Handling partial truth on type-2 similarity-based reasoning

Chung-Ming Own *

Department of Computer and Communication Engineering, St. John’s University, Tamsui, Taipei 62107, Taiwan, ROC

Abstract

Representation and manipulation of the vague concepts of partially true knowledge in the development of machine intelligence is a wide and challenging field of study. How to extract of approximate facts from vague and partially true statements has drawn significant attention from researchers in the fuzzy information processing. Furthermore, handling uncertainty from this incomplete information has its own necessity. This study theoretically examines a formal method for representing and manipulating partially true knowledge. This method is based on the similarity measure of type-2 fuzzy sets, which are directly used to handle rule uncertainties that type-1 fuzzy sets cannot. The proposed type-2 similarity-based reasoning method is theoretically defined and discussed herein, and the reasoning results are applied to show the usefulness with the comparison of the general fuzzy sets.

0. Introduction

Since much knowledge about many things and events in our daily life is necessarily incomplete, and since exceptions cannot be explicitly stated, people seek a mechanism that allows them to reach conclusions from incomplete information. Hence, they consistently reason approximately about incomplete knowledge with reasonably good results. Many approximate reasoning methods have been proposed. Zadeh (1965) first proposed the concept of approximate reasoning based on the theory of fuzzy sets and fuzzy logic. By Zadeh’s assumption, the imprecision in natural language implies possibility. So far, fuzzy approximate reasoning has applied extensively in various soft sciences such as biology, and psychology, in design of expert systems, in risk analysis and in areas of advanced technology (Lee, 1990; Mizumoto, 1985; Zadeh, 1970).

According to (Raha & Ray, 1999), some commonsense knowledge representations cannot be taken for granted but are partially true. That is, knowledge about exceptions may be incomplete, and so the set of exceptions cannot be denoted exclusively by using a default rule. Therefore, reasoning systems should be modified to accommodate such partially true knowledge. Previous surveys have used fuzzy sets to represent and manipulate such knowledge (Raha & Ray, 1997, 1999, 2000; Zwick, Carlstein, & Budescu, 1987).

In addition, According to the general inference methods in the type-1 fuzzy reasoning systems, there are several existing deficiencies. The process heavily relies on the computation of the conditional relation and the given fact in the computation of a meaningful conclusion (Tsang, Lee, & Yeung, 1995). In consonance, Zadeh’s max–min compositional inference rules have been widely criticized as too complicated with unclear semantics. Hence, in (Raha & Ray, 1997, 1999, 2000; Turksen & Zhong, 1988), they proposed similar methods as similarity-based reasoning. In such similarity-based reasoning methods, the desired conclusion is derived only using the similarity measure between the fact and the antecedent in a rule-based system. The conclusion becomes independent of the relational operator.

On the other hand, type-2 fuzzy sets, which were systematically proposed recently, are very appropriate for several applicable problems, including signal processing problems (Mendel, 2000). Type-2 fuzzy sets can effectively denote
and manage uncertain information. Meanwhile, much work has already been done into defining the degree of similarity between two standard fuzzy sets. However, the similarity between type-2 fuzzy sets has received little attention, and the historical similarity measures cannot be directly applied to measure type-2 fuzzy sets. Till in 2004, Hung and Yang (2004) proposed the axiom definition for a type-2 similarity measure based on Hausdorff distance. They compute the similarities by treating the secondary grades as having two opposite elements consisting of coexistence on peer secondary membership functions. Mitchell (2005a) introduced the similarity measure for measuring the similarity between type-2 fuzzy sets, and applied the proposed method to solve the problem of automatic evaluation of welded structures. Mitchell (2005b) also defined the correlation coefficient between two type-2 fuzzy sets, and adopted the embedded function model to interpret each type-2 fuzzy sets as a weighted ensemble of fuzzy sets. Besides, Mitchell (2006) adopted a statistical viewpoint and interpreted each type-2 fuzzy number to rank uncertainty for each intuitionistic fuzzy number. Lee, Lee, and Lee (2005) proposed a comparison method for discrete type-2 fuzzy values based on the concept of the satisfaction function. Their method focused on the handling the ambiguity in fuzzy comparisons.

In this study, such partially true knowledge is represented and manipulated using the type-2 fuzzy sets firstly. Thus, the author models the partially true knowledge as uncertainties in rule-base logic systems, and theoretically examines a formal method to represent the partially true statement as higher membership value in type-2 fuzzy sets. Accordingly, to consist with the advantage of similarity-based reasoning, the author formulates an approach to measure the similarity between type-2 fuzzy sets based on the improvement of Hung and Yang (2004) method, and proposes a type-2 similarity-based reasoning method as another type-2 inference method.

Section 1 introduces the representation of partially true knowledge. Section 2 briefly reviews essential concepts of type-2 fuzzy sets. Section 3 describes the proposed reasoning method with similarity between type-2 fuzzy sets. Section 4 presents an empirical example to illustrate the utility of the proposed models. Section 5 draws conclusions.

1. Representation of partially true knowledge

Reasoning systems are altered to deal with incomplete or partially true knowledge by splitting each such partially true statement into two components – a proposition component and an associated truth value component (Dubois & Prade, 1990). The truth value component provides a mechanism for modifying the significance of an original proposition. Generally, a proposition such as “x is F” is expressed as “x is F is τ”, where τ denotes a linguistic value of partial truth qualification, defined as the degree of compatibility of the situation with the proposition “x is F”.

Therefore, “x is F” is defined as a proposition component, and a linguistic truth value, τ, is defined as an associated truth value component. Equivalent statements in natural language include:

“‘David is healthy’ is quite true.”

“‘The speed is moderate’ is absolutely true.”

The unit interval [0, 1] is taken as the set of partially true values. Therefore, any vague definition related to truth can be represented by a fuzzy set on [0, 1].

A simple vague proposition about truth value, such as “This truth value represents ‘mostly true’”, can be translated into a rule of the form of general fuzzy sets,

“Truth value is mostly true” = ∑_x μ_{mostly true}(x)/x
= 0.4/0.65 + 0.5/0.7
+ 0.75/0.75 + 1/0.8
+ 0.75/0.9 + 0.26/0.95.

The value of μ_{mostly true}(x) does not change the meaning of the proposition, but represents the subjective opinion about the meaning of the proposition. That is, when μ_{mostly true}(x) = 0, the truth value certainly differs from x, and when μ_{mostly true}(x) = 1, the truth value equals x. Notably, μ_{mostly true}(x) reveals the uncertainty of the original knowledge.

Bellman and Zadeh assumed that this partial truth qualification is local, rather than absolute (Zadeh, 1979; Bellman & Zadeh, 1977). They obtained the true statement based on the partially true statement, and derived the corresponding fuzzy set as a representation of the statement. Baldwin proposed the implied statements, which consists the fuzzy truth value restricted to a Lukasiewicz logic implication related on a fuzzy truth space (Baldwin, 1979). Then, he applied set-theoretic considerations to obtain fuzzy truth value constraints on truth value restrictions from conditional fuzzy linguistic statements, by applying an inverse procedure for modifying truth functions.

Accordingly, Raha and Ray (1997, 1999, 2000) proposed a method for reasoning with the partial truth value associated with a vague sentence. The partial truth values are defined by fuzzy sets on the universe of discourse [0, 1], which is a unit interval. This vague proposition relates to a possibility distribution, such that each possibility distribution is assigned to and manipulated by a fuzzy set/relation accordingly.

The approach applied in this study differs from the above approaches. This study attempts to eliminate the deficiencies of the representation of partial true knowledge. Although previous methods, associate and manipulate the partial truth value in accordance to the proposition, they denote and compute the corresponding fuzzy set and partial truth qualification separately. That is, set-theoretic considerations cannot be used to derive partial true knowledge theoretically, and partial truth statements are not
associated with the existing proposition. Hence, the proposed approach to representing and manipulating such partially true knowledge is based on the type-2 fuzzy set theory.

2. Type-2 fuzzy sets

Type-2 fuzzy sets were initially defined by Zadeh (1975). A type-2 fuzzy set is characterized by a fuzzy membership. The membership value for each element of this set is a fuzzy set in [0,1], whereas the membership grade of a type-1 fuzzy set is a crisp value in [0,1]. To clarify statement, the fuzzy set ‘tall’ is represented as

\[
tall = \frac{0.95}{\text{Michael}} + \frac{0.4}{\text{Danny}} + \frac{0.6}{\text{Robert}}.
\]

Conversely, the interpretation of type-2 fuzzy set is

\[
tall = \frac{\text{High}}{\text{Michael}} + \frac{\text{Low}}{\text{Danny}} + \frac{\text{Medium}}{\text{Robert}}
\]

where membership functions of High, Low and Medium themselves are fuzzy sets. The former set is measured by one condition for one element, while the latter set is measured by several conditions for one element. Type-2 fuzzy sets are useful when the exact membership function for a type-1 fuzzy set cannot be easily determined, and are thus advantageous for incorporating uncertainties.

2.1. The definitions of type-2 fuzzy sets

According to (Mendel, 2001), type-2 fuzzy sets are defined as follows. For the sake of simplicity, the universe of discourse is assumed as a finite set, although the definition can be applied for infinite sets.

**Definition 2.1.** A type-2 fuzzy set, \( \tilde{A} \), is characterized by a type-2 membership function \( \mu_\tilde{A}(x) \), where \( X \) is the universal set, \( x \in X \) and \( u \in J_\tilde{A} \subseteq [0,1] \). \( J_\tilde{A} \) is the possible memberships collection for every \( x \). Meanwhile, the amplitude of a secondary membership function is called a secondary grade, and \( f_s(u) \) is a secondary grade. That is,

\[
\tilde{A} = \{(x, \mu_\tilde{A}(x))| x \in X \},
\]

or, as

\[
\tilde{A} = \sum_{x \in X} \mu_\tilde{A}(x) / x = \sum_{x \in X} \left[ \sum_{u \in J_\tilde{A}} f_s(u) / u \right] / x,
\]

\( J_\tilde{A} \subseteq [0,1], \)

In addition, because the membership grades of type-2 fuzzy sets are the values of type-1 fuzzy sets, performing operations like union and intersection on type-2 fuzzy sets is like performing t-conorm and t-norm operations between type-1 fuzzy sets. Hence, consider two type-2 fuzzy sets \( \tilde{A} \) and \( \tilde{B} \), such that

\[
\tilde{A} = \sum_{x \in X} \mu_\tilde{A}(x) / x = \sum_{x \in X} \left[ \sum_{u \in J_\tilde{A}^u} f_s(u) / u \right] / x,
\]

\( J_\tilde{A}^u \subseteq [0,1], \)

and

\[
\tilde{B} = \sum_{x \in X} \mu_\tilde{B}(x) / x = \sum_{x \in X} \left[ \sum_{w \in J_\tilde{B}^w} g_s(w) / w \right] / x,
\]

\( J_\tilde{B}^w \subseteq [0,1], \)

**Definition 2.2.** The union of \( \tilde{A} \) and \( \tilde{B} \) is another type-2 fuzzy set, which is defined as

\[
\tilde{A} \cup \tilde{B} = \sum_{x \in X} \left[ \sum_{u \in J_\tilde{A}^u} \sum_{w \in J_\tilde{B}^w} (f_s(u) \star g_s(w)) / (u \lor w) \right] / x,
\]

where \( \star \) denotes the general minimum operation. Similarly, the intersection of \( \tilde{A} \) and \( \tilde{B} \) is defined as

\[
\tilde{A} \cap \tilde{B} = \sum_{x \in X} \left[ \sum_{u \in J_\tilde{A}^u} \sum_{w \in J_\tilde{B}^w} (f_s(u) \star g_s(w)) / (u \land w) \right] / x,
\]

where \( \star \) denotes the general minimum operation.

The Cartesian production of type-2 fuzzy sets on different product spaces is defined as follows.

**Definition 2.3.** Let \( \tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n \) be type-2 fuzzy sets in universes of discourse \( X_1, X_2, \ldots, X_n \). The Cartesian product of \( \tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n \), denoted by \( \tilde{A}_1 \times \tilde{A}_2 \times \cdots \times \tilde{A}_n \), is a type-2 fuzzy set in the product space \( X_1 \times X_2 \times \cdots \times X_n \), given as

\[
\tilde{A}_1 \times \tilde{A}_2 \times \cdots \times \tilde{A}_n = \tilde{A}_1 \cap \tilde{A}_2 \cap \cdots \cap \tilde{A}_n = \sum_{x_1 \in X_1} \left[ \sum_{x_2 \in X_2} \cdots \sum_{x_n \in X_n} (f_1(u) \star \cdots \star f_n(w)) / (u \land \cdots \land w) \right] / (x_1, \ldots, x_n)
\]

where \( \star \) denotes the general minimum operation. In this study, a fuzzy relation is a fuzzy set defined in the Cartesian product. Accordingly, the projection of type-2 fuzzy sets is defined as follows.

**Definition 2.4.** Let \( \tilde{Q} \) denote a type-2 fuzzy relation in the product space \( X_1 \times X_2 \times \cdots \times X_n \), and let \( \{i_1, \ldots, i_n\} \) denote a subsequence of \( \{1,2,\ldots,n\} \). Then the projection of \( \tilde{Q} \) on \( X_{i_1} \times \cdots \times X_{i_k} \) is a fuzzy relation \( \tilde{Q}_{i_1,\ldots,i_k} \) in \( X_{i_1} \times \cdots \times X_{i_k} \) defined by

\[
\tilde{Q}_{i_1,\ldots,i_k} = \sum_{x_{i_1} \in X_{i_1}} \cdots \sum_{x_{i_k} \in X_{i_k}} \left[ \sup_{x_{i_{j-1}} \in X_{i_{j-1}} \ldots x_{i_{j-2}} \in X_{i_{j-2}}} \mu_\tilde{Q}(x_1, \ldots, x_n) \right] / (x_1, \ldots, x_n)
\]

\( \lor (x_1, \ldots, x_n) \)

where \( \lor \) denotes the general maximum operation.
where \( \{x_1, \ldots, x_{n+1}\} \) is the complement of \( \{x_1, \ldots, x_n\} \) with respect to \( \{x_1, \ldots, x_n\} \), and \( \bigcup \) is the operation of the fuzzy projection of type-2 fuzzy sets.

2.2. The similarity between type-2 fuzzy sets

Much work has already been conducted to define the degree of similarity between two general fuzzy sets. However, their similarity cannot be directly applied to handle partially true knowledge theoretically. Furthermore, Hung and Yang (2004) proposed the axiom definition for a type-2 similarity measure based on Hausdorff distance. They compute the similarities by treating the secondary values as having two opposite elements consisting of coexistence on peer secondary membership functions. However, according to Mendel’s definition (Mendel, 2001), elements in the primary memberships of \( x \) are not guaranteed to consist on the coexistence of peer two type-2 fuzzy sets. Thus, Hung’s method cannot compute the correctly answer of their definition.

Hence, in our study, the universe of discourse is assumed as a finite set, the type-2 fuzzy set \( A \) can be expressed as

\[
\tilde{A} = \sum_{x \in X} \left[ \sum_{u \in J_x} f_x(u)/u \right] / x = \sum_{i=1}^{N} \left[ \sum_{u \in J_{x_i}} f_{x_i}(u)/u \right] / x_i,
\]

\[
= \frac{\sum_{k=1}^{M_x} f_{x_k}(u_{ik})/u_{ik}}{x_1} + \cdots + \frac{\sum_{k=1}^{M_x} f_{x_k}(u_{nk})/u_{nk}}{x_N}.
\]

Observe that \( x \) has been discretized into \( N \) values, and that at each of these values \( u \) has been discretized into \( M_x \) values. Many choices are possible for the secondary membership functions. The secondary membership function can be treated as a type-1 membership function along each \( x \). Hence, the similarity between type-2 fuzzy sets can be defined as follows.

Definition 2.5. Let \( A \) and \( B \) be two type-2 fuzzy sets defined on the universe of discourse \( X \) for all sets having \( N \) elements. That is,

\[
\tilde{A} = \sum_{i=1}^{N} \mu_A(x_i)/x_i = \sum_{i=1}^{N} \left[ \sum_{u \in J_x} f_x(u)/u \right] / x_i, \quad u \in J_x,
\]

and

\[
\tilde{B} = \sum_{i=1}^{N} \mu_B(x_i)/x_i = \sum_{i=1}^{N} \left[ \sum_{u \in J_x} g_x(u)/u \right] / x_i, \quad u \in J_x,
\]

where \( J_x = J^A_x \cup J^B_x \), \( J^A_x \) and \( J^B_x \) are the primary memberships at the specific value \( x \) for \( A \) and \( B \). The similarity between a pair of type-2 fuzzy sets \( \{A, B\} \) is defined as

\[
\tilde{S}(\tilde{A}, \tilde{B}) = \frac{1}{N} \sum_{i=1}^{N} S(\mu_A(x_i), \mu_B(x_i)),
\]

where \( S(\cdot) \) can be any traditional similarity index for the general fuzzy sets. Note that, \( \mu_A(x_i) \) and \( \mu_B(x_i) \) are two secondary membership functions. For instance, the proposed similarity methods of Raha and Pappis (Raha, Pal, & Ray, 2002; Pappis & Karacapilidis, 1993) are

\[
S(A, B) = \frac{\sum_{x \in X} \{\mu_A(x) \cdot \mu_B(x)\}}{\sum_{x \in X} \max\{\mu_A(x)\} \cdot \mu_B(x)}/2,
\]

or

\[
S(A, B) = \frac{|A| \cdot |B| \cos(\theta)}{(\max(|A|, |B|))},
\]

where \( A \) and \( B \) are two type-1 fuzzy sets; \( |A| \) is the length of the vector \( A \), and \( \cos(\theta) \) is the cosine of the angle between the two vectors. An important consideration in this study is to select the similarity index of type-2 fuzzy sets such that the index exhibits the properties of similarity. Eq. (7) is adopted in this study, and the similarity index of type-2 fuzzy sets is formulated as follows:

\[
\tilde{S}(\tilde{A}, \tilde{B}) = \frac{1}{N} \sum_{i=1}^{N} S(\mu_A(x_i), \mu_B(x_i))
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \sum_{u \in J_x} \left[ f_x(u) \cdot g_x(u) \right] / \max\{\mu_A(x_i)\} \cdot \mu_B(x_i)/2,
\]

where \( x_i \in X \) and \( u \in J_x \). When \( \sum_{u} \max\{f_x(u), g_x(u)\} = 0 \), then \( S(\mu_A(x_i), \mu_B(x_i)) = 1 \), that is, \( \tilde{S}(\tilde{A}, \tilde{B}) = 1 \).

Note that, because the primary membership values may not be the same at a specific value of \( x \), that is \( J^A_x \neq J^B_x \), \( S(\mu_A(x_i), \mu_B(x_i)) \) cannot be exactly computed in some cases. Since the notion of a zero membership value generalizes in fuzzy set theory to the situation which a non-zero membership is not clearly stated, this concept has also been executed in this study. The value ‘0’ in the secondary grade denotes this item uselessly in the deterministic process. Hence, the value ‘0’ is applied as the appended secondary grade to those missing items in \( J^A_x \) or \( J^B_x \) for the similarity computation.

Example 2.1. Let \( U = \{x_1, x_2, x_3\} \), and

\[
\tilde{A} = \frac{(0.4/0.5) + (0.5/0.6)}{x_1} + \frac{(0.7/0.4) + (0.4/0.5)}{x_2} + \frac{(0.8/0.3) + (0.9/0.4)}{x_3},
\]

\[
\tilde{B} = \frac{(0.3/0.5) + (0.2/0.6)}{x_1} + \frac{(0.5/0.4) + (0.4/0.5)}{x_2} + \frac{(0.4/0.4)}{x_3}.
\]

Then, according to the previous statement, the missing item is appended to the \( \tilde{B} \) as
\[B = \frac{(0.3/0.5) + (0.2/0.6) + (0.5/0.4) + (0.4/0.5)}{x_1} + \frac{(0/0.3) + (0.4/0.4)}{x_3}.\]

Then, the following similarity is obtained
\[
\hat{S}(\tilde{A}, \tilde{B}) = \frac{1}{3} \sum_{i=1}^{3} \sum_{u \in \mathcal{X}} [\max\{f_u(x_i), g_u(x_i)\}]^2
= \frac{1}{3} \left(0.12 + 0.1 + 0.35 + 0.16 + 0.36 \right)
= 0.52.
\]

3. Reasoning with type-2 similarity

A type-1 fuzzy inference engine normally combines rules and provides a mapping from input type-1 fuzzy sets to output type-1 fuzzy sets. Additionally, the inference process is very similar in the type-2 inference case. The inference engine combines rules and provides a mapping from input type-2 fuzzy sets to output type-2 fuzzy sets. Another difference is in the defuzzification. In the type-2 case, the output sets are type-2; the extended defuzzification operation in the type-2 case gives type-1 fuzzy sets as the output. This operation is called a “type reducer”, and the type-1 fuzzy set is obtained as a “type reduced set”, which may then be defuzzified to obtain a single crisp number.

This study presents a new reasoning method, involving the similarity measure between type-2 fuzzy sets as an inference methodology. Consider two type-2 fuzzy sets \(A\) and \(A'\) defined on the same universe of discourse \(X\). Another two type-2 fuzzy sets \(B\) and \(B'\) are defined over the same universe of discourse \(Y\). Two corresponding linguistic variables \(x\) and \(y\) are also defined, then the typical propositions are given as

- **rule**: If \(x\) is \(\tilde{A}\) then \(y\) is \(\tilde{B}\).
- **fact**: \(x\) is \(\tilde{A}'\)
- **conclusion**: \(y\) is \(\tilde{B}'\).

Let \(\hat{S}(\tilde{A}, \tilde{A}')\) denote the similarity measure between two type-2 fuzzy sets \(A\) and \(A'\). Then, existing methods use the similarity measure to directly compute the inference without considering the induced relation. In the proposed method, the authors translate the conditional statement into a fuzzy relation. Then, the similarity between the fact and the antecedent of the rule is used to modify the derived relation. That is, every change in the conditional premise and the antecedent of the rule is used to modify the derived relation. The modified relation would be close to \(\hat{B}'\) less specific, then choosing the expansion form. Hence, with the decrease in similarity, which occurs at a significant difference between \(A\) and \(A'\), the inferred conclusion would be close to \(Y\). Conversely, when \(A' = A\), the inferred conclusion is obtained as \(\hat{B} = \hat{B}'\). Notably, when \(\hat{S}(A, A') = 0\), nothing can be concluded when \(\{A, A'\}\) are dissimilar, so \(\hat{B}' = \hat{B}'\) is yielded.

Subsequently, assume that \(k\) linguistic variables \(x_1, \ldots, x_k\), defined on the universe of discourses \(X_1, \ldots, X_k\), are considered. These typical propositions are listed as

- **rule**: if \(x_i\) is \(\tilde{A}\) and \(\ldots\) and \(x_k\) is \(\tilde{A}\) then \(y\) is \(\tilde{B}\)
- **fact**: \(x_i\) is \(\tilde{A}'\) and \(\ldots\) and \(x_k\) is \(\tilde{A}'\)
- **conclusion**: \(y\) is \(\tilde{B}\).

The proposed algorithm with multiple antecedents is stated as follows.

**Algorithm 3.1**

1. Select meaningful linguistic states for each variable, and express them by appropriate type-2 fuzzy sets; \(A_i, A'_i\) and \(\tilde{B}\), for \(i = 1, \ldots, k\). Compute the fuzzy relation \(\hat{Q}\) of \(A_1, \ldots, A_k\) and \(\tilde{B}\), according to Definition 2.3.
2. Compute similarities among type-2 fuzzy sets, \(\hat{S}(A_i, A'_i), \hat{S}(A_2, A'_2), \ldots, \hat{S}(A_k, A'_k)\).
3. Set \(\beta = \min\{\hat{S}(A_i, A'_i), \hat{S}(A_2, A'_2), \ldots, \hat{S}(A_k, A'_k)\}\).
4. Adjust \(\hat{Q}\) by applying \(\hat{S}(A_i, A'_i), \hat{S}(A_2, A'_2), \ldots, \hat{S}(A_k, A'_k)\) to obtain the conditional relation \(\hat{Q}'\) by

\[
\mu_{\hat{Q}}(x_1, \ldots, x_k, y) = \begin{cases} 
\sum_{x \in \mathcal{X}} \left(\frac{(\min(1, f_u(x_1, \ldots, x_k))}{\hat{S}(A, A')}/(\min(1, \hat{S}(A, A'))}\right), \\
\text{if } \hat{S}(\tilde{A}, \tilde{A}') \neq 0, \\
\mu_{\hat{Q}}(y), \\
\text{otherwise},
\end{cases}
\]
where \( x_1 \in X_1, \ldots, x_k \in X_k, y \in Y \).
Notably, \( \mu_{\tilde{Q}}(x_1, \ldots, x_k, y) = \sum_{u \in \mathcal{U}_{x_1, \ldots, x_k, y}} [(f_{x_1, \ldots, x_k, y}(u))/u] \).

Step 5. According to **Definition 2.4**, use sup-projection operation on \( \tilde{Q} \) to obtain \( \tilde{B} \) as
\[
\mu_{\tilde{B}}(y) = \sup_{x_1, x_2 \in X_1, \ldots, x_k \in X_k} \mu_{\tilde{Q}}(x_1, \ldots, x_k, y) = \bigcup_{x_1, x_2 \in X_1, \ldots, x_k \in X_k} \mu_{\tilde{Q}}(x_1, \ldots, x_k, y),
\]
where \( y \in Y \).

Following the previous proposition, a simple example is shown as follows.

**Example 3.1.** Let \( X = \{ x_1, x_2 \} \) and \( Y = \{ y_1, y_2 \} \). Additionally, let \( A, A' \) and \( B \) be type-2 fuzzy sets, that is
\[
\tilde{A} = \begin{pmatrix}
0.4/0.3 + 0.2/0.4 + 0.5/0.5 + 0.4/0.6 \\
0.4/0.3 + 0.2/0.4 + 0.7/0.5 + 0.8/0.6 \\
0.25/0.2 + 0.3/0.3 + 0.5/0.4 + 0.6/0.5
\end{pmatrix},
\]
\[
\tilde{A}' = \begin{pmatrix}
0.4/0.3 + 0.2/0.4 + 0.5/0.5 + 0.6/0.5 \\
0.4/0.3 + 0.2/0.4 + 0.5/0.4 + 0.6/0.6 \\
0.7/0.5 + 0.4/0.6 + 0.2/0.2 + 0.3/0.3 \\
0.25/0.2 + 0.3/0.3 + 0.2/0.2 + 0.2/0.3
\end{pmatrix},
\]
\[
\tilde{B} = \begin{pmatrix}
0.4/0.3 + 0.2/0.4 + 0.5/0.4 + 0.6/0.5 \\
0.4/0.3 + 0.2/0.4 + 0.5/0.4 + 0.6/0.5 \\
0.25/0.2 + 0.3/0.3 + 0.2/0.2 + 0.2/0.3
\end{pmatrix}.
\]

Accordingly, the fuzzy relation \( \tilde{Q} \) is computed as follows:
\[
\tilde{Q} = \begin{pmatrix}
0.4/0.3 + 0.2/0.4 + 0.5/0.5 + 0.4/0.6 \\
0.4/0.3 + 0.2/0.4 + 0.7/0.5 + 0.8/0.6 \\
0.25/0.2 + 0.3/0.3 + 0.5/0.4 + 0.6/0.5
\end{pmatrix}.
\]

Furthermore, adjust \( \tilde{Q} \) with \( \tilde{S}(\tilde{A}, \tilde{A}') \) to obtain the conditional relation
\[
\tilde{Q}' = \begin{pmatrix}
(0.4/0.35 + 0.35/0.35) + (0.4/0.35 + 0.35/0.35) \\
(0.4/0.35 + 0.35/0.35) + (0.4/0.35 + 0.35/0.35) \\
(0.4/0.35 + 0.35/0.35) + (0.4/0.35 + 0.35/0.35)
\end{pmatrix}.
\]

Consequently, \( \tilde{Q}' \) is projected to obtain \( \tilde{B}' \) by
\[
\tilde{B}' = \begin{pmatrix}
0.4/0.3 + 0.2/0.4 + 0.5/0.5 + 0.4/0.6 \\
0.4/0.3 + 0.2/0.4 + 0.7/0.5 + 0.8/0.6 \\
0.25/0.2 + 0.3/0.3 + 0.5/0.4 + 0.6/0.5
\end{pmatrix}.
\]

In contrast to type-1 inference, a recalling example is reviewed as follows:
\[
A = \begin{pmatrix}
0.4/0.6 \\
0.3/0.7 \\
0.2/0.5
\end{pmatrix},
\]
\[
A' = \begin{pmatrix}
0.4/0.6 \\
0.3/0.7 \\
0.2/0.5
\end{pmatrix},
\]
\[
B = \begin{pmatrix}
0.4/0.6 \\
0.3/0.7 \\
0.2/0.5
\end{pmatrix}.
\]

where \( A, A' \) and \( B \) are type-1 fuzzy sets. Based on the type-1 fuzzy inference according to the Mamdani implication, the conclusion \( B' \) is derived as a type-1 fuzzy set, shown as follows:
\[
B' = \begin{pmatrix}
0.2/0.3 + 0.6/0.6 \\
0.3/0.7 + 0.7/0.7 \\
0.2/0.5 + 0.5/0.5
\end{pmatrix}.
\]

The similarity between type-2 fuzzy sets is computed as
\[
\tilde{S}(\tilde{A}, \tilde{A}') = \frac{1}{2} \sum_{i=1}^{2} \tilde{S}(\tilde{Q}^i),
\]
\[
= \frac{1}{2} \left( \begin{array}{c}
\begin{pmatrix}
0.4/0.4 + 0.2/0.6 \\
0.4/0.4 + 0.2/0.6
\end{pmatrix} + \begin{pmatrix}
0.5/0.7 + 0.4/0.8 \\
0.5/0.7 + 0.4/0.8
\end{pmatrix}
\end{array} \right)
= 0.57.
\]
\[
\tilde{B}' = \frac{0.53/0.53 + 0.7/0.88}{y_1 + y_2}.
\]
Moreover, in the low-level truth linguistic problem, the conclusion of \(\tilde{B}'\) is modeled as
\[
\tilde{B}' = \frac{0.44/0.35 + 0.7/0.7}{y_1 + y_2}.
\]
Hence, type-2 fuzzy sets are able to handle uncertainties, as typified by the adage, “Words can mean different things to different people (Mendel, 1999).”

4. Truth-qualified proposition

In this study, the partially truth-qualified statements of the form are illustrated as follows.

“• David is healthy” is quite true,” or
“• The temperature is moderate” is mostly true.”

Simple statements are of the general propositional form,

“\(x\) is \(F\); \(t\) is \(Q\),”

where \(x\) and \(t\) are two linguistic variables, and \(t\) denotes the truth value. \(F\) represents the vague descriptions of the object \(x\), and \(Q\) denotes the truth of proposition “\(x\) is \(F\)”. That is, \(F\) and \(Q\) are type-1 fuzzy sets. Fig. 1 shows an example of the propositional form. Fig. 1a illustrates the membership function \(\mu_F(x)\) for \(X=\{0, 1, 2, 3, 4, 5\}\). Fig. 1b illustrates an example of truth qualification, which translates the obtained vague description into a truth value.

Consequently, the previous general propositional form can be translated into a type-2 fuzzy statement,

“\(x\) is \(\tilde{F}\),”

where
\[
\tilde{F} = \sum_{x \in X} \frac{\mu_F(x)}{\mu_F(x) / \mu_Q(x)} / x, \quad u \in J, \subset U = [0, 1].
\]

Notably, a secondary grade, \(\mu_Q(u)\), is applied to state the truth value. More specifically, \(\tilde{F}\) also can be considered as
\[
\tilde{F} = \sum_{x \in X} \frac{\mu_F(x)}{\mu_F(x) / \mu_Q(x)} / x, \quad F(x) \subset U = [0, 1].
\]

Hence, the partially truth-qualified statement is represented as the type-2 fuzzy set. Fig. 2 illustrates an example of the truth-qualified propositional form represented by a type-2 fuzzy set. For each \(x\), the corresponding secondary grade is derived from the truth value in Fig. 1b. Consequently, the truth value of a composite proposition is computed as follows.

\[
(x \text{ is } F; \; t \text{ is } Q) \land (x \text{ is } G; \; t \text{ is } R) = (x \text{ is } \tilde{F}) \land (x \text{ is } \tilde{G}) \Rightarrow \mu_{\tilde{F}}(x) \land \mu_{\tilde{G}}(x),
\]
\[
(x \text{ is } F; \; t \text{ is } Q) \lor (x \text{ is } G; \; t \text{ is } R) = (x \text{ is } \tilde{F}) \lor (x \text{ is } \tilde{G}) \Rightarrow \mu_{\tilde{F}}(x) \lor \mu_{\tilde{G}}(x),
\]
\[
(\neg(x \text{ is } F; \; t \text{ is } Q) = (x \text{ is } \tilde{F}^c) \Rightarrow \mu_{\tilde{F}^c}(x).
\]

Accordingly, the inference techniques and deductive processes are introduced based on the similarity measure among type-2 fuzzy sets. In the following, \(k\) linguistic variables \(x_1, \ldots, x_k\), defined on the universe of discourses \(X_1, \ldots, X_k\), are considered, where \(t\) denotes only the truth of the proposition. These typical propositions are listed as

\[
\text{rule: if } x_i \text{ is } A_i \text{ and } \cdots \text{ and } x_k \text{ is } A_k \text{ then } y \text{ is } B; \text{t is C_true}
\]
\[
\text{fact: } x_i \text{ is } A_i' \text{ and } \cdots \text{ and } x_k \text{ is } A_k' \text{;t is C_true}
\]
\[
\text{conclusion: } y \text{ is } B; \text{t is C_true.}
\]

Subsequently, the partially true proposition is represented by the statement of type-2 fuzzy sets,

\[
\text{rule: if } x_i \text{ is } A_i \text{ and } \cdots \text{ and } x_k \text{ is } A_k \text{ then } y \text{ is } \tilde{B}
\]
\[
\text{fact: } x_i \text{ is } A_i' \text{ and } \cdots \text{ and } x_k \text{ is } A_k'
\]
\[
\text{conclusion: } y \text{ is } \tilde{B}.
\]
Thus, at any particular time, the humidity of air is normalized in the choice of universe. Similarity, tolerance index ‘1.0’ means “feeling comfortable”; anything less than ‘1.0’ means “partially comfortable”, and ‘0.0’ means “absolutely uncomfortable”. Hence, the definitions of type-2 fuzzy sets are listed as follows.

Percentile humidity ∈ [0, 1].
Percentile tolerance index ∈ [0, 1].
Truth value ∈ [0, 1].

Then, the purpose is to represent the inexact concepts in the propositions as type-2 fuzzy sets based on an appropriate universe of discourses. Let the universe of discourses be denoted as follows:

\[
\text{Percentile humidity} = \frac{0.3 \times 0.25 + 0.6 \times 0.5 + 0.8 \times 0.75 + 1/1}{1} \leq 0.5 + 0.6 + 0.75 + 1/1 \\
\text{Percentile tolerance index} = \frac{0.9 \times 0.8 + 0.7 \times 0.65 + 0.6 \times 0.55 + 0.4 \times 0.35}{0.25 + 0.25 + 0.5 + 0.5} + \frac{0.3 \times 0.2 + 0.2 \times 0.1 + 0.2 \times 0.1}{1} \\
\text{Truth value} = \frac{0.6 \times 0.5 + 0.9 \times 0.75 + 1/1 + 0.9 \times 0.75}{1.0} \\
\]

Accordingly, similarity is given by

\[
\tilde{S}(\text{high, moderate}) = \frac{1}{4} \left( 0.3 \times 0.5 + 0.6 \times 0.5 + 0.5 \times 0.65 + 0.6 \times 0.5 + 0.5 \times 0.9 + 0.6 \times 0.5 + 0.5 \times 0.9 \\
+ 0.8 \times 0.5 + 0.5 \times 1 + 0.8 \times 0.5 + 0.5 \times 1 + 1 \times 0.5 + 0.9 \times 0.5 \\
+ 1 \times 0.5 + 0.9 \times 0.5 \\
\right) \\
= 0.61 \\
\]

and, the fuzzy relation \( \tilde{Q} \) is computed as follows:

\[
\tilde{Q} = \frac{0.3 \times 0.5 + 0.3 \times 0.5 + 0.3 \times 0.5 + 0.3 \times 0.35}{(0.25, 0.0) + (0.25, 0.125) + (0.25, 0.25) + (0.25, 0.5)} + \frac{0.3 \times 0.2 + 0.2 \times 0.1 + 0.6 \times 0.5 + 0.6 \times 0.5}{(0.25, 0.75) + (0.25, 1.0) + (0.5, 0.0) + (0.5, 0.125)} + \frac{0.6 \times 0.4 + 0.4 \times 0.35 + 0.3 \times 0.2 + 0.2 \times 0.1}{(0.5, 0.25) + (0.5, 0.5) + (0.5, 0.75) + (0.5, 1.0)} + \frac{0.8 \times 0.75 + 0.7 \times 0.65 + 0.6 \times 0.55}{(0.75, 0.0) + (0.75, 0.125) + (0.75, 0.25)} + \frac{0.4 \times 0.35 + 0.3 \times 0.2 + 0.2 \times 0.1 + 0.9 \times 0.8}{(0.75, 0.5) + (0.75, 0.75) + (0.75, 1.0) + (1.0, 0.0)} + \frac{0.7 \times 0.65 + 0.6 \times 0.55 + 0.4 \times 0.35 + 0.3 \times 0.2}{(1.0, 0.125) + (1.0, 0.25) + (1.0, 0.5) + (1.0, 0.75)} + \frac{0.2 \times 0.1}{(1.0, 1.0)} \\
\]
Furthermore, the relation $\hat{Q}$ is adjusted by

$$
\hat{Q'} = \frac{0.49/0.82 + 0.49/0.82 + 0.49/0.82 + 0.49/0.57}{(0.25, 0.0) + (0.25, 0.125) + (0.25, 0.25) + (0.25, 0.5)}
+ \frac{0.49/0.33 + 0.33/0.16 + 0.98/0.82}{(0.25, 0.75) + (0.25, 1.0) + (0.5, 0.0) + (0.5, 0.125)}
+ \frac{0.98/0.82 + 0.66/0.57 + 0.49/0.33 + 0.33/0.16}{(0.5, 0.25) + (0.5, 0.5) + (0.5, 0.75) + (0.5, 1.0)}
+ \frac{1/1}{(0.75, 0.0) + (0.75, 0.125) + (0.75, 0.25) + (0.75, 0.5)}
+ \frac{0.98/0.9}{0.66/0.57 + 0.49/0.33 + 0.33/0.16 + 1/1}
+ \frac{(0.75, 0.5) + (0.75, 0.75) + (0.75, 1.0) + (1.0, 0.0)}{1/1 + 0.98/0.9 + 0.66/0.57 + 0.49/0.33}
+ \frac{(1.0, 0.125) + (1.0, 0.25) + (1.0, 0.5) + (1.0, 0.75)}{0.33/0.16 + (1.0, 1.0)}.
$$

Consequently, $\hat{Q}$ is projected to obtain the conclusion according to

$$
\hat{B'} = \frac{(0.49/0.82)\square(0.98/0.82)\square(1/1)\square(1/1)}{0.0}
+ \frac{(0.49/0.82)\square(0.98/0.82)\square(1/1)\square(1/1)}{0.125}
+ \frac{(0.49/0.82)\square(0.98/0.82)\square(0.98/0.9)\square(0.98/0.9)}{0.25}
+ \frac{(0.49/0.57)\square(0.66/0.57)\square(0.66/0.57)\square(0.66/0.57)}{0.5}
+ \frac{(0.49/0.33)\square(0.49/0.33)\square(0.49/0.33)\square(0.49/0.33)}{0.75}
+ \frac{(0.33/0.16)\square(0.33/0.16)\square(0.33/0.16)\square(0.33/0.16)}{1.0}
+ \frac{1/1}{0.0 + 0.125 + 0.25 + 0.5 + 0.75 + 0.98/0.9 + 0.66/0.57 + 0.49/0.33}
+ \frac{0.33/0.16}{1.0}.
$$

Hence, the conclusion describes the tolerance when the humidity is moderate and the truth condition is true. According to the derivation, when the humidity is moderate, the people will feel less uncomfortable due to the secondary grades rising after the tolerance index “0.25”. Moreover, this case was also applied by the general fuzzy implication in (Raha & Ray, 1999), wherein the results was obtained as

$$
\hat{B'} = \frac{1}{0.0 + 0.125 + 0.25 + 0.5 + 0.75 + 1.0},
$$

and the truth function was given as

$$
\text{truth} = \frac{0.25 + 0.5 + 0.75 + 1 + 0.95}{0.75 + 0.8 + 0.85 + 0.9 + 0.95}.
$$

The corresponding results show that the general fuzzy implication is not sufficient to handle fuzzy proposition with the truth function, because the truth function is independent from the fuzzy processing, and the results of the truth function are hard to associate to the original proposition. Conversely, the partial truth statements are associated with the proposition in our proposed method, the set-theoretic considerations can be used to derive partial true knowledge theoretically.

### 5. Conclusions

This study has marked a new direction in approximate reasoning based on vague knowledge, which is associated with partial or incomplete truth values. The proposed method can be used to handle vague quantities by converting this partial truth value into a precisely quantified statement based on the type-2 fuzzy inference system. The membership functions of type-2 fuzzy sets have more parameters than those of type-1 fuzzy sets. Hence, type-2 fuzzy sets provide with more design degrees of freedom than type-2 fuzzy sets. Therefore, type-2 fuzzy sets may outperform type-1 fuzzy sets, especially in uncertain environments. For this reason, this study proposes a new indexing method for measuring the similarity between a pair of type-2 fuzzy sets, and has introduced a technique for reasoning with vague concepts based on these measures. The proposed methodology can perform reasoning with incomplete knowledge; help to yield meaningful resolutions using fuzzy sentential logic, and systematically compute uncertainty.

### References


Mendel, J. M. (1999). Computing with words when words can mean different things to different people. In International ICSC congress computational intelligence: Methods and applications. 3rd Annual symposium on fuzzy logic application.


