{\mu_i})\alpha_i^* \alpha_i + \frac{1}{\mu_i} a_i a_i^*,$$
$$T = \sum_i \frac{1}{2}((1+\frac{1}{\mu_i})\partial\alpha_i^* \alpha_i - (1-\frac{1}{\mu_i})\alpha_i^* \partial\alpha_i) + (1-\frac{1}{2\mu_i})\partial a_i a_i^* - \frac{1}{2\mu_i} a_i \partial a_i^* \quad (107)$$

and it is easy to see that

$$J[n]|\frac{1}{2}\mathbf{v}>=0,n>0,\ \ L[n]|\frac{1}{2}\mathbf{v}>=0,n>0,$$
$$G^{+}[n]|\frac{1}{2}\mathbf{v}>=0,n>0,\ \ G^{-}[n]]|\frac{1}{2}\mathbf{v}>=0,n>-1. \tag{108}$$

Hence $|\frac{1}{2}\mathbf{v}>$ is the standard Ramound vacuum. Note that $G^{-}[0]$ is acting on the space of states generated from this vector similar to de Rham differential action on the space of differential forms. It is easy to see that the space of states generated by (104) from $|\frac{1}{2}\mathbf{v}>$ together with $N=2$ Virasoro algebra action (107) has chiral de Rham complex structure introduced for any smooth manifold by [26]. In our particular case this manifold is C^3.

The screening charges Q_i^- can be expressed in terms of the fields (104) as

$$Q_i^- = \oint dz\alpha_i a_i^{\mu_i-1}(z), \tag{109}$$

so $\sum_i Q_i^-$ is Koszul differential associated with LG potential $W(a_i)=\sum_i a_i^{\mu_i}$ [25], [31], [32]. Therefore $\sum_i Q_i^-$ cohomolgy calculation (the second step) corresponds to LG potential $\sum_i a_i(z)^3$ is switched on.

Adding the twisted sectors $U^{(n+\frac{1}{2})\mathbf{v}}$ and making $J[0]$ projection converts the chiral de Rham complex over C^3 into chiral de Rham complex on the orbifold [28] of C^3 by some discret group$G$. This group contains Z_3 because the spectrum is $J[0]$ projected and twisted sectors generated by spectral flow operator $U^{n\mathbf{v}}$ are added. But the open string spectrum (102) contains also the twisted sectors generated by $U^{(1,2,0)}$ and $U^{(2,1,0)}$ operators which are not related to GSO projection. One can explaine the appearence of these sectors as the result of B-type boundary conditions. Indeed the charge conjugation condition (87) extracts the states which are invariant modulo spectral flow shift. The charge conjugation $(\mathbf{h},\mathbf{t})\rightarrow(\mathbf{h},\boldsymbol{\mu}+\mathbf{h}-\mathbf{t})$ does not commute to the GSO projection (simple current extension) giving thereby the extention [?], [47] of the orbifold group . Therefore in B-type case one obtains the orbifold C^3/Z_3^2 which is mirror [47] to the orbifold C^3/Z_3.

Note that there are no physical exitations in the spectrum (102) which are created by powers $a_i^{n_i}[0]$ hence this boundary state can be identifyed to the skyscrapper sheaf $C[a_1,...,a_3]/(a_1,...,a_3)$ at the origin of the orbifold. This interpretation is not unique however due to "field identification" automorphysms (32), (30) which are quasi-isomorphisms of butterfly resolutions.

On the equal footing one can consider first the $\sum_i Q_i^-$ cohomology. It is given by "dual" $bc\beta\gamma$-system of fields

$$b_i(z)=\exp{(\frac{1}{\mu_i}\mathbf{s}_iX^*(z))},\ \ b_i^*(z)=(\mu_i\mathbf{s}_i^*\partial X-\mathbf{s}_i\psi^*\mathbf{s}_i^*\psi)\exp{(-\frac{1}{\mu_i}\mathbf{s}_iX^*(z))},$$
$$\beta_i(z)=\frac{1}{\mu_i}\mathbf{s}_i\psi^*\exp{(\frac{1}{\mu_i}\mathbf{s}_iX^*(z))},\ \ \beta_i^*(z)=\mu_i\mathbf{s}_i^*\psi\exp{(-\frac{1}{\mu_i}\mathbf{s}_iX^*(z))}. \tag{110}$$

The Virasoro currents are given by

$$G_i^-=(1-\frac{1}{\mu_i})\beta_i^*\partial b_i-\frac{1}{\mu_i}b_i\partial\beta_i^*,\ \ G_i^+=\beta_ib_i^*,$$
$$J_i=-(1-\frac{1}{\mu_i})\beta_i^*\beta_i-\frac{1}{\mu_i}b_ib_i^*,$$
$$T_i=\frac{1}{2}((1+\frac{1}{\mu_i})\partial\beta_i^*\beta_i-(1-\frac{1}{\mu_i})\beta_i^*\partial\beta_i)+(1-\frac{1}{2\mu_i})\partial b_ib_i^*-\frac{1}{2\mu_i}b_i\partial b_i^*. \tag{111}$$

It is easy to see that twisted sectors generated by the operators $U^{\mathbf{t}}$ for these fields coincide to the twisted sectors for the fields (104). It means in particular that the vector $|\frac{1}{2}\mathbf{v}>$ satisfy the same annihilation conditions (106), (108) with respect to the dual fields (110). Thus it is the standard Ramound vacuum again. Therefore the space of states generated by dual fieleds from $|\frac{1}{2}\mathbf{v}>$ together with $N=2$ Virasoro algebra action (111) is chiral de Rham complex on C^3.

In the dual picture the $BRST$ operator $\sum_i Q^+_i$ plays the role of Kozsul differential

$$Q^+_i = \oint dz \beta_i b_i^{\mu_i - 1}(z). \tag{112}$$

so we have the same LG potential in the dual coordinates. Taking into account the $J[0]$ projection and twisted sectors one can conclude that we have open string spectrum on the same LG orbifold.

It is important to note here that going from $bc\beta\gamma$ fields (104) to its dual (110) we change the role of fermionic $N=2$ Virasoro superalgebra currents $G^{\pm}(z)$. Initialy $G^-[0]$ operator played the role of de Rham differential, while in dual coordinates $G^+[0]$ becomes de Rham differential so that the chiral and anti-chiral rings are exchanged also [27], [29]. This is a kind of mirror symmetry in the open string chanal which has to be important in intersection index [3] calculations.

The "mirror" symmetry of the spectrum can be explained by toric geometry of the chiral de Rham complex on the orbifold. The set of screening charges Q^+_i forming the differential at the first step of cohomolgy calculations can be represented as the polytope in lattice $P \in \Pi$ whose vertices are given by the origin $(0,0,0)$ and vectors $\mathbf{s}_i$ from $\Re$. The volume of this polytope mesured in the units of the lattice P is 3^3 and hence the group Z_3^3 acting in the product of minimal models can be represented as factor of P, by the sublattice $P_{\Re}$ generated by the screening charge vectors $\Re$. The $J[0]$ projection is determined by the vector $\mathbf{d} - \mathbf{d}^* \in \Pi$ and defines the sublattice

$$\Pi_{int} = \{(\mathbf{p}, \mathbf{p}^*) \in \Pi | < \mathbf{p} + \mathbf{p}^*, \mathbf{d} - \mathbf{d}^* > \in Z\}. \tag{113}$$

The intersection $P_{int} \equiv \Pi_{int} \cap P$ can be represented

$$P_{int} = \{\sum_i n_i \mathbf{w}_i; n_i \in Z\}$$
$$\mathbf{w}_1 = \mathbf{d}, \mathbf{w}_{i+1} = \mathbf{e}_{i+1} - \mathbf{e}_i,\ i = 1,2. \tag{114}$$

The intersection of the polytope $\Re$ with the height one plane

$$\Omega = \{v \in E; < v, \mathbf{d}^* >= 1\} \tag{115}$$

gives the following set of lattice points

$$\Sigma = \{\sigma_0 = \mathbf{w}_1, \sigma_1 = \mathbf{w}_1 + \mathbf{w}_2, \sigma_2 = \mathbf{w}_1 + \mathbf{w}_3, \sigma_3 = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3, \sigma_4 = \mathbf{w}_1 - \mathbf{w}_2,$$
$$\sigma_5 = \mathbf{w}_1 - \mathbf{w}_3, \sigma_6 = \mathbf{w}_1 - \mathbf{w}_2 - \mathbf{w}_3,$$
$$\sigma_7 = \mathbf{s}_1, \sigma_8 = \mathbf{s}_2, \sigma_9 = \mathbf{s}_3\} \tag{116}$$

The lattice P^* is dual to the lattice $P_{\Re}$. The basic cone $K = \{\sum_i k_i \mathbf{s}^*_i; k_i \geq 0\} \in P^*$ defines the set of monomials $a_1^{k_1}...a_3^{k_3}$ on C^3 and hence the vertex algebra generated by the fields (104) corresponds to the cone K [27]. Taking into account $J[0]$ projection one can see that toric manifold determined by mutualy dual lattices $P_{\Re}$, P^* and the polytope $\Re$ is singular with the orbifold singularity Z_3^2. Thus the operators $U^{\mathbf{v}}$, $U^{(1,2,0)}$, $U^{(2,1,0)}$ can be associated to the divisors $\sigma_0, \sigma_1, \sigma_2$.

The basis (118) corresponds to the $J[0]$ invariant $bc\beta\gamma$ system of fields

$$\begin{aligned} x_i(z) &= \exp\left(\mathbf{w}_i^* X(z)\right), \\ x_i^*(z) &= (\mathbf{w}_i \partial X^* - \mathbf{w}_i^* \psi \mathbf{w}_i \psi^*) \exp\left(-\mathbf{w}_i^* X(z)\right), \\ \epsilon_i(z) &= \mathbf{w}_i^* \psi \exp\left(\mathbf{w}_i^* X(z)\right), \\ \epsilon_i^*(z) &= \mathbf{w}_i \psi^* \exp\left(-\mathbf{w}_i^* X(z)\right). \end{aligned} \tag{117}$$

and the monomials $x_1^{n_1}...x_3^{n_3}$ for integer n_i generate the ring of holomorphic functions on the orbifold C^3/Z_3 if $n_1\mathbf{w}^1 + ... + n_3\mathbf{w}_3^* \in K$.

Analogously one can associate the polytope $\Re^*$ in the lattice P^* to the set of screening charges Q_i^-. Let us denote by $P^*_{\Re^*}$ the sublattice in P^* generated by $\Re^*$. The factor $P^*/P^*_{\Re^*}$ gives the group Z_3^3 of symmetries of the minimal models. In the lattice $P^*_{int} \equiv P^* \cap \Pi_{int}$ we fix the basis $\mathbf{w}_i^*$

$$\mathbf{w}_1^* = \mathbf{d}^*,\ \mathbf{w}_2^* = \mathbf{s}_3^* - \mathbf{s}_1^*,\ \mathbf{w}_3^* = \mathbf{s}_2^* - \mathbf{s}_3^*, \tag{118}$$

The intersection of the polytope $\Re^*$ with the height one plane

$$\Omega^* = \{v^* \in E^*; < v^*, \mathbf{d} >= 1\} \tag{119}$$

gives the following set of lattice points from P^*_{int}

$$\begin{aligned} \Sigma^* = \{\sigma_0^* = \mathbf{w}_1^*, \sigma_1 = \mathbf{w}_1^* + \mathbf{w}_2^*, \sigma_2 = \mathbf{w}_1^* + \mathbf{w}_3, \sigma_3^* = \mathbf{w}_1^* + \mathbf{w}_2^* + \mathbf{w}_3^*, \sigma_4^* = \mathbf{w}_1^* - \mathbf{w}_2^*, \\ \sigma_5 = \mathbf{w}_1^* - \mathbf{w}_3^*, \sigma_6 = \mathbf{w}_1^* - \mathbf{w}_2^* - \mathbf{w}_3^*, \\ \sigma_7^* = 3\mathbf{s}_1^*, \sigma_8^* = 3\mathbf{s}_2^*, \sigma_9^* = 3\mathbf{s}_3^*\} \end{aligned} \tag{120}$$

The lattice P is dual to the lattice $P^*_{\Re^*}$. The cone $K^* = \{\sum_i k_i \mathbf{s}_i; k_i \geq 0\} \in P$ defines the set of monomials $b_1^{k_1}...b_3^{k_3}$ on C^3 and hence the vertex algebra generated by the fields (110) corresponds to the cone K^* [27]. Taking into account $J[0]$ projection we can see the toric manifold determined by mutualy dual lattices P, $P^*_{\Re^*}$ and the polytope $\Re^*$ is singular with the orbifold singularity Z_3^2. Thus the operators $U^{\mathbf{v}}$, $U^{(1,2,0)}$, $U^{(2,1,0)}$ can be associated to the divisors $\sigma_0^*, \sigma_2^*, \sigma_4^*$. Thus the dual toric manifold determined by the polytope $\Re^*$ has the same type of the singularity as $\Re$. Hence the "mirror" background for the open string is the same.

Now we consider the open string spectrum between A-type boundary states. As is well known they corresponds in the large volume limit to the special Lagrangian submanifolds [48] which are real submanifolds in CY manifold. In the free field representation this fact manifests in the relation between left-moving and right-moving degrees of freedom for A-type boundary conditions. It rises the question if one can apply chiral de Rham complex on C^3 in the open string spectrum investigation? We leave this problem beyond the scope of the paper and use formaly chiral de Rham complex in the analysis of the spectrum.

In $\boldsymbol{\mu} = (3, 3, 3)$ model the spectral flow invariant A-type boundary states are labeled (101) and the set Δ_{CY} is given by

$$[\mathbf{t}] = \{[0, 0, 0], [1, 2, 0], [2, 1, 0]\}. \tag{121}$$

Let us consider first the transition amplitude $Z^A_{[\boldsymbol{\lambda}],[\boldsymbol{\lambda}]}$, for spectral flow invariant boundary state. From (78) we obtain

$$Z^A_{[\boldsymbol{\lambda}],[\boldsymbol{\lambda}]}(\tau) = \\ << [\boldsymbol{\lambda}], \eta, A|(-1)^g \exp(\imath 2\pi\tau(L[0] - \frac{1}{8}))|[\boldsymbol{\lambda}], \eta, A >>= ch_{[0,0,0]}(-\frac{1}{\tau}). \tag{122}$$

Analogously to the B-type case we see that in R sector $\sum_i Q^+_i$ cohomology are given by $bc\beta\gamma$ system of fields (104) and the space of states generated by these fields from the Ramound vacuum $|\frac{1}{2}\mathbf{v} >$ has the chiral de Rham complex structure on C^3, while $\sum_i Q^+_i$ is the Koszul differential associated to the potential $W = \sum_i a_i^3$. Adding the twisted sectors $U^{(n+\frac{1}{2})\mathbf{v}}$ and making $J[0]$ projection converts the chiral de Rham complex over C^3 into chiral de Rham complex over the orbifold C^3/Z_3.

The transition amplitude between different spectral flow invariant boundary states gives additional twisted sectors. Let us take, for example $[0, 0, 0,]$ and $[1, 2, 0]$. Then we obtain

$$<< [0, 0, 0], \eta, A|(-1)^g \exp(\imath 2\pi\tau(L[0] - \frac{1}{8}))|[1, 2, 0], \eta, A >>= ch_{[1,2,0]}(-\frac{1}{\tau}). \tag{123}$$

Before we take $\sum_i Q^-_i$ cohomology the spectrum of states is generated by the fields (104) from the twisted vacuum vectors $|(n + \frac{1}{2})\mathbf{v} + \mathbf{t} >$, where $\mathbf{t} = (1, 2, 0)$. We interprate these additional twisted states as the corresponding to the open string ending on different D-branes in C^3/Z_3 orbifold such that twisting determines the angle. The angles can be read off easely from (105)

$$\begin{aligned} a_1(e^{\imath 2\pi} z) = a_1(z),\ a_2(e^{\imath 2\pi} z) = e^{\imath\frac{2\pi}{3}} a_2(z),\ a_3(e^{\imath 2\pi} z) = e^{-\imath\frac{2\pi}{3}} a_3(z), \\ a^*_1(e^{\imath 2\pi} z) = a^*_1(z),\ a^*_2(e^{\imath 2\pi} z) = e^{-\imath\frac{2\pi}{3}} a^*_2(z),\ a^*_3(e^{\imath 2\pi} z) = e^{\imath\frac{2\pi}{3}} a^*_3(z), \\ \alpha_1(e^{\imath 2\pi} z) = \alpha_1(z),\ \alpha_2(e^{\imath 2\pi} z) = e^{+\imath\frac{2\pi}{3}} \alpha_2(z),\ \alpha_3(e^{\imath 2\pi} z) = e^{-\imath\frac{2\pi}{3}} \alpha_3(z), \\ \alpha^*_1(e^{\imath 2\pi} z) = \alpha^*_1(z),\ \alpha^*_2(e^{\imath 2\pi} z) = e^{-\imath\frac{2\pi}{3}} \alpha^*_2(z),\ \alpha^*_3(e^{\imath 2\pi} z) = e^{\imath\frac{2\pi}{3}} \alpha^*_3(z). \end{aligned} \tag{124}$$

In this example one can see the general property of A-type branes that the operator $\exp(-\imath 2\pi \sum_i \phi_i J_i[0])$ realy rotates the D-branes on the orbifold and $\boldsymbol{\lambda}$ parametrize the angles. One can check this by the direct caculation

Let us find the boundary condition for the spectral flow operator $U^{\mathbf{t}'}$. It belongs to (left-moving) CY Hilbert space if it has integer $J[0]$ charge. We have

$$\begin{aligned} U^{\mathbf{t}'}|[\boldsymbol{\Lambda}, \boldsymbol{\lambda}], \eta, A >>= \exp(-\imath 2\pi \sum_i \lambda_i J_i[0]) \exp(\imath 2\pi \sum_i \frac{2t'_i\lambda_i}{\mu_i}) U^{\mathbf{t}'}|[\boldsymbol{\Lambda}, 0], \eta, A >>= \\ \exp(\imath 2\pi \sum_i \frac{2t'_i\lambda_i}{\mu_i}) \bar{U}^{-\mathbf{t}'} \exp(-\imath 2\pi \sum_i \lambda_i J_i[0]) \\ \frac{1}{\kappa^2} \sum_{(\mathbf{h},\mathbf{t})\in\tilde{\Delta}} W^{\mathbf{h},\mathbf{t}}_{\boldsymbol{\Lambda},0} \sum_{m,n}^{\kappa-1} \exp(\imath 2\pi n(J[0] + \sum_i \frac{c_i}{3} t'_i)) U^{m\mathbf{v}} \bar{U}^{m\mathbf{v}} U^{\mathbf{t}'-\mathbf{t}} \bar{U}^{\mathbf{t}'-\mathbf{t}} |I_{\mathbf{h}}, \eta, A >>= \\ \exp(-\imath 2\pi \sum_i \frac{t'_i(\Lambda_i - 2\lambda_i)}{\mu_i}) \bar{U}^{-\mathbf{t}'}|[\boldsymbol{\Lambda}, \boldsymbol{\lambda}], \eta, A >>, \end{aligned} \tag{125}$$

where it is implied that $J[0]$ charge of the state $U^{\mathbf{t}'}$ is integer. We have used here the quasi-isomorphism property of spectral flow operator. In other words we identified to each other the

Ishibashi states which differ by the action of spectral flow operator $\prod_{i=1}^{r} U_i^{n_i\mu_i}\bar{U}_i^{n_i\mu_i}$, where n_i are integers. The reason why free field realized Ishibashi and boundary states are not invariant under the spectral flow operators is that the butterfly resolutions are not invariant (see section 1). They are not invariant also with respect to field identification automorphisms (32). It is obvious that chain homotopies of butterfly resolutions gives equivalent Ishibashi and boundary states also. Taking into account these identifications we find that A-type boundary state is localized at the angles $\theta_i = -2\pi\frac{[\mathbf{\Lambda}-2\boldsymbol{\lambda}]_i}{\mu_i}$.

In conclusion of this section two remarks are in order. The first one is related to the intersection index and D-branes charges calculation in the boundary state approach [3], [4]- [7]. This calculation in boundary state formalizm is given by two steps. At first step we calculate the closed string transition amplitude which is given by the linear combination of characters. At the second step one has to take a limit of these expression when $Im\tau$ goes to infinity. As a result only chiral prymary fields in the open string chanal give a contribution to the intersection index. As we have argued in this section the open string spectrum has the chiral de Rham complex structure. It was shown by Malikov Schechtman and Vaintrob [26] that chiral de Rham complex is a (loop-coherent) sheaf of vertex algebras [27]. Then one can give geometric interpretation for the first step as a calculation of the index of the Dirac operator [16], [49]- [52], [53] of the sheaf. So it is natural to suggest that it gives thereby string generalization of the bilinear form on the K-theory classes.

As a second remark one has to note that BPS condition of the open string spectrum which was very important for the geometric interpretation has been satisfyed due to spectral flow invariance of boundary states. Namely $J[0]$ projection in open string chanal allowed to relate the open string spectrum in these models to the open string spectrum of fractional branes on the LG orbifolds C^r/Z_r^{r-1}. In other words, GSO projection fixes the background.

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