Family Gromov-Witten Invariants for Kähler Surfaces

Junho Lee
University of Minnesota
Minneapolis, MN, 55454

Abstract

The usual Gromov-Witten invariants are zero for Kähler surfaces with $p_g \geq 1$. In this paper we use analytic methods to define Family Gromov-Witten Invariants for Kähler surfaces. We prove that these are well-defined invariants of the deformation class of the Kähler structure and develop methods for computing them, including a version of the TRR formula and the symplectic sum formula. Finally, we explicitly compute some of these family GW invariants for elliptic surfaces including K3 surfaces.

Gromov-Witten invariants are counts of holomorphic curves in a symplectic manifold X. To define them using the analytic approach one chooses an almost complex structure J compatible with the symplectic structure and considers the set of maps $f : \Sigma \to X$ from Riemann surfaces Σ which satisfy the (nonlinear elliptic) J-holomorphic map equation

$$\overline{\partial}_J f = 0. \tag{0.1}$$

After compactifying the moduli space of such maps, one imposes constraints, requiring, for example, that the image of the map passes through specified points. With the right number of constraints and a generic J, the number of such maps is finite. That number is a GW invariant; it depends only on the symplectic structure of X.

There are some beautiful conjectures about what the counts of holomorphic curves on Kähler surfaces *ought to be* ([V],[KP],[YZ],[G]). However, as currently defined, the corresponding GW invariants of Kähler surfaces with $p_g \geq 1$ are all zero! This discrepancy occurs because GW invariants count curves for *generic* almost complex structures J, whereas Kähler structures are very special — Donaldson details this in [D]. They can have whole families of curves which disappear when the Kähler J is perturbed to a generic J. For example, a generic $K3$ surface ($p_g = 1$) has no holomorphic curves at all, whereas *algebraic* K3 surfaces do admit holomorphic curves.

Clearly a new version of the invariants is needed — one which counts the relevant holomorphic curves. Work in that direction is just beginning. Bryan and Leung ([BL1],[BL2]) defined such invariants for K3 and abelian surfaces by using the Twistor family; they were also able to calculate their invariants in important cases. In a preprint to appear shortly, Behrend-Fantechi [BF] have define invariants for a more general class of algebraic surfaces using algebraic geometry, but have not yet made calculations. We approach the same issues using the geometric analysis approach to GW invariants.

Given a Kähler manifold (X, ω, J, g) we constructs a $2p_g$-dimensional family of elements $K_J(f, \alpha)$ in $\Omega^{0,1}(f^*TX)$, where α is a real part of a holomorphic 2 form. We then modifies the J-holomorphic map equation (1) by considering the pairs (f, α) satisfying

$$\overline{\partial}_J f = K_J(f, \alpha). \tag{0.2}$$

The solutions of this equation form a moduli space whose dimension is $2p_g$ larger than the dimension of the usual GW moduli space.

Because α range over a vector space compactness is an issue. Here things get interesting because there are instances when the moduli space for (0.2) is *not* compact. In fact, when the map represents a component of a canonical divisor the moduli space is *never* compact. Nevertheless, there is a simple analytic criterion — the uniform boundedness of the energy of the map and the L^2 norm of α — that ensures that the moduli space is compact.

Theorem 0.1 *Let (X, J) be a Kähler surface and fix a genus g and a class $A \in H_2(X, \mathbb{Z})$. Denote by $C(J)$ the supremum of $E(f) + ||\alpha||_{L^2}$ over all (J, α)-holomorphic maps from genus g curves into X which represent A. If $C(J)$ is finite, then the family GW invariants*

$$GW_{g,k}^{J,\mathcal{H}}(X, A)$$

are well-defined. They are invariant under deformations $\{J_t\}$ of the Kähler structure with $C(J_t)$ bounded. Furthermore, if A is a $(1,1)$ class then all the maps which contribute to these invariants are in fact J-holomorphic.

The last sentence of Theorem 0.1 means that the invariants for $(1,1)$ classes are counts of holomorphic curves in (X, J). That is not the same as saying the invariants are enumerative, since there is no claim that each curve is counted with multiplicity one. But it does mean that the family GW invariants, which *a priori* are counts of maps which are holomorphic with respect to families of almost complex structures on X, are in fact calculable from the complex geometry of (X, J) alone.

Theorem 0.1 yields well-defined family GW invariants provided there is a finite energy bound $C(J)$. Following the Kodaira classification of surfaces, we verify the energy bound case-by-case using geometric arguments. That yields the following cases where the family GW invariants are well-defined.

Proposition 0.2 *The moduli space for a class A is compact, and hence the family GW invariants are well-defined, when (X, J) is*

(a) a K3 or abelian surface with $A \neq 0$,

(b) a minimal elliptic surface $\pi : E \to C$ with Kodaira dimension 1 with $-A \cdot$ (fiber class) $\neq$ deg $\pi_*(A)$, and

(c) a minimal surface of general type and A is in a certain subspace of the (1,1) classes (see Proposition 4.6).

The second half of this paper develops computational methods. We extend several existing techniques for calculating GW invariants to the family GW invariants. In particular, the 'TRR formula' applies to the family invariants, and at least some special cases of the symplectic sum formula [IP3] apply, with appropriate minor modifications to the formula. Those formulas enable us to enumerate the curves in the elliptic surfaces $E(n)$ for the class A= section plus multiples of the fiber.

Theorem 0.3 *Let $E(n)$ be a standard elliptic surface with a section s of self-intersection $-n$. Denote by S and F the homology class of the section and the fiber. Then the $g=0$ family GW invariants for the classes $A = S + dF$ are well-defined and are given by the generating function*

$$\sum_{d\geq 0} GW_{0,0}^{\mathcal{H}_n}\left(E(n), S+dF\right) t^d = \prod_{d\geq 0}\left(\frac{1}{1-t^d}\right)^{12n}. \tag{0.3}$$

Bryan and Leung used algebraic methods to show (0.3) for $K3$ surfaces (i.e. $n=2$) [BL1]. This provided a verification of the well-known Yau-Zaslow Conjecture [YZ] for those cases when the homology class A is primitive. On the other hand, the above formula for $n=1$ gives the ordinary GW-invariants of rational elliptic surface $E(1)$, which was shown by Ionel and Parker [IP3]. They related TRR formula and their sum formula for the relative invariants to obtain a quasi-modular form as in (0.3). We follow the same argument — relating TRR formula and sum formula — to show Theorem 0.3.

Section 1 gives the definition of a (J,α)-holomorphic map and some of the analytic consequences of that definition, most notably an expression for the energy in terms of pullback of the symplectic form and the form α. Section 2 begins by describing the relation between a complete linear system $|C|$ — or more generally a Severi variety — and the moduli space of (J,α)-holomorphic maps. That leads us to consider the family of (J,α)-holomorphic maps in which α is the real part of holomorphic 2-form; the corresponding family moduli space should be an analytic version of the Severi variety. As partial justification of that view, we prove the last statement of Theorem 0.1: any (J,α)-holomorphic map which represents a (1,1) class is in fact holomorphic (theorem 2.4).

Section 3 summarizes the analytic results which lead to the definition of the family GW-invariants. That involves constructing the virtual moduli cycle by adapting the method of Li and Tian [LT]. Thus defined, the family invariants satisfy a Divisor Axiom and a Composition Law analogous to those of ordinary GW-invariants.

Section 4 contains examples of Kähler surfaces with $p_g \geq 1$ with well-defined family invariants. We focus on minimal surfaces and establish the results summarized in Proposition 0.2 above. For the case of K3 and Abelian surfaces we prove that our family GW-invariants agree with the invariants defined by Bryan and Leung. That is done in the course of the proof of Theorem 4.3 by relating the holomorphic 2-forms to the twistor family.

Turning to the computations, we give an overview of the proof of Theorem 0.3 in Section 5. This argument is an extension of the elegant argument used by Ionel and Parker to compute the GW-invariants of $E(1)$ [IP3]. It involves computing the generating function for the invariants in two ways, first using the so-called TRR formula, and second using a symplectic sum formula as in [IP3]. Roughly, the only modification needed is a shift in the dimension counts. The extended TRR formula is proved in Section 6 and the sum formula is established in the last three sections.

Section 7 gives an alternative definition of the family invariants for $E(n)$ based on the idea of perturbing the (J,α)-holomorphic map equations as in [RT1] and [RT2]. This alternative definition is better suited to adapt the analytic arguments in [IP2] and [IP3] to a family version of sum formula. The proof of the sum formula begins by studying holomorphic maps into a degeneration of $E(n)$. Because $E(n)$ is a Kähler surface we are able to degenerate within a holomorphic family, rather than the symplectic family used in [IP3].

The degeneration family Z is described in Section 8. It is a family $\lambda : Z \to D^2$ whose fiber Z_λ at $\lambda \neq 0$ is a copy of $E(n)$ and whose center fiber is a union of $E(n)$ with $T^2 \times S^2$ along a fixed elliptic fiber V. As $\lambda \to 0$ maps into Z_λ converge to maps into Z_0, and by bumping α to zero along the fiber V we can ensure that the limits satisfy a simple matching condition along V (there is a single matching condition for the classes A that we consider). Conversely, if a map into Z_0 satisfies the matching condition then it can be smoothed to produce a map into Z_λ for small λ. That smoothing is the Gluing Theorem in [IP3] which we use in Section 9 to prove the required sum formular for the family invariants of $E(n)$.

The appendix contains a brief discussion of how the family GW invariants defined here relate to those defined by Behrend and Fantachi in [BF].

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1 (J,α)-holomorphic maps

A J-holomorphic map into an almost complex manifold (X,J) is a map $f : \Sigma \to X$ from a complex curve Σ (a closed Riemann surface with complex structure j) whose differential is complex linear. Equivalently, f is a solution of the J-holomorphic map equation

$$\overline{\partial}_J f = 0 \qquad \text{where} \qquad \overline{\partial}_J f = \frac{1}{2}(df + J df j).$$

In this section we will show that when X is four-dimensional there is natural infinite-dimensional family of almost complex structures parameterized the J-anti-invariant 2-forms on X.

Let (X,J) be a 4-dimensional almost Kähler manifold with the hermitian triple (ω,J,g). Using J, we can decompose $\alpha \in \Omega^2(X)$ as $\alpha = \alpha_+ + \alpha_-$ where

$$\alpha_+(u,v) = \frac{\alpha(u,v) + \alpha(Ju,Jv)}{2} \qquad \alpha_-(u,v) = \frac{\alpha(u,v) - \alpha(Ju,Jv)}{2} \tag{1.1}$$

Definition 1.1 *A 2-form α is called J-anti-invariant if $\alpha = \alpha_-$. Denote the set of all J-anti-invariant 2-forms by $\Omega^-_J(X)$. Each $\alpha \in \Omega^-_J(X)$ defines an endomorphism K_α of TX by the equation*

$$\langle u, K_\alpha v\rangle = \alpha(u,v). \tag{1.2}$$

It follows that

$$\langle K_\alpha u, v\rangle = -\langle u, K_\alpha v\rangle, \quad JK_\alpha = -K_\alpha J, \quad \text{and} \quad \langle Ju, K_\alpha u\rangle = 0. \tag{1.3}$$

Definition 1.2 *For $\alpha \in \Omega_J^-(X)$, a map $f : \Sigma \to X$ is called (J, α)-holomorphic if*

$$\overline{\partial}_J f = K_J(f, \alpha) \tag{1.4}$$

where $K_J(f, \alpha) = K_\alpha(\partial f \circ j) = \frac{1}{2} K_\alpha(df - Jdfj)j$.

The next proposition and its corollary list some pointwise relations involving the quantities that appear in the (J, α)-holomorphic equation. We state these first for general C^1 maps, then specialize to (J, α)-holomorphic maps.

Proposition 1.3 *Fix a metric within the conformal class j and let dv be the associated volume form. Then for any C^1 map f we have the pointwise equalities*

(a) $|\overline{\partial}_J f|^2 \; dv = \frac{1}{2}\, |df|^2 \; dv - f^*\omega,$ *(b)* $\langle \overline{\partial}_J f \, , \, K_J(f, \alpha)\rangle \; dv = f^*\alpha,$

(c) $K_\alpha^2 = -|\alpha|^2\, I,$ *(d)* $|K_J(f, \alpha)|^2 \; dv = |\alpha|^2 \left(\frac{1}{2}|df|^2 \; dv + f^*\omega\right).$

Proof. Fix a point $p \in \Sigma$ and an orthonormal basis $\{e_1, e_2 = je_1\}$ of $T_p\Sigma$. Setting $v_1 = df(e_1)$ and $v_2 = df(e_2)$, we have $2\overline{\partial}_J f(e_1) = v_1 + Jv_2$ and $2K_J(f, \alpha)(e_1) = K_\alpha v_2 - JK_\alpha v_1$, and similarly $2\overline{\partial}_J f(e_2) = v_2 - Jv_1$ and $2K_J(f, \alpha)(e_2) = -K_\alpha v_1 - JK_\alpha v_2$. Therefore,

$$\begin{aligned} 4|\overline{\partial}_J f|^2 = |v_1 + Jv_2|^2 + |v_2 - Jv_1|^2 &= 2(|v_1|^2 + |v_2|^2) + 4\langle v_1, Jv_2\rangle \\ &= 2|df|^2 - 4f^*\omega(e_1, e_2). \end{aligned}$$

That gives (a), and (b) follows from the similar computation

$$\begin{aligned} 4\langle \overline{\partial}_J f, K(f, \alpha)\rangle &= \langle v_1 + Jv_2, K_\alpha v_2 - JK_\alpha v_1\rangle + \langle v_2 - Jv_1, -K_\alpha v_1 - JK_\alpha v_2\rangle \\ &= \langle v_1, K_\alpha v_2\rangle - \langle v_1, JK_\alpha v_1\rangle + \langle Jv_2, K_\alpha\, v_2\rangle - \langle Jv_2, JK_\alpha v_1\rangle \\ &\qquad -\langle v_2, K_\alpha v_1\rangle - \langle v_2, JK_\alpha v_2\rangle + \langle Jv_1, K_\alpha v_1\rangle + \langle Jv_1, JK_\alpha v_2\rangle \\ &= 4\langle v_1, K_\alpha v_2\rangle \\ &= 4f^*\alpha(e_1, e_2). \end{aligned}$$

Next fix $x \in X$ and an orthonormal basis $\{w^1, w^2, w^3, w^4\}$ of T_x^*X with $w^2 = -Jw_1$ and $w^4 = -Jw^3$. Then the six forms

$$w^1 \wedge w^2 \pm w^3 \wedge w^4, \quad w^1 \wedge w^3 \pm w^2 \wedge w^4, \quad w^1 \wedge w^4 \pm w^2 \wedge w^3$$

give an orthonormal basis of $\Lambda^2(T_x^*X)$, and two of these span the subspace of J anti-invariant forms. Hence

$$\alpha = a(w^1 \wedge w^3 - w^2 \wedge w^4) + b(w^1 \wedge w^4 + w^2 \wedge w^3)$$

for some a and b, and in this basis K_α is the matrix

$$\begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & b & -a \\ -a & -b & 0 & 0 \\ -b & a & 0 & 0 \end{pmatrix}$$

Consequently, $K_\alpha^2 = -(a^2+b^2)I = -|\alpha|^2 I$. Lastly, since K_α is skew-adjoint, (c) shows that

$$|K_J(f,\alpha)|^2 \;=\; -\langle \partial f \circ j,\; K_\alpha^2(\partial f \circ j)\rangle \;=\; |\alpha|^2\, |\partial f|^2.$$

Equation (d) then follows from (a) because $|df|^2 = |\partial f|^2 + |\overline{\partial}_J f|^2$. □

Corollary 1.4 *Suppose the map $f : \Sigma \to X$ is (J,α)-holomorphic. Then*

(a) $|\overline{\partial}_J f|^2 \; dv = f^*\alpha,$

(b) $(1-|\alpha|^2)\; |df|^2 \; dv = 2\,(1+|\alpha|^2)\, f^*\omega,$ *and*

(c) $|\alpha|^2\, |df|^2 = (1+|\alpha|^2)\, |\overline{\partial}_J f|^2.$

Proof. Since f is (J,α)-holomorphic, $|\overline{\partial}_J f|^2 = \langle \overline{\partial}_J f, K_J(f,\alpha)\rangle = |K_J(f,\alpha)|^2$, so (a) follows from Proposition 1.3b while (b) and (c) follow from Proposition 1.3 (a) and (d). □

There is a second way of writing the (J,α)-holomorphic equation (1.4). For each $\alpha \in \Omega_J^-(X)$, $I + JK_\alpha$ is invertible since JK_α is skew-adjoint. Hence

$$J_\alpha = (I + JK_\alpha)^{-1} J (I + JK_\alpha) \tag{1.5}$$

is an almost complex structure. A map $f : \Sigma \to X$ is (J,α)-holomorphic if and only if f is J_α-holomorphic, i.e. satisfies

$$\overline{\partial}_{J_\alpha} f \;=\; \frac{1}{2}\,(df + J_\alpha df j) \;=\; 0. \tag{1.6}$$

From this perspective, a solution of the (J,α)-holomorphic equation is a J_α holomorphic map with J_α lying in the family (1.5) parameterized by $\alpha \in \Omega_J^-(X)$. In particular, we see from (1.6) that the (J,α)-holomorphic equation is elliptic.

Proposition 1.5 *For any $\alpha \in \Omega_-^J(X)$, the almost complex structure J_α on X satisfies*

$$\langle J_\alpha u, J_\alpha v\rangle = \langle u, v\rangle \qquad \textit{and} \qquad J_\alpha = \frac{1-|\alpha|^2}{1+|\alpha|^2}\; J - \frac{2}{1+|\alpha|^2}\; K_\alpha \tag{1.7}$$

Proof. From (1.3), the endomorphisms $A_+ = I + JK_\alpha$ and $A_- = I - JK_\alpha$ are transposes, and $A_+J = JA_-$ and $A_+K_\alpha = K_\alpha A_-$. Consequently, A_+^{-1} and A_-^{-1} are transposes, with $A_-^{-1}J = JA_+^{-1}$ and $A_-^{-1}K_\alpha = K_\alpha A_+^{-1}$ and therefore $A_-^{-1}A_+ = A_+A_-^{-1}$. Consequently,

$$\begin{aligned}\langle J_\alpha u, J_\alpha v\rangle &= \langle A_+^{-1}JA_+u, A_+^{-1}JA_+v\rangle = \langle JA_-^{-1}A_+u, JA_-^{-1}A_+v\rangle \\ &= \langle A_-^{-1}A_+u, A_-^{-1}A_+v\rangle = \langle u, A_-A_+^{-1}A_-^{-1}A_+v\rangle \\ &= \langle u, v\rangle.\end{aligned}$$

On the other hand, noting that $K_\alpha^2 = -|\alpha|^2 I$, it is easy to verify that

$$(I + JK_\alpha)^{-1} = \frac{1}{1+|\alpha|^2}\, I - \frac{1}{1+|\alpha|^2}\, JK_\alpha. \tag{1.8}$$

With that, the second part of (1.7) follows from the definition of J_α. □

In summary, (J,α)-holomorphic maps can be regarded as solutions of the J_α-holomorphic map equation $\overline{\partial}_{J_\alpha} f = 0$ for a family of almost complex structures parameterized by α as in (1.6). We will frequently move between these two viewpoints.

2 Curves and Canonical Families of (J,α) Maps

Given a Kähler surface X, we would like to use (J,α)-holomorphic curves to solve the following problem in enumerative geometry:

Enumerative Problem Give a $(1,1)$ homology class A, count the curves in X that represent A, have a specified genus g, and pass through the appropriate number of generic points.

We begin this section with some dimension counts which show that in order to interpret this problem in terms of holomorphic maps we need to consider families of maps of dimension p_g. We then show that there is a very natural family of (J,α)-holomorphic maps with exactly that many parameters. We conclude the section with a theorem showing that such maps do indeed represent holomorphic curves in X.

One can phrase the above enumerative problem in terms of the *Severi variety* $V_g(C) \subset |C|$, which is defined to be the closure of the set of all curves with geometric genus g. Assuming that $C - K$ is ample, it follows from the Riemann-Roch theorem that the dimension of the complete linear system $|C|$ is given in terms of $p_g = \dim_{\mathbb{C}} H^{0,2}(X)$ and $q = \dim_{\mathbb{C}} H^{0,1}(X)$ by

$$\dim_{\mathbb{C}}|C| \;=\; \frac{C^2 - C\cdot K}{2} \;+\; p_g \;-\; q$$

and the formal dimension of the Severi variety is

$$\dim_{\mathbb{C}} V_g(C) \;=\; -K\cdot C \;+\; g-1 \;+\; p_g \;-\; q. \tag{2.1}$$

The right-hand side of (2.1) is the 'appropriate number' of point constraints to impose; the set of curves in $V_g(C)$ through that many generic points should be finite, making the enumerative problem well-defined.

Now let $\mathcal{M}_g(X, A)$ be the moduli space of holomorphic maps from Riemann surfaces of genus g, which represent homology class A. Then its virtual dimension is given by

$$\dim{}_{\mathbb{C}}\mathcal{M}_g(X, A) \;=\; -K\cdot A \;+\; g-1. \tag{2.2}$$

The image of a map in $\overline{\mathcal{M}}_g(X,[C])$ might be not a divisor in $|C|$, instead it is a divisor in some other complete linear system $|C'|$ with $[C'] = [C]$. As in [BL1], we define the parameterized Severi variety

$$V_g([C]) = \coprod_{[C']=[C]} V_g(C')$$

Its expected dimension is now given by

$$\dim_{\mathbb{C}} V_g([C]) \;=\; -K\cdot C \;+\; g-1 \;+\; p_g. \tag{2.3}$$

We still have p_g dimensional difference between (2.3) and (2.2). Therefore, the cut-down moduli space by (2.3) many point constraints is empty when $p_g \geq 1$. This implies that the corresponding Gromov-Witten invariants is zero, whenever $p_g \geq 1$.

We show that there is a natural — in fact obvious — p_g-dimensional family of (J,α)-holomorphic maps associated with every Kähler surface.

Definition 2.1 *Given a Kähler surface on X, define the parameter space $\mathcal{H}$ by*

$$\mathcal{H} \;=\Big\{\alpha+\overline{\alpha}\;\Big|\;\alpha\in H^{2,0}(X)\Big\} \tag{2.4}$$

Here $H^{2,0}(X)$ means the set of holomorphic $(2,0)$ forms on X. Note that all forms $\alpha \in H^{2,0}(X)$ are closed since $d\alpha = \partial\alpha + \overline{\partial}\alpha = \partial\alpha$ is a $(3,0)$ form and hence vanishes because X is a complex surface. Thus $\mathcal{H} \subset \Omega^-_J(X)$ is a $2p_g$-dimensional real vector space of closed forms. We give it the (real) inner product defined by the L^2 inner product of forms:

$$\langle\alpha,\;\beta\rangle \;=\; \int_X \alpha\wedge\beta. \tag{2.5}$$

We can use the forms $\alpha \in \mathcal{H}$ to parameterize the right-hand side of the (J,α)-holomorphic map equation (1.2).

Definition 2.2 *Henceforth the term '(J,α)-holomorphic map' means a map satisfying (1.2) for α in the above family $\mathcal{H}$.*

Lemma 2.3 *The zero divisor $Z(\alpha)$ of each nonzero $\alpha \in \mathcal{H}$ represents the canonical class.*

Proof. Write $\alpha = \beta + \overline{\beta}$ with $\beta \in H^{2,0}(X)$. Since β is a section of the canonical bundle, this means that $Z(\alpha) = Z(\beta)$ represents the canonical divisor with appropriate multiplicities. □

Next, using this $2p_g$ dimensional parameter space $\mathcal{H}$, we define the family moduli space

$$\overline{\mathcal{M}}_g^{\mathcal{H}}(X,[C]) \;=\; \{\ (f,\alpha) \mid \overline{\partial}_{J_\alpha} f = 0,\ [\mathrm{Im}\, f] = [C],\ \text{and}\ \alpha \in \mathcal{H}\ \}$$

Since we just parameterize the $\overline{\partial}$-operator by $2p_g$ dimensional parameter space, the formal dimension of the family moduli space is given by

$$(\,\text{Formal}\,)\ \dim_{\mathbb{C}} \overline{\mathcal{M}}_g^{\mathcal{H}}(X,[C]) \;=\; -K\cdot C \;+\; g-1 \;+\; p_g$$

On the other hand, we define *a component of the canonical class* to be a homology class of a component of some canonical divisor.

Theorem 2.4 *If f is a (J,α)-holomorphic map which represents a class $A \in H^{1,1}(X)$. Then f is, in fact, holomorphic. Furthermore, if A is not a linear combination of components of the canonical class, then $\alpha = 0$.*

Proof. Since $\alpha \in H^{2,0}(X) \oplus H^{2,0}(X)$ is closed and $A \in H^{1,1}(X)$, it follows from Corollary 1.4a that

$$\int_\Sigma |\overline{\partial}_J f|^2 = \alpha(A) = 0.$$

Thus f is holomorphic, that is, $\overline{\partial}_J f \equiv 0$. Consequently, $|\alpha|^2 |df|^2 \equiv 0$ by Corollary 1.4c. Since df has at most finitely many zeros, we can conclude that $\alpha = 0$ along the image of f. Hence $\alpha = 0$, otherwise it contradicts to the assumption on A by Lemma 2.3. □

3 Family GW-Invariants

Let X be a complex surface with a Kähler structure (ω, J, g). In this section we will define the Family Gromov-Witten Invariants associated to (X, J) and the parameter space $\mathcal{H}$ of (2.4). We also state some properties of these invariants.

Our approach is the same analytic arguments as that of Li and Tian [LT] to show that the moduli space of (J,α)-holomorphic maps carries a virtual fundamental class whenever it is compact. While compactness is automatic for the usual Gromov-Witten invariants, it must be verified case-by-case for the family GW invariants (see Example 3.5). Thus compactness appears as a hypothesis in the results of this section.

First, we recall the notion of C^ℓ stable maps as defined in [LT]. Fix an integer $l \geq 0$ and consider pairs $(f; \Sigma, x_1, \cdots, x_k)$ consisting of

1. a connected nodal curve $\Sigma = \bigcup_{i=1}^{m} \Sigma_i$ of arithmetic genus g with distinct smooth marked points $x_1, \cdots, x_k$, and

2. a continuous map $f : \Sigma \to X$ so that each restriction $f_i = f_{|\Sigma_i}$ lifts to a C^l-map from the normalization $\tilde{\Sigma}_i$ of Σ into X.

Definition 3.1 *A stable* C^l map *of genus* g *with* k *marked points is a pair* $(f; \Sigma, x_1, \cdots, x_k)$ *as above which satisfies the stability condition:*

- *If the homology class* $[f_i] \in H_2(X, \mathbb{Q})$ *is trivial, then the number of marked points in* Σ_i *plus the arithmetic genus of* Σ_i *is at least three.*

Two stable maps $(f, \Sigma; x_1, \cdots, x_k)$ and $(f', \Sigma'; x'_1, \cdots, x'_k)$ are equivalent if there is a biholomorphic map $\sigma : \Sigma \mapsto \Sigma'$ such that $\sigma(x_i) = x'_i$ for $1 \le i \le k$ and $f' = f \circ \sigma$. We denote by

$$\overline{\mathcal{F}}^l_{g,k}(X, A)$$

the space of all equivalence classes $[f; \Sigma, x_1, \cdots, x_k]$ of C^l-stable maps of genus g with k marked points and with total homology class A. The topology of $\overline{\mathcal{F}}^l_{g,k}(X, A)$ is defined by sequential convergence as in section 2 of [LT]. There are two continuous maps from $\overline{\mathcal{F}}^l$. First, there is an evaluation map

$$ev : \overline{\mathcal{F}}^l_{g,k}(X, A) \to X^k \tag{3.1}$$

which records the images of the marked points. Second, for $2g + k \ge 3$, collapsing the unstable components of the domain gives a stabilization map

$$st : \overline{\mathcal{F}}^l_{g,k}(X, A) \to \overline{\mathcal{M}}_{g,k} \tag{3.2}$$

to the compactified Deligne-Mumford space of genus g curves with k marked points. For $2g+k < 3$ we define $\overline{\mathcal{M}}_{g,k}$ to be the topological space of consisting of a single point and define (3.2) to be the map to that point.

We next construct a 'generalized bundle' E over $\overline{\mathcal{F}}^l_{g,k}(X, A) \times \mathcal{H}$, again following [LT]. Recall that each $\alpha \in \mathcal{H}$ defines an almost complex structure J_α on X by (1.5). Denote by Reg(Σ) the set of all smooth points of Σ. For each $([f; \Sigma, x_1, \cdots, x_k], \alpha)$, define

$$\Lambda_{j_\Sigma J_\alpha}(f^*TX)$$

to be the set of all continuous sections ν of $\mathrm{Hom}(T\mathrm{Reg}(\Sigma), f^*TX)$ with $\nu \circ j_\Sigma = -J_\alpha \circ \nu$ such that ν extends continuously across the nodes of Σ. We take E to be the bundle whose fiber over $([f, \Sigma; x_1, \cdots, x_k], \alpha)$ is $\Lambda_{j_\Sigma J_\alpha}(f^*TX)$ and give E the continuous topology as in section 2 of [LT]. We then define a section $\Phi : \overline{\mathcal{F}}^l_{g,k}(X, A) \times \mathcal{H} \to E$ by

$$\Phi([f, \Sigma; x_1, \cdots, x_k], \alpha) = df + J_\alpha df j_\Sigma. \tag{3.3}$$

The right-hand side of (3.3) vanishes for J_α-holomorphic maps. Thus $\Phi^{-1}(0)$ is the moduli space of (J, α)-holomorphic maps. The following is a family version of Proposition 2.2 in [LT].

Proposition 3.2 *Suppose that the set* $\Phi^{-1}(0)$ *is compact. Then the section* Φ *gives rise to a generalized Fredholm orbifold bundle with a natural orientation and with index*

$$r \;=\; 2c_1(X)[A] + 2(g-1) + 2k + \dim \mathcal{H}. \tag{3.4}$$

By Theorem 1.2 of [LT], the bundle E has a rational homology "Euler class" in $\overline{\mathcal{F}}^l_{g,k}(X,A)\times\mathcal{H}$; in fact, since $\mathcal{H}$ is contractible this Euler class lies in $H_r(\overline{\mathcal{F}}^l_{g,k}(X,A);\mathbb{Q})$ where r is the index (3.4). We call this class the *virtual fundamental cycle* of the moduli space of family holomorphic maps parameterized by $\mathcal{H}$ and denote it by

$$[\mathcal{M}^{J,\mathcal{H}}_{g,k}(X,A)]^{\mathrm{vir}}. \tag{3.5}$$

In particular,

$$\dim\,[\mathcal{M}^{J,\mathcal{H}}_{g,k}(X,A)]^{\mathrm{vir}} \;=\; 2c_1(X)[A]+2(g-1)+2k+2p_g. \tag{3.6}$$

The next issue is whether the virtual fundamental cycle is independent of the Kähler structure on X. The next proposition is analogous to the Proposition 2.3 in [LT]. It shows that the virtual fundamental cycle depends only on certain deformation class of the Kähler structure.

.

Proposition 3.3 *Let (ω_t,J_t,g_t), $0\le t\le 1$, be a continuous family of Kähler structures on X. Let $\mathcal{H}_t$ be the corresponding continuous family of finite subspaces defined by (2.4) and let Φ_t be the corresponding family of sections of E_t over $\overline{\mathcal{F}}^l_{g,k}(X,A)\times\mathcal{H}_t$. If $\Phi_t^{-1}(0)$ is compact for all $0\le t\le 1$, then*

$$[\mathcal{M}^{J_0,\mathcal{H}_0}_{g,k}(X,A)]^{\mathrm{vir}}=[\mathcal{M}^{J_1,\mathcal{H}_1}_{g,k}(X,A)]^{\mathrm{vir}}.$$

The family GW invariants can now be defined by pulling back cohomology classes by the evaluation and stabilization maps and integrating over the virtual fundamental cycle. That of course requires that the virtual fundamental cycle exists, so we must assume that we are in a situation where $\Phi_t^{-1}(0)$ is compact.

Definition 3.4 *Whenever the virtual fundamental cycle $[\mathcal{M}^{J,\mathcal{H}}_{g,k}(X,A)]^{\mathrm{vir}}$ exists, we define the family GW invariants of (X,J) to be the map*

$$GW^{J,\mathcal{H}}_{g,k}(X,A):[H^*(X;\mathbb{Q})]^k\times H^*(\overline{\mathcal{M}}_{g,k};\mathbb{Q})\mapsto\mathbb{Q}$$

defined on $\alpha_1,\cdots,\alpha_k\in H^*(X;\mathbb{Q})$ and $\beta\in H^*(\overline{\mathcal{M}}_{g,k};\mathbb{Q})$ by

$$GW^{J,\mathcal{H}}_{g,k}(X,A)(\beta;\alpha_1,\cdots,\alpha_k)=[\mathcal{M}^{J,\mathcal{H}}_{g,k}(X,A)]^{\mathrm{vir}}\cap\left(st^*(\beta)\cup ev^*(\pi_1^*\alpha_1\wedge\cdots\wedge\pi_k^*\alpha_k)\right).$$

We will use the shorter notation

$$GW^{J,\mathcal{H}}_{g,k}(X,A)(\alpha_1,\cdots,\alpha_k)$$

for the special case when $\beta=1\in H^0(\overline{\mathcal{M}}_{g,k})$; this corresponds to imposing no constraints on the complex structure of the domain.

The condition that $\Phi^{-1}(0)$ is compact must be checked "by hand". In general, $\Phi^{-1}(0)$ is compact for some choices of A, but not for others.

Example 3.5 Let (X, J) be a Kähler surface with $p_g > 1$. Then there is a non-zero element $\beta \in H^{2,0}$ whose zero set $Z(\beta)$ is non-empty, represents the canonical class K, and whose irreducible components can be parameterized by holomorphic maps. Fix a parameterization $f : \Sigma \to X$ of one such component; this represents a non-zero class $A \in H_2(X, \mathbb{Z})$. Then $\alpha = \beta + \overline{\beta}$ lies in the space $\mathcal{H}$ of (2.1) and $\Phi(f, \lambda\alpha) = 0$ for all real λ. Thus on any Kähler surface with $p_g > 1$, the set $\Phi^{-1}(0)$ is not compact for an component of the canonical class A.

On the other hand, in the next section we will give examples of classes A in Kähler surfaces with $p_g > 1$ for which $\Phi^{-1}(0)$ *is* compact.

Theorem 3.6 *If there is a constant C, depending only on (X, ω, J, g) such that $E(f) + ||\alpha|| < C$ for all (J, α)-holomorphic maps into (X, J), then $\Phi^{-1}(0)$ is compact and hence the family GW invariants are well-defined.*

Proof. Consider a sequence (f_n, α_n) of J_α-holomorphic maps. The uniform bound on $||\alpha_n||$ implies that the J_α lie in a compact family. Since $E(f_n) < C$ the proof of Gromov's Compactness Theorem (see [PW] and [IS]) shows that $\{(f_n, \alpha_n)\}$ has a convergent subsequence. Consequently, $\Phi^{-1}(0)$ is compact as in the hypothesis of Proposition 3.2. That means that the virtual fundamental cycle (3.5) is well-defined. The family GW invariants are then given by Definition 3.4. $\square$

We conclude this section by listing two important properties of the family GW invariants. These are analogous to divisor axiom and composition laws of ordinary GW invariants.

Proposition 3.7 (Divisor Axiom) *If $\alpha_k \in H^2(X, Z)$ then*

$$GW^{J,\mathcal{H}}_{g,k}(X, A)(\alpha_1, \cdots, \alpha_k) = \alpha_k(A)\ GW^{J,\mathcal{H}}_{g,k-1}(X, A)(\alpha_1, \cdots, \alpha_{k-1}). \tag{3.7}$$

The second property generalizes the composition law of ordinary Gromov-Witten invariants. For that we consider maps from a domain Σ with node p and relate them to maps whose domain is the normalization of Σ at p. When the node is separating the genus and the number of marked points decompose as $g = g_1 + g_2$ and $k = k_1 + k_2$ and is a natural map

$$\sigma : \overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1} \mapsto \overline{\mathcal{M}}_{g,k} \tag{3.8}$$

defined by gluing $(k_1 + 1)$-th marked point of the first component to the first marked point of the second component. We denote by $PD(\sigma)$ the Poincaré dual of the image of this map σ.

Given any decomposition $A = A_1 + A_2$, $g = g_1 + g_2$, and $k = k_1 + k_2$ let $E_1 \oplus E_2^t$ be the generalized bundle over

$$\overline{\mathcal{F}}_{g_1,k_1+1}(X, A_1) \times \overline{\mathcal{F}}_{g_2,k_2+1}(X, A_2) \times \mathcal{H}$$

whose fiber over $([f_1, \Sigma_1; \{x_i\}], [f_2, \Sigma_2; \{y_j\}], \alpha)$ is $\Lambda^{0,1}_{j_{\Sigma_1} J_\alpha} \oplus \Lambda^{0,1}_{j_{\Sigma_2} J_{t\alpha}}$. The formula

$$\Psi_t([f_1, \Sigma_1; \{x_i\}], [f_2, \Sigma_2; \{y_j\}], \alpha) = (df_1 + J_\alpha df_1 j_{\Sigma_1}, df_2 + J_{t\alpha} df_2 j_{\Sigma_2}) \tag{3.9}$$

defines a section of $E_1 \oplus E_2^t$.

On the other hand, for non-separating nodes there is another natural map

$$\theta : \overline{\mathcal{M}}_{g-1,k+2} \mapsto \overline{\mathcal{M}}_{g,k} \tag{3.10}$$

defined by gluing the last two marked points. We also write $PD(\theta)$ for the Poincaré dual of the image of θ. The composition law is then the following two formulas.

Proposition 3.8 (Composition Law) *Let $\{H_\gamma\}$ be any basis of $H^*(X;Z)$ and $\{H^\gamma\}$ be its dual basis and suppose that $GW^{J,\mathcal{H}}_{g,k}(X,A)$ is defined.*

(a) Given any decomposition of (A,g,k), if the set $\Psi_t^{-1}(0)$ is compact for all $0 \le t \le 1$, then

$$\begin{aligned} &GW^{J,\mathcal{H}}_{g,k}(X,A)(PD(\sigma);\alpha_1,\cdots,\alpha_k) \\ &= \sum_{A=A_1+A_2}\sum_\gamma GW^{J,\mathcal{H}}_{g_1,k_1+1}(X,A_1)(\alpha_1,\cdots,\alpha_{k_1},H_\gamma)GW_{g_2,k_2+1}(X,A_2)(H^\gamma,\alpha_{k_1+1},\cdots,\alpha_k)\end{aligned}$$

(b) $GW^{J,\mathcal{H}}_{g,k}(X,A)(PD(\theta);\alpha_1,\cdots,\alpha_k) = \sum_\gamma GW^{J,\mathcal{H}}_{g-1,k+2}(X,A)(\alpha_1,\cdots,\alpha_k,H_\gamma,H^\gamma)$

That completes our overview of the family GW invariants. We next look at some examples, namely the various types of minimal Kähler surfaces. There we can use the specific geometry of the space to verify that the moduli space is compact and hence the family GW invariants are well-defined.

4 Kähler surfaces with $p_g \ge 1$

In this section we will focus on the family GW-invariants for minimal Kähler surfaces X with $p_g \ge 1$. The Enriques-Kodaira Classification [BPV] separates such surfaces into the following three types.

1. X is K3 or Abelian surface with canonical class $K = 0$. In this case, $p_g = 1$.
2. X is an elliptic surface $\pi : X \to C$ with Kodaira dimension 1. If the multiple fibers B_i have multiplicity m_i, then a canonical divisor is

$$K = \pi^* D + \sum (m_i - 1) B_i \qquad \text{where} \quad \deg D = 2g(C) - 2 + \chi(\mathcal{O}_X) \tag{4.1}$$

3. X is a surface of general type with $K^2 > 0$.

We will examine these cases one at a time. For each we will show that the family invariants $GW^{J,\mathcal{H}}_{g,k}(X,A)$ are well-defined. By Theorem 3.6 the key issue is bounding the energy $E(f)$ and the pointwise norm $|\alpha|$ uniformly for all (J,α)-holomorphic maps into X.

K3 and Abelian Surfaces

Let (X, J) be a K3 or Abelian surface. Since the canonical class is trivial, Yau's proof of the Calabi conjecture implies that (X, J) has a Kähler structure (ω, J, g) whose metric g is Ricci flat. For such a structure all holomorphic $(0, 2)$ forms are parallel, and hence have pointwise constant norm (see [B]). Thus $\mathcal{H} \cong \mathbb{C}$ consists of closed forms α with $|\alpha|$ constant. Furthermore, the structure is also hyperkähler, meaning that there is a three-dimensional space of Kähler structures which is isomorphic as an algebra to the imaginary quaternions. The unit two-sphere in that space is the so-called *Twistor Family* of complex structures.

Consider the set $\mathcal{T}_0 = \{\, J_\alpha \mid \alpha \in \mathcal{H}\}$. Since α has no zeros, equation (1.7) shows that $J_\alpha \to -J$ uniformly as $|\alpha| \to \infty$. We can therefore compactify $\mathcal{T}_0$ to $\mathcal{T} \cong \mathbb{P}^1$ by adding $-J$ at infinity.

Proposition 4.1 *$\mathcal{T}$ is the Twistor Family induced from the hyperkähler metric g.*

Proof. Let $\alpha \in \mathcal{H}$ with $|\alpha| = 1$. It then follows from Proposition 1.5 that $J_\alpha = -K_\alpha$ and (α, J_α, g) is a Kähler structure on X. On the other hand, we define α' by $\alpha'(u, v) = \alpha(u, Jv)$. Then $|\alpha'| = 1$ and $\alpha' \in \mathcal{H}$ since β' is holomorphic for each holomorphic 2-form β. Moreover, by definition we have

$$J_{\alpha'} = -K_{\alpha'} = -JK_\alpha = JJ_\alpha.$$

Since $(\alpha', J_{\alpha'}, g)$ is also Kähler and $JJ_\alpha J_{\alpha'} = -Id$, the Kähler structures $\{J, J_\alpha, J_{\alpha'}\}$ multiply as unit imaginary quaternions. It follows that $\mathcal{T}$ is the Twistor Family induced from the hyperkhler metric g. $\Box$

Lemma 4.2 *Let A be a nontrivial homology class with $\omega(A) \geq 0$. Then there exits a constant C_A such that every (J, α)-holomorphic map $f : C \to X$ representing A with $\alpha \in \mathcal{H}$ satisfies*

$$E(f) = \frac{1}{2}\int_\Sigma |df|^2 < \omega(A) + C_A \qquad \textit{and} \qquad |\alpha| \leq 1.$$

Proof. Since $|\alpha|$ is a constant, we can integrate Corollary 1.4b to conclude that $|\alpha| \leq 1$. Let C_A be an upper bound for the function $\alpha \mapsto |\alpha(A)|$ on the set of $\alpha \in \mathcal{H}$ with $|\alpha| \leq 1$. Because α is closed, Proposition 1.3a and Corollary 1.4a imply that

$$E(f) = \frac{1}{2}\int_C |df|^2 = \int_\Sigma f^*(\omega + \alpha) = \omega(A) + \alpha(A) \leq \omega(A) + C_A. \quad \Box$$

Theorem 4.3 *Let (X, J) be a K3 or Abelian surface. For each non-trivial $A \in H_2(X, \mathbb{Z})$, the invariants $GW^{J,\mathcal{H}}_{g,k}(X, A)$ are well-defined and independent of J. Furthermore, if $A = mB$ and $A' = mB'$ where B and B' are primitive with the same square, then*

$$GW^{J,\mathcal{H}}_{g,k}(X, A) = GW^{J,\mathcal{H}}_{g,k}(X, A').$$

Proof. For any nontrivial homology class A, we can choose a Ricci flat Kähler structure (ω, J, g) such that $\omega(A) \geq 0$ (if $\omega(A) < 0$, then we choose $(-\omega, -J, g)$). It then follows from Lemma 4.2 and Theorem 3.6 that $GW^{J,\mathcal{H}}_{g,k}(X, A)$ is well-defined.

Bryan and Leung have applied the machinery of Li and Tian to define family GW invariants associated to the Twistor Family $\mathcal{T}$ [BL1, BL2]. Their invariants, which we denote by

$$\Phi^{\mathcal{T}}_{g,k}(X, A),$$

are actually independent of the Twistor Family since the moduli space of complex structures on X is connected. On the other hand, if $A = mB$ and $A' = mB'$ where B and B' are primitive with the same square, then there is an orientation preserving diffeomorphism of X which sends the class B to the class B'. That implies that $\Phi^{\mathcal{T}}_{g,k}(X, A) = \Phi^{\mathcal{T}}_{g,k}(X, A')$.

To complete the proof it suffices to show that

$$GW^{J,\mathcal{H}}_{g,k}(X, A) = \Phi^{\mathcal{T}}_{g,k}(X, A). \tag{4.2}$$

For that, recall from Theorem 1.2 of [LT] that the moduli cycle is defined from a section s of a generalized Fredholm orbifold bundle $E \to B$ and is represented by a cycle that lies in an arbitrarily small neighborhood of $s^{-1}(0)$. Both sides of (4.2) are defined in that way using the same Fredholm bundle E over the space of Kähler structures. In the first case B is $\{J_\alpha \,|\, \alpha \in \mathcal{H}\}$ and $s^{-1}(0)$ is the set of all (f, α) where f is a J_α-holomorphic map, and in the second case $B = \mathcal{T}$ is the Twistor Family and $s^{-1}(0)$ is the set of J_α-holomorphic maps for $J_\alpha \in \mathcal{T}$. By Proposition 4.1 $\{J_\alpha \,|\, \alpha \in \mathcal{H}\}$ parameterizes the Twistor Family after adding a point at infinity to $\mathcal{H}$. But since $\omega(A) \geq 0$, Lemma 4.2 shows that $|\alpha| \leq 1$ for all J_α holomorphic maps representing the homology class A with $\alpha \in \mathcal{H}$. Thus the moduli cycle is bounded away from the point at infinity, so the two definitions of the moduli cycle are exactly equal. That gives (4.2) $\square$

Elliptic Surfaces

First, we recall the well-known facts about minimal elliptic surfaces X with Kodaira dimension 1 [FM].

1. X is elliptic in a unique way.
2. Every deformation equivalence is through elliptic surfaces.

Therefore, there is a unique elliptic structure $\pi : (X, J) \to C$. Moreover, for the fiber class F and any homology class $A \in H_2(X; Z)$, the integer

$$F \cdot A + \deg(\pi_* A) \tag{4.3}$$

is well-defined for each complex structure J and it is invariant under the deformation of complex structure J.

Let (ω, J, g) be a Kähler structure on X and $\mathcal{H}$ be as in (2.4). For $\alpha \in \mathcal{H}$, let $\|\alpha\|$ denote the L^2 norm as in (2.5).

Lemma 4.4 *Let $A \in H_2(X; Z)$ such that the integer (4.3) is positive. Then, there exit uniform constants E_0 and N such that for any J_α-holomorphic map $f : \Sigma \to X$, representing homology class A, with $\alpha \in \mathcal{H}$, we have*

$$E(f) = \frac{1}{2} \int_\Sigma |df|^2 \leq E_A\,, \qquad ||\alpha|| \leq N.$$

Proof. It follows from (4.1) and Lemma 2.3 that for any nonzero $\alpha \in \mathcal{H}$, the zero set of α lies in the union of fibers F_i. Let $N(\alpha)$ be a (non-empty) union of ε-tubular neighborhoods of the F_i. Denote by $\mathcal{S}$ the unit sphere in $\mathcal{H}$ and set

$$m(J) = \min_{\alpha\in\mathcal{S}} \min_{x\in X\setminus N(\alpha)} |\alpha| \quad \text{and} \quad N = \frac{2}{m(J)}.$$

We can always choose a smooth fiber $F \subset X \setminus N(\alpha)$ such that f is transversal to F. Let $f^{-1}(F) = \{p_1, \cdots, p_n\}$ and for each i fix a small holomorphic disk D_i normal to F at $f(p_i)$. We can further assume that f is transversal to each D_i at $f(p_i)$.

Define $\mathrm{sgn}(r)$ to be the sign of a real number r if $r \neq 0$, and 0 if $r = 0$. Denote by $I(S, f)_p$ the local intersection number of the map f and a submanifold $S \hookrightarrow X$ at $f(p)$. In terms of bases $\{e_1,\, e_2 = j\, e_1\}$ of $T_{p_i}\Sigma$, $\{v_1,\, v_2 = j\, v_1\}$ of $T_{f(p_i)}F$, and $\{v_3,\, v_4 = j\, v_3\}$ of $T_{f(p_i)}D_i$ we have

$$\begin{aligned} I(F, f)_{p_i} &= \mathrm{sgn}\left((v^1 \wedge v^2 \wedge v^3 \wedge v^4)(v_1, v_2, f_*e_1, f_*e_2)\right) = \mathrm{sgn}\left((v^3 \wedge v^4)(f_*e_1, f_*e_2)\right), \\ I(D_i, f)_{p_i} &= \mathrm{sgn}\left((v^1 \wedge v^2 \wedge v^3 \wedge v^4)(f_*e_1, f_*e_2, v_3, v_4)\right) = \mathrm{sgn}\left((v^1 \wedge v^2)(f_*e_1, f_*e_2)\right). \end{aligned}$$

Comparing with $\mathrm{sgn}\, f^*\omega(e_1, e_2) = \mathrm{sgn}\left((v^1 \wedge v^2)(f_*e_1, f_*e_2) + (v^3 \wedge v^4)(f_*e_1, f_*e_2)\right)$ shows that

$$I(F, f)_{p_i} + I(D_i, f)_{p_i} \;=\; \mathrm{sgn}\ (f^*\omega)(e_1, e_2). \tag{4.4}$$

Now suppose $m(J)||\alpha|| \geq 2$. Then $|\alpha| \geq 2$ along each F_i, so by (4.4) and Corollary 1.4b

$$\sum_i \left(I(f, F)_{p_i} + I(f, D_i)_{p_i}\right) < 0.$$

This contradicts to our assumption $A \cdot f + \deg(\pi_* A) > 0$ since by definition $\sum_i I(f, F)_{p_i} = A \cdot f$ and $\sum_i I(f, D_i)_{p_i} = \deg(\pi_* A)$. Therefore $||\alpha|| < N$ with N as above. The energy bound follows exactly same arguments as in the proof of Lemma 4.2. □

Proposition 4.5 *For any homology class A with (4.3) positive, the invariants $GW^{J,\mathcal{H}}_{g,k}(X, A)$ are well-defined and depend only on the deformation class of (X, J).*

Proof. It follows from Lemma 4.4 and Theorem 3.6 that the invariants $GW^{J,\mathcal{H}}_{g,k}(X, A)$ are well-defined. On the other hand, (4.3) is invariant under the deformation of J. Therefore, applying Proposition 3.3, we can conclude that the invariants only depends on the deformation equivalence class of J. □

Surfaces of General Type

Let (X, J) be a surface of general type.

Proposition 4.6 *If A is of type (1,1) and is not a linear combination of components of the canonical class, then we can define the invariant $GW^{J,\mathcal{H}}_{g,k}(X, A)$. They are invariant under the deformations of complex structures which preserve (1,1)-type of A.*

Proof. Lemma 2.4 and Theorem 3.6 imply that the invariants $GW^{\mathcal{H}}_{g,k}(X, A)$ are well-defined under the assumption that A is type $(1, 1)$. On the other hand, Proposition 3.3 also implies that the invariants $GW^{\mathcal{H}}_{g,k}(X, A)$ are invariant under deformations of the complex structure which preserve the $(1, 1)$ type of A. □

5 The Invariants of $E(n)$ — Outline

Let $\pi : E(n) \to \mathbb{P}^1$ be a standard elliptic surface with a section s of self-intersection number $-n$. Denote by S and F the homology class of the section and the fiber. We will compute family GW-invariants for the class $S+dF$ with $2p_g = 2(n-1)$ dimensional parameter space $\mathcal{H}_n$ defined as in (2.4). These invariants $GW^{\mathcal{H}_n}_{g,k}(S+dF)$ are unchanged under deformations of Kähler structure. For convenience we assemble these into the generating function

$$F(t) = \sum_{d\geq 0} GW^{\mathcal{H}_n}_{0,0}\,(S+dF)\; t^d. \tag{5.1}$$

In this and the following four sections we will calculate the invariants $GW^{\mathcal{H}_n}_{g,k}(S+dF)$ by deriving the formula for $F(t)$ stated in Theorem 0.3. Thus our aim it to prove:

Proposition 5.1 *For $n \geq 1$,*

$$F(t) \;=\; \prod_{d\geq 0}\left(\frac{1}{1-t^d}\right)^{12n} \tag{5.2}$$

As mentioned in the introduction, the cases $n = 1, 2$ have been proven by Ionel-Parker and Bryan-Leung, respectively.

This section shows how Proposition 5.1 follows from two formulas, equations (5.4) and (5.5) below, that are proved in later sections. Our proof parallels the proof of Ionel and Parker [IP3] with two changes. First, we replace the use of the τ class by ψ class; that makes the argument conceptually a bit easier. Second, we must extend the TRR formula and the Symplectic gluing formula of [IP3] to family invariants.

Here is the outline the proof of (5.2). Let $G(t)$ be the generating function for the function for the sum of divisors function $\sigma(n) = \sum_{d|n}$:

$$G(t) \;=\; \sum_{d\geq 0}\sigma(d)\; t^d \;=\; \sum_{d\geq 0}\frac{dt^d}{1-t^d}$$

Following [IP3] we also consider the generating function for a genus 1 invariant, namely

$$H(t) \;=\; \sum_{d\geq 0} GW^{\mathcal{H}_n}_{1,4}\,(S+dF)\,(\psi_{(1,4);4}; F^4)\, t^d \tag{5.3}$$

where $\psi_{(g,k);i}$ denotes the first Chern class of the line bundle $L_{(g,k);i} \to \overline{\mathcal{M}}_{g,k}$ whose geometric fiber over $(C; x_1, \cdots, x_n)$ is $T^*_{x_i}C$.

We can compute $H(t)$ in two different ways. In section 6, we show how to combine the composition law together with the relation between ψ class and the divisor classes in $\overline{\mathcal{M}}_{1,4}$ to obtain the formula

$$H(t) = \frac{1}{12}tF'(t) - \frac{1}{12}F(t) + (2-n)F(t)G(t) \tag{5.4}$$

Then, in sections 7–9 we establish a family version of the sum formula

$$H(t) = -\frac{1}{12}F(t) + 2F(t)G(t) \tag{5.5}$$

(see Proposition 9.4). Equations (5.4) and (5.5) give rise to the ODE with

$$tF'(t) = 12nG(t)F(t) \tag{5.6}$$

and we show in Proposition 7.6 that the initial condition is $F(0) = 1$. It is well-known that the solution of this ODE is given by

$$F(t) = \prod_{d\geq 0}\left(\frac{1}{1-t^d}\right)^{12n}.$$

That completes the proof of Proposition 5.1 and hence of the main Theorem 0.3 of the introduction.

6 The Topological Recursion Relation (TRR)

A pinched torus can be regarded as a two-sphere with two points identified. Consequently, maps from a pinched torus are a special class of maps from the two-sphere. That observation allows one to express certain $g = 1$ GW invariants in terms of $g = 0$ invariants. Such formulas are called topological recursion relations or TRR formulas. In this section we will prove formula (5.4), which is a TRR formula for the family GW invariants.

We begin by recalling the notion of the dual graph associated with a stable curve [G]. Given a stable genus g curve with n marked points $(\Sigma; x_i, \cdots, x_k)$, its dual graph is defined as follows. Let $\pi : \tilde{\Sigma} \to \Sigma$ be the normalization of Σ. The dual graph G has one vertex for each component of $\tilde{\Sigma}$, and the edges of G correspond to nodal points of Σ; if two points on $\tilde{\Sigma}$ maps to a node, then the edge, corresponding to that node, are attached to the vertices associated to the components of $\tilde{\Sigma}$ on which the two points lie. The legs (half-edge) of G correspond to marked points of Σ, and these are indexed in an obvious way.

We denote by $\mathcal{M}(G)$ the moduli space of all genus g curves with n marked points whose dual graph is G. We also denote by δ_G the orbifold fundamental class of $\overline{\mathcal{M}(G)}$, that is, the fundamental class divided by the order of the automorphisms of a general element of $\mathcal{M}(G)$. Graphs with one edge correspond to degree two classes. There are two types of such graphs. One is the graph G_{irr} with one vertex of genus $g-1$. The other types are the graphs $G_{a,I}$, which have two vertices, one of genus a, with attached the legs indexed by I, and one of genus $g-a$, with attached the legs indexed by $\{1, \cdots, k\} \setminus I$.

For any $i \in \{1, \cdots, k\}$, we have

$$\psi_{(1,k);i} = \frac{1}{12}\delta_{G_{irr}} + \sum_{\substack{i\in I\\ |I|\geq 2}} \delta_{G_{0,I}} \qquad \text{in} \quad H^2(\overline{\mathcal{M}}_{1,k}; \mathbb{Q}). \tag{6.1}$$

For the proof of (6.1), see [AC] and [G].

Proposition 6.1 *The generating function (5.3) satisfies*

$$H(t) = \frac{1}{12}tF'(t) - \frac{1}{12}F(t) + (2-n)F(t)G(t)$$

Proof. It follows from (6.1) that the coefficients $GW^{\mathcal{H}_n}_{1,4}(S+dF)(\psi_{(1,4);4}; F^4)$ of $H(t)$ is

$$\frac{1}{12}GW^{\mathcal{H}_n}_{1,4}(S+dF)(\delta_{G_{irr}}; F^4) + \sum_{\substack{i\in I\\ |I|\geq 2}} GW^{\mathcal{H}_n}_{1,4}(S+dF)(\delta_{G_{0,I}}; F^4). \tag{6.2}$$

We will apply the Second Composition Law to the first term in (6.2) and the First Composition Law to the second term in (6.2).

Note that $PD(\mathrm{Im}(\theta)) = 2\delta_{G_{irr}}$ where $\theta : \overline{\mathcal{M}}_{0,6} \to \overline{\mathcal{M}}_{1,4}$ as in (3.10). It then follows from the Second Composition Law and Proposition 3.7 that

$$\begin{aligned}\frac{1}{12}GW^{\mathcal{H}_n}_{1,4}(S+dF)(\delta_{G_{irr}}; F^4) &= \frac{1}{24}\sum_{\gamma} GW^{\mathcal{H}_n}_{0,6}(S+dF)(F^4, H_\gamma, H^\gamma)\\ &= \frac{1}{24}\sum_{\gamma}(H_\gamma(S+dF))\,(H^\gamma(S+dF))\,GW^{\mathcal{H}_n}_{0,4}(S+dF)(F^4)\\ &= \frac{2d-n}{24}GW^{\mathcal{H}_n}_{0,0}(S+dF). \end{aligned}\tag{6.3}$$

Recalling Proposition 3.8, the only possible decompositions of the class $S+dF$, which can appear when we apply the First Composition Law, are $S+d_1F$ and d_2F with $d_1+d_2=d$. It then follows from a dimension count and the First Composition Law that

$$\begin{aligned}&GW^{\mathcal{H}_n}_{1,4}(S+dF)(\delta_{G_{0,I}}; F^4)\\ &= \sum_{d_1+d_2=d}\sum_{\gamma} GW^{\mathcal{H}_n}_{0,|I|+1}(S+d_1F)(F^{|I|}, H_\gamma)\,GW_{1,5-|I|}(d_2F)(F^{4-|I|}, H^\gamma).\end{aligned}\tag{6.4}$$

where $\{H_\gamma\}$ and $\{H^\gamma\}$ are bases of $H^*(E(n))$ dual by the intersection form. It also follows from Proposition 3.7 that, if $I = \{1, \cdots, 4\}$, then (6.4) becomes

$$\begin{aligned}&\sum_{\substack{d_1+d_2=d\\ d_2>0}}\sum_{\gamma}(H_\gamma\cdot(S+d_1F))\,(H^\gamma\cdot d_2F)GW^{\mathcal{H}_n}_{0,0}(S+d_1F)\,GW_{1,0}(d_2F)\\ &+\sum_{\gamma}(H_\gamma\cdot(S+dF))\,GW^{\mathcal{H}_n}_{0,0}(S+dF)\,GW_{1,1}(0)(H^\gamma).\end{aligned}\tag{6.5}$$

Otherwise, (6.4) vanishes. Since $\sum_\gamma (H_\gamma A)(H^\gamma B) = AB$ and $kGW_{1,0}(kF) = (2-n)\sigma(k)$ (see [IP1]), the first sum in (6.5) becomes $(2-n)\sum_{k\geq 1} GW^{\mathcal{H}_n}_{0,0}(S+(d-k)F)\,\sigma(k)$. On the other hand, $GW_{1,1}(0)(H^\gamma) = \frac{1}{24}(KH^\gamma)$ (see [IP3]), where $K = (n-2)F$ is the canonical class. This implies that the second sum in (6.5) becomes $\frac{n-2}{24}GW^{\mathcal{H}_n}_{0,0}(S+dF)$. In summary, we have

$$\begin{aligned}&\sum_{\substack{i\in I\\ |I|\geq 2}} GW^{\mathcal{H}_n}_{1,4}(S+dF)(\delta_{G_{0,I}}; F^4)\\ &= (2-n)\sum_{k\geq 1} GW^{\mathcal{H}_n}_{0,0}(S+(d-k)F)\,\sigma(k) + \frac{n-2}{24}GW^{\mathcal{H}_n}_{0,0}(S+dF).\end{aligned}\tag{6.6}$$

The proof follows from (6.2), (6.3), (6.6) and the definition of $F(t)$ and $H(t)$. $\Box$

7 Ruan-Tian Invariants of $E(n)$

Instead of constructing virtual moduli cycle directly from the moduli space of stable J-holomorphic maps, Ruan and Tian [RT1, RT2] perturbed the equation (1.6) to $\overline{\partial}_J f = \nu$ where the inhomogeneous term ν can be chosen generically. For generic (J, ν), the moduli space of stable (J, ν)-holomorphic maps is then a compact smooth orbifold with all lower strata having codimension at least two. Ruan and Tian defined GW-invariants from this (perturbed) moduli space.

We can follow as similar procedure for the family invariants by introducing an inhomogeneous term into the (J, α)-holomorphic equation and vary ν. In taking that approach, we immediately face the main problem: compactness and the dimension of lower strata. However, for generic perturbation, the moduli space of perturbed (J, α)-holomorphic maps representing a homology class $S + dF$ in $E(n)$ with fixed Kähler structure J is still compact with all lower strata having codimension at least two. Therefore, we can define invariants from this moduli space in the same way as for ordinary GW-invariants. This alternative definition of invariants is more geometric. In particular, using this definition of invariants we can follow the analytic arguments of Ionel and Parker in [IP2, IP3] to show sum formula (5.5) for the case at hand: the class $S + dF$ in $E(n)$.

To simplify notation in this section we will set $X = E(n)$ and $A = S + dF$.

The construction of invariants starts from the perturbed equation $\overline{\partial}_J f = \nu$. Using Prym structures defined as in [L], we can lift Deligne-Mumford space $\overline{\mathcal{M}}_{g,k}$ to a finite cover

$$p_\mu : \overline{\mathcal{M}}^\mu_{g,k} \to \overline{\mathcal{M}}_{g,k}. \tag{7.1}$$

This finite cover is now a smooth manifold and has a universal family

$$\pi_\mu : \overline{\mathcal{U}}^\mu_{g,k} \to \overline{\mathcal{M}}^\mu_{g,k}$$

which is projective. Moreover, for each $b \in \overline{\mathcal{M}}^\mu_{g,k}$, $\pi_\mu^{-1}(b)$ is a stable curve isomorphic to $p_\mu(b)$.

We fix, once and for all, an embedding of $\overline{\mathcal{U}}^\mu_{g,k}$ into some $\mathbb{P}^N$. An inhomogeneous term ν is then defined as a section of the bundle $\mathrm{Hom}(\pi_1^*(T\mathbb{P}^N), \pi_2^* TX)$ which is anti-J-linear :

$$\nu(j_{\mathbb{P}}(v)) = -J(\nu(v)) \qquad \text{for any } v \in T\mathbb{P}^N \tag{7.2}$$

where $j_{\mathbb{P}}$ is the complex structure on $\mathbb{P}^N$.

For each stable map $f : \Sigma \to X$, we can specify one element $j \in p_\mu^{-1}(st(\Sigma))$. Then $\pi_\mu^{-1}(j)$ is isomorphic to the stable curve $st(\Sigma)$. In this way, we can define a map

$$\phi : \Sigma \to st(\Sigma) \cong \ \pi_\mu^{-1}(b) \subset \overline{\mathcal{U}}^\mu_{g,k} \hookrightarrow \mathbb{P}^N. \tag{7.3}$$

Definition 7.1 *A stable (J, ν, α)-holomorphic map is a stable map $f : (\Sigma, \phi) \to X$ satisfying*

$$(\, df + J_\alpha df\, j_\Sigma\,)(p) = \nu_\alpha(\phi(p), f(p))$$

where ϕ is defined as in (7.3), and $\nu_\alpha = (I + JK_\alpha)^{-1}\nu$. $\Box$

Two stable (J, ν, α)-holomorphic maps $(f, (\phi, \Sigma); x_1, \cdots, x_k)$ and $(f', (\phi', \Sigma'); x'_1, \cdots, x'_k)$ are equivalent if

$$d_H(\phi(\Sigma), \phi'(\Sigma'))) + d_H(f(\Sigma), f'(\Sigma')) + \sum d(f(x_i), f'(x'_i)) = 0$$

where d_H is the Hausdorff distance. We then define the moduli space

$$\overline{\mathcal{M}}_{g,k}(X, A, \nu, \mathcal{H}, \mu)$$

as the set of all pairs $([f, (\phi, \Sigma); x_1, \cdots, x_k], \alpha)$, where $\alpha \in \mathcal{H}$ and $[f, (\phi, \Sigma); x_1, \cdots, x_k]$ is the equivalence class of (J, ν, α)-holomorphic maps with $[f(\Sigma)] = A \in H_2(X; Z)$. We denote by

$$\mathcal{M}_{g,k}(X, A, \nu, \mathcal{H}, \mu)$$

the set of $([f, (\phi, \Sigma); x_1, \cdots, x_k], \alpha)$ with a smooth domain Σ. We will often abuse notation by writing (f, Σ, α), (f, j, α) or simply (f, α), instead of $(\, f,\, (\phi, \Sigma),\, \alpha\,)$.

Lemma 7.2 *There exist uniform constants E_A and N such that for any $(f, \Sigma, \alpha) \in \overline{\mathcal{M}}_{g,k}(X, A, \nu, \mathcal{H})$*

$$E(f) = \frac{1}{2} \int_\Sigma |df|^2 \leq E_A \quad \text{and} \quad ||\alpha|| \leq N.$$

Proof. Similarly to Corollary 1.4, we have

$$\int_\Sigma |\overline{\partial}_J f|^2 = \int_\Sigma f^*\alpha + 2 \int_\Sigma \langle \overline{\partial}_J f, \nu \rangle \tag{7.4}$$

$$(1 + |\alpha|^2) f^*\omega \, dv = \frac{1}{2}(1 - |\alpha|^2)|df|^2 \, dv - 4\langle \overline{\partial}_J f, \nu \rangle \, dv + 4|\nu|^2 \, dv \tag{7.5}$$

Note that f represent homology class $A = S + dF$ which is of type (1,1) with respect to the complex structure J. Therefore, it follows from (7.4) and Proposition 1.3a that

$$\frac{1}{2} \int_\Sigma |df|^2 \leq \omega(A) + 2 \left(\int_\Sigma |df|^2 \right)^{\frac{1}{2}} \left(\int_\Sigma |\nu|^2 \right)^{\frac{1}{2}}$$

We then have a uniform energy bound by using the inequality $2ab \leq \varepsilon\, a^2 + \varepsilon^{-1}\, b^2$ on the last term and absorbing the $|df|$ term on the left-hand side.

Next, we will show uniform bound of $||\alpha||$. This proof is similar to those of Lemma 4.4 except for using (7.5) instead of Corollary 1.4b. Let $\pi : X \to \mathbb{CP}^1$ be the elliptic structure for J on X and $N(\alpha)$, $m(J)$, and N be as in the proof of Lemma 4.4. If there is a holomorphic fiber $F \subset X \setminus N(\alpha)$ such that

(i) f is transversal to F,

(ii) at each $p \in f^{-1}(F)$, f is transversal to a holomorphic disk $D_{f(p)}$ normal to F at $f(p)$, and

(iii) $4\,|df|\,|\nu| + 4|\nu|^2 \leq \frac{1}{2}\,|df|^2$ on $f^{-1}(F)$

then the proof follows exactly as in the proof of Lemma 4.4. We can clearly find fibers satisfying (i) and (ii), so we need only verify that we can also obtain (iii). For that we consider the set Σ_0

of all points in Σ where $4\,|df|\,|\nu|+4|\nu|^2 \;>\; \frac{1}{2}\,|df|^2$. Then $|df|^2 \le 16|\nu|$ on Σ_0, since both $|df|$ and $|\nu|_\infty$ are less than 1. Therefore

$$\int_{\Sigma_0} |\, d\pi \circ df \,|^2 \;\le\; 16\,\mathrm{Area}(\, st(\Sigma)\,)\,|\, d\pi\,|_\infty^2 |\,\nu\,|_\infty \tag{7.6}$$

We can thus assume that (7.6) $\le \frac{1}{3}\mathrm{Area}(\mathbb{CP}^1)$ for sufficiently small $|\nu|_\infty$. On the other hand, from the definition of $N(\alpha)$, we can also assume that $\mathrm{Area}(\,\pi(N(\alpha))\,) \le \frac{1}{3}\mathrm{Area}(\mathbb{CP}^1)$. Therefore, we can always choose a holomorphic fiber $F = \pi^{-1}(q)$ as in the above claim with $q \in \mathbb{CP}^1 \setminus (\;\pi(N(\alpha)) \cup \pi \circ f(\Sigma_0)\,)$. $\square$

Consider the following stabilization and evaluation maps

$$\mathcal{M}_{g,k}(X, A, \nu, \mathcal{H}, \mu) \xrightarrow{st^\mu \times ev^\mu} \overline{\mathcal{M}}^\mu_{g,k} \times X^k. \tag{7.7}$$

Its Frontier is defined to be the set

$$\{\, r \in \overline{\mathcal{M}}^\mu_{g,k} \times X^k \,|\, r = \lim(st^\mu \times ev^\mu)(f_n, \alpha_n) \text{ and } (f_n, \alpha_n) \text{ has no convergent subsequence}\,\}.$$

We denote by $\mathcal{Y}_0$ the space of all ν with $|\nu|_\infty$ is sufficiently small. Now, we are ready to state "Structure Theorem" for the moduli space.

Theorem 7.3 (Structure Theorem) *For generic $\nu \in \mathcal{Y}_0$,*

(a) $\mathcal{M}_{g,k}(X, A, \nu, \mathcal{H}, \mu)$ is an oriented smooth manifold of dimension

$$-\,2\,KA + 2(g-1) + 2k + dim(\mathcal{H}) = 2(g+k) \tag{7.8}$$

(b) the Frontier of the smooth map

$$\mathcal{M}_{g,k}(X, A, \nu, \mathcal{H}, \mu) \xrightarrow{st^\mu \times ev^\mu} \overline{\mathcal{M}}^\mu_{g,k} \times X^k$$

lies in dimension 2 less than $2(g+k)$.

Proof. This proof is similar to the proof of Proposition 2.3 in [RT2]. We will sketch the proof, without specifying Sobolev norms.

Let $\mathcal{E}^\mu \to \mathcal{M}^\mu_{g,k} \times \mathcal{H} \times \mathcal{Y}_0$ be the vector bundle whose fiber over (f, j, α, ν) is $\Omega^{0,1}_{j,J_\alpha}(f^*TX)$. The (J, ν, α)-holomorphic equation then defines a section Φ of $\mathcal{E}^\mu$. This section is transverse to the zero section because of ν-term. Therefore, the universal moduli space

$$\mathcal{U}^\mu_{g,k}(X, A) = \{\, (f, j, \alpha, \nu) \in \mathcal{M}^\mu_{g,k} \times \mathcal{H} \times \mathcal{Y}_0 \mid \Phi(f, j, \alpha, \nu) = 0\,\}$$

is smooth.

On the other hand, it follows from standard argument that the differential $d\pi$ of the projection $\pi : \mathcal{U}^\mu_{g,k}(X, A) \to \mathcal{Y}_0$ is Fredholm of the same index as the index of the linearization $D\Phi$. Applying Sard-Smale Theorem, we can thus conclude that for generic $\nu \in \mathcal{Y}_0$, the moduli space

$$\pi^{-1}(\nu) = \mathcal{M}_{g,k}(X, A, \nu, \mathcal{H}, \mu)$$

is a smooth manifold. The dimension formula follows from the Index Theorem.

For generic ν, the tangent space $T_{(f,j,\alpha)}\mathcal{M}_{g,k}(X,A,\nu,\mathcal{H},\mu) = \mathrm{Ker}(D\Phi_{(f,j,\alpha)})$. On the other hand, similarly to the linearization of J-holomorphic maps, the linearization $D\Phi_{(f,j,\alpha)}$ is also homotopic through Fredholm operators to a complex linear operator. So, its determinant line $\det\left(D\Phi_{(f,j,\alpha)}\right)$ carries a natural orientation and this determines an orientation of the tangent space.

In order to prove (b), we first consider the well-defined stabilization and evaluation map

$$\overline{\mathcal{M}}_{g,k}(X,A,\nu,\mathcal{H},\mu) \xrightarrow{st^\mu \times ev^\mu} \overline{\mathcal{M}}^\mu_{g,k} \times X^k. \tag{7.9}$$

It then follows from Theorem 3.6 and Lemma 7.2 that (7.9) extends (7.7) continuously.

Next, we reduce the moduli space by (i) collapsing all ghost bubbles, (ii) replacing each multiple maps from a bubble by its reduced map, and (iii) identifying those bubble components which have the same image. (7.9) now descends to this reduced moduli space $\overline{\mathcal{M}}^r_{g,k}(X,A,\nu,\mathcal{H},\mu)$ and by definition we have

$$\mathrm{Fr}(\, st^\mu \times ev^\mu \,) \subset st^\mu \times ev^\mu \left(\overline{\mathcal{M}}^r_{g,k}(X,A,\nu,\mathcal{H},\mu) \setminus \mathcal{M}_{g,k}(X,A,\nu,\mathcal{H},\mu)\right).$$

It remains to show that those strata consisting of (f,α) with domain more than two components has a dimension at least 2 less than $2(g+k)$. Similarly to the moduli space of (J,ν)-holomorphic maps, the strata corresponding to the domain with no bubble component has a dimension at least 2 less than $2(g+k)$ for generic ν.

On the other hand, it follows from Lemma 7.2, Theorem 3.6, and Gromov Compactness Theorem that the restriction of (f,α) to any component of domain should represents one of the following homology classes

$$S, \quad S + d_1 F, \quad d_2 F \quad \text{with} \quad 0 \;<\; d_1, d_2 \;\leq d.$$

Since the inhomogeneous term ν vanishes on bubble components, it follows from Theorem 2.4 that each bubble component maps into either a section or a singular fiber.

Now, suppose (f,α) has some bubble components. Then either $\alpha \equiv 0$ or the zero divisor $Z(\alpha)$ contains some singular fibers. Since there's no fixed component in the complete linear space of a canonical divisor of $E(n)$, the parameter α lies in the proper subspace of $\mathcal{H}$. This reduces the dimension of the strata containing (f,α) at least 2. □

Now, we are ready to define invariant. Instead of using intersection theory as in [RT1, RT2], we will follow the approach in [IP2]. The above Structure Theorem and Proposition 4.2 of [KM] assert that the image

$$st^\mu \times ev^\mu\,(\mathcal{M}_{g,k}(X,A,\nu,\mathcal{H},\mu)_I)$$

gives rise to a rational homology class. We denote it by

$$[\,\overline{\mathcal{M}}_{g,k}(X,A,\nu,\mathcal{H},\mu)\,] \in H_*(\overline{\mathcal{M}}^\mu_{g,k};\mathbb{Q}) \otimes H_*(X^k;\mathbb{Q}). \tag{7.10}$$

Definition 7.4 *For $2g+k \geq 3$, $\beta \in H^*(\overline{\mathcal{M}}_{g,k};\mathbb{Q})$, and $\alpha_1, \cdots, \alpha_k \in H^*(X^k;\mathbb{Q})$, we define invariants by*

$$\Phi_{g,k}(X,A,\mathcal{H})(\beta;\alpha_1,\cdots,\alpha_k) = \frac{1}{\lambda_\mu}\,(\,\beta\otimes(\alpha_1\wedge\cdots\wedge\alpha_k)\,)\cap[\,\overline{\mathcal{M}}_{g,k}(X,A,\nu,\mathcal{H},\mu)\,]$$

where λ_μ is the order of the finite cover in (7.1).

By repeating the arguments used in [RT] for the ordinary GW-invariants, we can prove that these invariants $\Phi_{g,k}(X,A,\mathcal{H})$ are independent of the inhomogeneous term ν, the finite cover p_μ, and the projective embedding $\overline{\mathcal{U}}^{\mu}_{g,k} \hookrightarrow \mathbb{P}^N$. Alternatively, we can simply observe that those three facts emerge as corollaries of the following proposition.

Proposition 7.5 $\Phi_{g,k}(X,A,\mathcal{H}) = GW^{\mathcal{H}}_{g,k}(X,A)$

Proof. As in section 3, we define $\overline{\mathcal{F}}^{l}_{g,k}(X,A,\mu)$ to be the set of all equivalence classes of the stable maps of the form $(\,f,(\Sigma,\phi)\,)$, where ϕ is defined as in (7.3); two stable maps $(\,f,(\Sigma,\phi)\,)$ and $(\,f',(\Sigma',\phi')\,)$ are equivalent if there is a marked points preserving biholomorphic map $\sigma : \Sigma \to \Sigma'$ such that $f = f'\circ\sigma$ and $\phi = \phi'\circ\sigma$. Note that $\overline{\mathcal{F}}^{l}_{g,k}(X,A,\mu)$ is a finite cover of $\overline{\mathcal{F}}^{l}_{g,k}(X,A)$. Similarly, we define a generalized bundle E^μ over $\overline{\mathcal{F}}^{l}_{g,k}(X,A,\mu)\times\mathcal{H}$ and a section Φ^μ by $(\,f,(\Sigma,\phi),\alpha) \to df + J_\alpha df\, j$. It follows from Lemma 7.2 that the zero set of Φ^μ is compact. Therefore, by Proposition 3.2 there is a virtual moduli cycle which satisfying

$$\pi_*[\,\mathcal{M}^{\mathcal{H}_n}_{g,k}(X,A,\mu)\,] = \lambda_\mu[\,\mathcal{M}^{\mathcal{H}_n}_{g,k}(X,A,)\,] \tag{7.11}$$

where $\pi : \overline{\mathcal{F}}^{l}_{g,k}(X,A,\mu) \to \overline{\mathcal{F}}^{l}_{g,k}(X,A)$ and λ_μ is the order of p_μ.

Now, fix a generic ν as in Theorem 7.3. It follows from Proposition 3.3 and Lemma 7.2 that we still have the same virtual moduli cycle as in (7.11) when we change the section Φ^μ by adding $-\nu$ in an obvious way. We still use the same notation Φ^μ for this new section. We set

$$\mathcal{M} = \mathcal{M}(X,A,\nu,\mathcal{H},\mu) \subset (\Phi^\mu)^{-1}(0).$$

Let $n = \dim(\overline{\mathcal{M}}_{g,k}\times X^k)$ and $d = 2(g+k) = \dim(\,\mathcal{M}(X,A,\nu,\mathcal{H},\mu)\,)$. Since the Frontier of $st^\mu \times ev^\mu$ lies in dimension $d-2$, there is an arbitrary small neighborhood V of $Fr(st^\mu\times ev^\mu)$ such that every homology class in $H_{n-d}(\overline{\mathcal{M}}^{\mu}_{g,k}\times X^k;\mathbb{Q})$ has a representative disjoint from $\overline{V}$.

On the other hand, from the proof of Proposition 2.2 in [LT], we can assume that $(\Phi^\mu)^{-1}(0)$ can be covered by finitely many smooth approximation $\{U_i\}$ such that (i) $\{U_i\}_{i\neq 1}$ cover the complement of $\mathcal{M}$ in $(\Phi^\mu)^{-1}(0)$ and (ii) $\{U_i\}_{i\neq1}$ lies in $(st^\mu\times ev^\mu)^{-1}(\,V\,)$. It then follows from the proof of Theorem 1.2 in [LT] that we can construct a representative Z for virtual moduli cycle such that

$$(st^\mu\times ev^\mu)\,(\,Z\,)\bigcap\left(\overline{\mathcal{M}}^{\mu}_{g,k}\times X^k\setminus\overline{V}\right) = (st^\mu\times ev^\mu)\,(\,\mathcal{M}\,)\bigcap\left(\overline{\mathcal{M}}^{\mu}_{g,k}\times X^k\setminus\overline{V}\right).$$

This implies that two invariants are same. □

In the below, we will not distinguish two invariants and use the same notation $GW^{\mathcal{H}_n}_{g,k}(X,A)$ for them. The following proposition shows $F(0) = 1$ which provides the initial condition for (5.6).

Proposition 7.6 $GW^{\mathcal{H}_n}_{0,3}(X,S)(F^3) = 1.$

Proof. Fix $\nu = 0$. Since the section class S is of type $(1,1)$, Theorem 2.4 implies that for any (J,α)-holomorphic map (f,α) with $[f] = S$, f is holomorphic and $\alpha = 0$. In fact, there is a unique such f since $S^2 = -n$. Now, consider the linearization of (f,α)-holomorphic equation $L_f \oplus Jdf \oplus L_0$ as in appendix. Propositions A.1 and A.2 of the appendix show, quite generally, that L_f is a $\overline{\partial}_J$ operator and L_0 defines a map

$$L_0 : \mathcal{H} \to \mathrm{Coker}(L_f \oplus Jdf)$$

which is injective if and only if the family moduli space $\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X,A)$ is compact. But we just showed the moduli space is a single point, and hence compact.

On the other hand, $\mathrm{Ker}(L_f \oplus Jdf)$ is same as $H^0(f^*N)$, where N is the normal bundle of the section in $E(n)$. It is trivial since the Chern number of N is $S\cdot S = -n < 0$. Therefore,

$$\dim \mathrm{Coker}(\, L_f \oplus Jdf\,) = -\mathrm{Index}(L_f \oplus Jdf) = -2(\, c_1(f^*TX) - 1\,) = 2(n-1)$$

Since L_0 is injective and $\dim(\mathcal{H}) = 2(n-1)$, $L_f \oplus Jdf \oplus L_0$ is onto. That implies $\nu = 0$ is generic in the sense of Theorem 7.3. Consequently, the invariant is ± 1. In this case, the sign is determined by L_f and L_f is $\overline{\partial}_J$-operator, the invariant is 1. $\square$

8 Degeneration of $E(n)$

Throughout this section, X always denotes the standard elliptic surface $E(n) \to \mathbb{CP}^1$ and Y always denotes $T^2 \times S^2$ with a product complex structure.

In this section, we describe a degeneration of X into a singular surface which is a union of X and Y with $V = T^2$ intersection. We then define the parameter space and inhomogeneous terms corresponding to this degeneration. The sum formula (5.5) will be formulated from this degeneration

We fix a small constant $\epsilon > 0$ and let $D(\epsilon) \subset \mathbb{C}$ be a disk of radius ϵ. Choose a smooth fiber V in X. We then define $p : Z \to X \times D(\epsilon)$ to be the blow-up of $X \times D(\epsilon)$ along $V \times \{0\}$ and let

$$\lambda : Z \xrightarrow{p} X \times D(\epsilon) \to D(\epsilon)$$

be the composition, where the second map is the projection of the second factor. The central fiber $Z_0 = \lambda^{-1}(0)$ is a singular surface $X \cup_V Y$ and the fiber Z_λ with $\lambda \neq 0$ is isomorphic to X as a complex surface. Since Z is a blow-up of a Kähler manifold, it is also Kähler. Denote by (ω_Z, J_Z, g_Z) the Kähler structure on Z induced from the blow-up. We also denote by $(\omega_\lambda, J_\lambda, g_\lambda)$ the induced Kähler structure on each Z_λ with $\lambda \neq 0$.

Let U be a neighborhood of V in X which does not contain any singular fibers. Choose a bumf function β on U which satisfies $\beta = 1$ on $X \setminus U$ and $\beta = 0$ near V in U. Let $\mathcal{H}$ be the parameter space of X defined as in (2.4). We consider each $\alpha \in \mathcal{H}$ as a 2-form on $X \times D(\epsilon)$. Then each $p^*\alpha$ is also closed and J_Z-anti-invariant.

Definition 8.1 *We define the parameter space of the fibration* $\lambda : Z \to D(\epsilon)$ *by*

$$\mathcal{H}_Z \;=\; \{\; p^*(\beta\alpha) \mid \alpha \in \mathcal{H} \;\} \quad \textit{and} \quad \mathcal{H}_\lambda \;=\; \{\; \alpha_\lambda = \alpha_{|Z_\lambda} \mid \alpha \in \mathcal{H}_Z \;\} \;\; \textit{when} \;\; \lambda \neq 0$$

We can consider X as a Kähler submanifold of Z. It then follows from the above definition that

$$\mathcal{H}_X = \{\; \beta\alpha \mid \alpha \in \mathcal{H} \;\} = \{\; \alpha_{|X} \mid \alpha \in \mathcal{H}_Z \;\}. \tag{8.1}$$

Lemma 8.2 *There exit uniform constants* E_A *and* N*, which does not depend on* λ*, such that*

$$E(f) = \frac{1}{2}\int_\Sigma |df|^2 \leq E_A \quad \textit{and} \quad ||\alpha_\lambda||_2 \leq N$$

for any $(f, \Sigma, \alpha_\lambda) \in \overline{\mathcal{M}}_{g,k}(Z_\lambda, A, \nu, \mathcal{H}_\lambda)$*, where* $|\nu|_\infty$ *is sufficiently small and* $A = S + dF$*.*

Proof. The proof of the uniform bound of α_λ is similar to the proof of Lemma 7.2. We define $N(\alpha_\lambda)$ as an open neighborhood of zero set of α_λ and define $m(J_\lambda)$ as in the proof of Lemma 4.4. Since each α_λ is supported on the $p^{-1}(X \setminus U \times \{\lambda\})$, there is a constant $c > 0$ such that $m(J_\lambda) > c$ for any λ. Then, the same argument as in Lemma 7.2 shows that $N = 2/c > 2/m(J_\lambda)$.

It remains to show the uniform energy bound. Note that $\alpha_\lambda = p^*(\beta\alpha)$ for some $\alpha \in \mathcal{H}$. For each $p \in \Sigma$, let $\{\, e_1(p), e_2(p) = j\, e_1(p) \,\}$ be an orthonormal basis of $T_p\Sigma$. We set

$$\Sigma_- = \{\; p \in \Sigma \mid f^*p^*\alpha(\, e_1(p), e_2(p)\,) < 0 \;\}$$

Since $|\overline{\partial}_{J_\lambda} f|^2\, dv = f^*p^*(\beta\alpha) + 2\langle \overline{\partial}_{J_\lambda} f, \nu\rangle\, dv$, we have $|\overline{\partial}_{J_\lambda} f| \leq 2|\nu|$ on Σ_-. This implies

$$-f^*p^*\alpha(\, e_1(p), e_2(p)\,) \leq M|df||\nu|$$

where $p \in \Sigma_-$ and $M = \max\{\; |p^*\alpha| \mid ||p^*\alpha||_2 \leq N \;\}$. Therefore, we can conclude that

$$\begin{aligned} \frac{1}{2}\int_\Sigma |df|^2 &\leq \int_\Sigma f^*\, p^*(\beta\alpha) + \int_\Sigma |df||\nu| + \omega_\lambda(A) \\ &\leq -\int_{\Sigma_-} f^*p^*\alpha + \int_\Sigma |df||\nu| + \omega_Z(A) \\ &\leq (1+M)\left(\int_\Sigma |\nu|^2\right)^{\frac{1}{2}} \left(\int_\Sigma |df|^2\right)^{\frac{1}{2}} + \omega_Z(A) \end{aligned}$$

This implies the uniform energy bound independent of λ. □

Next, following Definition 3.2 of [IP2], we define inhomogeneous terms on the fibration $\lambda : Z \to D(\epsilon)$. As in section 7, we fix a finite cover $\overline{\mathcal{M}}^\mu_{g,k}$, universal family $\overline{\mathcal{U}}^\mu_{g,k}$ over it, and a projective embedding $\overline{\mathcal{U}}^\mu_{g,k} \hookrightarrow \mathbb{P}^N$. We also denote by the orthogonal projection onto the normal bundle N_X (N_Y) of V by $\xi \to \xi^N$.

Definition 8.3 *We define an inhomogeneous term* ν *of the fibration* $\lambda : Z \to D(\epsilon)$ *to be a section of the bundle* $Hom(T\mathbb{P}^N, TZ)$ *over* $\mathbb{P}^N \times Z$ *which satisfies* *(i)* ν *is anti-*J_Z*-linear, i.e.* $\nu\, j_{\mathbb{P}^N} = -J_Z\, \nu$,

(ii) *the restriction of* ν *to* Z_λ*, denoted by* ν_λ*, is an inhomogeneous term on* Z_λ

(iii) *the restriction* ν_X *(resp.* ν_Y*) of* ν *to* X *(resp.* Y*) is also an inhomogeneous term on* X *(resp.* Y*) such that the normal component* ν_X^N *(resp.* ν_Y^N*) on* V *vanishes, and*

(iv) *for all* $\xi \in N_X$ *(resp.* N_Y*) and* $v \in TV$

$$[\,(J\nabla_{\nu_X} J)\xi\,]^N = 0 \ \Big(resp.\ [\,(J\nabla_{\nu_Y} J)\xi\,]^N = 0\Big)$$

We denote by $\mathcal{Y}^V$ *the set of all inhomogeneous terms on* Z*.*

Proposition 8.4 *For generic* $\nu \in \mathcal{Y}^V$*,*

(a) $\mathcal{M}_{g,k}(Z_\lambda, A, \mathcal{H}_\lambda, \nu_\lambda)$ *is an orientable smooth manifold of dimension* $2(g+k)$*, and*

(b) *the Frontier of the smooth map*

$$\mathcal{M}_{g,k}(Z_\lambda, A, \mathcal{H}_\lambda, \nu_\lambda) \xrightarrow{st\times ev} \overline{\mathcal{M}}_{g,k} \times Z_\lambda^k.$$

lies in dimension 2 less than $g+k$*.*

Proof. The proof of (a) is same as the proof of Theorem 7.3a. On the other hand, note that any bubble component of (f,α) maps into either a singular fiber or the section by Corollary 1.4a. Therefore, we can conclude (b) using the same argument as in the proof of Theorem 7.3b. □

We end this section with the splitting argument as in [IP3]. This shows how maps into $X = E(n)$ split along the degeneration of $E(n)$. It is also a key observation for gluing of maps into X and Y, which leads to the sum formula (5.5).

Lemma 8.5 *Let* $(f_n, \Sigma_n, \alpha_n)$ *be any sequence of* (J_Z, ν, α_n)*-holomorphic maps such that (i) each* f_n *maps into* Z_{λ_n}*, (ii) each* f_n *represent the homology class* $S+dF$*, and (iii)* $\lambda_n \to 0$ *as* $n \to \infty$*. Then we have*

(i) f_n *converges to a limit* $f : \Sigma \to Z_0 = X \cup_V Y$ *and* α_n *converges to* α*, after passing to some subsequence.*

(ii) *the limit map* f *can be decomposed as*

$$f_1 : \Sigma_1 \to X,\ \ f_2 : \Sigma_2 \to Y,\ \ and\ f_3 : \Sigma_3 \to V$$

where f_1 *is a stable* (J_X, ν_X, α)*-holomorphic, and* f_2 *and* f_3 *are stable* (J_Y, ν_Y)*-holomorphic and* (J_V, ν_V)*)-holomorphic, respectively,*

(iii) *for* $i = 1, 2$*, each* f_i *transverse to* V *with* $f_i^{-1}(V) = \{\,p\,\}$*, where each* p *is a node of* Σ_i*.*

Proof. (i) follows from Gromov Convergence Theorem and Lemma 8.2. Note that $\alpha = 0$ near $V \subset Z$ when $\alpha \in \mathcal{H}_Z$. Hence, $J_{\alpha_n} = J$ near $V \subset Z$. Therefore, we can apply Contact Lemma in [IP2] to conclude (ii). Lastly, (iii) follows from Contact lemma in [IP2] and lemma 3.3 in [IP3]. □

9 Sum Formula

This section shows the sum formula (5.5) using Gluing Theorem(Theorem 10.1 of [IP3]). This Gluing Theorem relates invariants of $Z_\lambda = E(n)$ with *relative invariants* of $X = E(n)$ and $Y = T^2 \times S^2$. In our case, *rim tori* in $X \setminus V$ disappear in the symplectic gluing $X \sharp Y$ along the fiber V, which is deformation equivalent to Z_λ. Together with the simple matching condition as in Lemma 8.5, this leads to the simple definition of relative invariants (see Remark 5.3 of [IP2] and Appendix in [IP3]). In fact, relative invariants in our case is same as absolute invariants.

Following [IP2], we first introduce relative invariants for $X = E(n)$ relative to $V = T^2$ for the class $A = S + dF$. As in section 7, we fix the complex structure on X. In the below, we will not specify complex structures on X in the notation. We also assume that we always work with a finite good cover p_μ as in (7.1) without specifying it.

For each $\nu \in \mathcal{Y}^V$ we define the relative moduli space as

$$\mathcal{M}^V_{g,k+1}(X, A, \mathcal{H}_X, \nu_X) = \{ (\, f, j, \alpha\,) \in \mathcal{M}_{g,k+1}(X, A, \mathcal{H}_X, \nu_X) \,|\, f(x_{k+1}) \in V \,\} \tag{9.1}$$

The following proposition is the structure theorem for the above relative moduli space.

Proposition 9.1 *For generic $\nu \in \mathcal{Y}^V$ and $g \leq 1$,*

(a) $\mathcal{M}^V_{g,k+1}(X, A, \mathcal{H}_X, \nu_X)$ is an oriented smooth manifold of dimension $2(g + k)$, and

(b) The Frontier of the map

$$\mathcal{M}^V_{g,k+1}(X, A, \mathcal{H}_X, \nu_X) \xrightarrow{st \times ev \times h} \overline{\mathcal{M}}_{g,k+1} \times X^k \times V \tag{9.2}$$

is contained in dimension 2 less than $2(g + k)$, where ev is the evaluation map of the first k marked points and h is the evaluation map of the last marked point.

Proof. Since for each $\alpha \in \mathcal{H}_X$, $\alpha = 0$ in some neighborhood of $V \subset X$, $J_\alpha = J$ on that neighborhood. Therefore, (a) follows from Lemma 4.2 of [IP2].

On the other hand, the Frontier of (9.2) is the image of

$$\mathcal{C}\mathcal{M}^V_{g,k+1}(X, A, \mathcal{H}_X, \nu_X) \subset \overline{\mathcal{M}}_{g,k+1}(X, A, \mathcal{H}_X, \nu_X) \tag{9.3}$$

under stabilization and evaluation maps, where (9.3) is the closure of $\mathcal{M}^V_{g,k+1}(X, A, \mathcal{H}_X, \nu_X)$ in $\overline{\mathcal{M}}_{g,k+1}(X, A, \mathcal{H}_X, \nu_X)$. We reduce the closure (9.3) under the reduction as in section 7. Then the dimension count is same as in the proof Theorem 7.3b except for (f, α) with some components which map entirely into V. Those strata corresponding to such (f, α) are empty by Lemma 6.6 of [IP2]. Therefore, we can conclude (b). □

It follows from the above proposition and Proposition 4.2 of [KM] that the image of (9.2) gives rise to a rational homology class. We denote it by

$$[\, \mathcal{M}^V_{g,k+1}(X, A, \mathcal{H}_X)\,] \in H_*(\overline{\mathcal{M}}_{g,k+1}; \mathbb{Q}) \otimes H_*(X^k; \mathbb{Q}) \otimes H_*(V; \mathbb{Q})$$

Definition 9.2 *For $g \leq 1$ with $2g+k \geq 3$, β in $H^*(\overline{\mathcal{M}}_{g,k+1};\mathbb{Q})$, $\alpha_1, \cdots, \alpha_k$ in $H^*(X^k;\mathbb{Q})$, and γ in $H_*(V;\mathbb{Q})$, we define relative invariants by*

$$GW^V_{g,k+1}(X,A,\mathcal{H}_X)(\,\beta;\alpha_1,\cdots,\alpha_k;C(\gamma)\,) = (\,\beta \otimes (\alpha_1 \wedge \cdots \wedge \alpha_k) \otimes C(\gamma)\,) \cap [\,\mathcal{M}^V_{g,k+1}(X,A,\mathcal{H}_X)\,]$$

where $C(\gamma)$ is the Poincaré dual of γ in V.

Similarly as above, for $Y = T^2 \times S^2$ and $V = T^2$ we set

$$\mathcal{M}^V_{g,k+1}(Y,A,\nu_Y) = \{\ (\,f,j\,) \in \mathcal{M}_{g,k+1}(Y,A,\nu_Y) \mid f(y_{k+1}) \in V, \ \ \} \tag{9.4}$$

$$\mathcal{M}^V_{g,k+1}(Y,A,\nu_Y) \xrightarrow{st\times ev\times h} \overline{\mathcal{M}}_{g,k+1} \times Y^k \times V \tag{9.5}$$

where h is the evaluation map of the first marked point and ev is the evaluation map of the last k marked points. Repeating the same arguments as above, we can define *relative invariants* from (9.4) and (9.5).

Remark 9.3 It follows from similar arguments as in Lemma 8.2 and Proposition 8.4 that for generic ν_X the moduli space

$$\mathcal{M}_{g,k+1}(X,A,\mathcal{H}_X,\nu_X) \tag{9.6}$$

carries a well-defined homology class. By Proposition 3.3 and with some appropriate one parameter family of bump functions $\{\beta_t\}$ as in Definition 8.1, we can also show that (9.6) gives the same absolute invariants as in Definition 7.4. Therefore, by Proposition 14.9 of [IP3] we have

$$GW^V_{g,k+1}(X,A,\mathcal{H}_X)(\,\beta;\alpha_1,\cdots,\alpha_k;C(\gamma)\,) = GW^{\mathcal{H}_n}_{g,k+1}(X,A)(\,\beta;\alpha_1,\cdots,\alpha_k,i_*\gamma\,),$$

where $i : V \hookrightarrow X$ is the inclusion. Similarly, we also have

$$GW^V_{g,k+1}(Y,A)(\,\beta;\alpha_1,\cdots,\alpha_k;C(\gamma)\,) = GW_{g,k+1}(X,A)(\,\beta;\alpha_1,\cdots,\alpha_k,j_*\gamma\,),$$

where $j : V \hookrightarrow Y$ is the inclusion.

We are ready to prove the sum formula (5.5).

Proposition 9.4

$$H(t) = -\frac{1}{12}F(t) + 2F(t)G(t)$$

Proof. By definition of generating functions $H(t)$, $F(t)$, and $G(t)$, it suffices to show that

$$GW^{\mathcal{H}_n}_{1,4}(S+dF)(\psi_{(1,4);4};F^4) = \sum_{d_1+d_2=d} GW^{\mathcal{H}_n}_{0,0}(S+d_1F)\,\left(2\sigma(d_2) - \frac{1}{12}\right)$$

where $\sigma(d_2) = \sum_{k|d_2}$ as in section 5.

We can choose a submanifold $F_i \subset Z$ for $i = 1,\cdots,4$ satisfying (i) each $F_i \cap Z_\lambda$ represents a fiber class in $Z_\lambda = E(n)$, (ii) for $i = 1,2$ $F_i \cap X$ (resp. $F_{i+2} \cap Y$) represents a fiber class in

X (resp. Y), and (iii) those $F_i \cap Z_\lambda$, $F_i \cap X$, and $F_{i+2} \cap Y$ are all in general position with respect to the corresponding evaluation maps. On the other hand, there is a pseudo-submanifold K in $\overline{\mathcal{M}}_{1,4}$ representing Poincaré dual of $\psi_{(1,4);4}$. We can assume that K is in general position with respect to the corresponding stabilization maps and the following gluing maps:

$$\sigma_1 : \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{1,3} \to \overline{\mathcal{M}}_{1,4},\ \sigma_2 : \overline{\mathcal{M}}_{1,3} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{1,4},\ \text{and}\ \ \sigma_3 : \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{1,4}.$$

Now, consider the cut-down moduli space

$$\mathcal{M}_\lambda = \{\ (f,(j;\{x_i\}),\alpha) \mid f(x_i) \in F_i,\ st(j) \in K\ \}\ \subset\ \mathcal{M}_{1,4}(Z_\lambda, A, \mathcal{H}_\lambda)$$

It then follows from Remark 9.3 that

$$[\, st \times ev(\mathcal{M}_\lambda)\,] = GW^{\mathcal{H}_n}_{1,4}(S+dF)(\psi_{(1,4);4}; F^4). \tag{9.7}$$

Similarly, let $\mathcal{M}_0$ be the cut-down moduli space which consists of all

$$\Big(\, \Big(f_1,(j_1,\{x_i\}),\alpha\Big), \Big(f_2,(j_2,\{y_k\})\Big)\,\Big)$$

satisfying (i) $f_1(x_3) = f_2(y_3)$, (ii) for $i = 1,2$, $f_1(x_i)$ in F_i and $f_2(y_i)$ in F_{i+2}, and (iii) $\sigma(j_1,j_2)$ in K, where σ is one of σ_1 and σ_2. Since $\sigma_2^*(\psi_{(1,4);4}) = 0$, we have

$$\mathcal{M}_0 \subset \bigcup_{d_1+d_2=d} \mathcal{M}^V_{0,3}(X, S+d_1F, \mathcal{H}_X) \underset{h}{\times} \mathcal{M}^V_{1,3}(Y, S+d_2F). \tag{9.8}$$

Together with routine dimension count, (9.8) implies that

$$\begin{aligned}
&[\, st \times ev\,(\mathcal{M}_0)\,] \\
&= \sum_{d_1+d_2=d} GW^{V;\mathcal{H}_X}_{0,3}(S+d_1F)(\,F^2; C(V)\,)\, GW^V_{1,3}(S+d_2F)(\,\psi_{(1,3);3}; F^2; C(pt)\,) \\
&= \sum_{d_1+d_2=d} GW^{\mathcal{H}_n}_{0,3}(S+d_1F)(\,F^3\,)\, GW_{1,3}(S+d_2F)(\,\psi_{(1,3);3}; F^2, pt\,) \\
&= \sum_{d_1+d_2=d} GW^{\mathcal{H}_n}_{0,0}(S+d_1F)\left(2\sigma(d_2) - \frac{1}{12}\right)
\end{aligned} \tag{9.9}$$

where the second equality follows from Remark 9.3, while the third equality follows from TRR for $T^2 \times S^2$.

It remains to show that

$$\pi_{0_*}[\, st' \times ev\,(\mathcal{M}_0)\,] = \pi_{\lambda_*}[\, st \times ev\,(\mathcal{M}_\lambda)\,]\ \text{ in }\ H_0(\,\overline{\mathcal{M}}_{1,4} \times Z_0^4\,) \tag{9.10}$$

where $\pi_0 : X \cup Y \hookrightarrow Z_0$ and $\pi_\lambda : Z_\lambda \to Z_0$ are collapsing maps (see section 2 of [IP3]). By Lemma 8.5, as $\lambda \to 0$ any sequence $(f_\lambda, j_\lambda, \alpha_\lambda) \in \mathcal{M}_\lambda$ converges to a limit (f, j, α) such that j lies on among the images of the gluing maps σ_1, σ_2, and σ_3. Since j also lies on K and both $\sigma_2^*(\psi_{(1,4);4})$ and $\sigma_3^*(\psi_{(1,4);4})$ are trivial, j lies on the image of σ_1. This implies the limit (f, j, α) lies in $\mathcal{M}_0$. Now, the proof follows from Theorem 10.1 of [IP3]. □

11 Appendix – Relations with the Behrend-Fantechi Approach

Behrend and Fantechi [BF] have defined modified GW invariants for Kähler surfaces using algebraic geometry. While their techniques are completely different from ours, the definitions seem to be, at their core, equivalent. In this appendix we make several observations which relate their approach to ours. This is necessarily tentative because the paper [BF] is not yet available; we are relying on the terse description given in [BL3].

In algebraic geometry, the virtual fundamental class $[\overline{\mathcal{M}}_{g,k}(X,A)]^{\text{vir}}$ is obtained from the relative tangent-obstruction spaces together with the tangent-obstruction spaces of Deligne-Mumford space $\overline{\mathcal{M}}_{g,k}$. Behrend and Fantechi modified their machinery, intrinsic normal cone and obstruction complex, by replacing the relative obstruction space $H^1(f^*TX)$ by the kernel of the map

$$H^1(f^*TX) \to H^2(X,\mathcal{O}) \tag{A.1}$$

defined by dualizing of the composition

$$H^0(X,\Omega^2) \to H^0(f^*\Omega^2) \to H^0(f^*\Omega^1 \otimes f^*\Omega^1) \to H^0(f^*\Omega^1 \otimes \Omega^1). \tag{A.2}$$

In order for their machinery to work, the map (A.1) is of constant rank — in particular surjective — for every f in $\overline{\mathcal{M}}_{g,k}(X,A)$ [BL3]. Composing (A.2) with the Kodaira-Serre dual map, we have

$$H^0(X,\Omega) \to H^0(f^*\Omega^1 \otimes \Omega^1) \to H^1(f^*TX). \tag{A.3}$$

This map is given by $\beta \to K_\beta \, df\, j$.

Proposition A.1 *Let (X,J) be a Kähler surface and $A \in H^{1,1}(X,\mathbb{Z})$. Then the family moduli space $\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X,A)$ is compact if and only if the map (A.1) is surjective for every f in $\overline{\mathcal{M}}_{g,k}(X,A)$.*

Proof. By Theorem 2.4 $\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X,A)$ consists of pairs (f,α) with $f \in \overline{\mathcal{M}}_{g,k}(X,A)$ and with the image of f contained in the zero set of α; the latter condition means that $K_\alpha = 0$ along the image, so $K_\alpha \, df\, j = 0$ for all (f,α). As usual, $\overline{\mathcal{M}}_{g,k}(X,A)$ is compact by the Gromov Compactness Theorem.

Now, suppose (A.1) is surjective. Then by duality (A.3) is injective. This implies $\alpha = 0$ and hence $\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X,A) = \overline{\mathcal{M}}_{g,k}(X,A)$ is compact. Conversely, suppose for some $f \in \overline{\mathcal{M}}_{g,k}(X,A)$ there is a β in the kernel of (A.3). Then setting $\alpha = \beta + \bar\beta$ we have $\overline{\partial}_J f = tK_\alpha df j = 0$ — and hence $(f,t\alpha) \in \overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X,A)$ — for all real t. That means that $\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X,A)$ is compact only when (A.3) is injective or equivalently when (A.1) is surjective. □

The map (A.3) is directly related to the linearization operator of the (J,α)-holomorphic map equation.

Suppose that A is $(1,1)$ and that the family moduli space $\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X,A)$ is compact as in Proposition A.1. Consider the linearization of the (J,α)-holomorphic map equation at (f,j,α). Since J is Kähler, the linearization reduces to

$$L_f \oplus Jdf \oplus L_0 : \Omega^0(f^*TX) \oplus T_j\overline{\mathcal{M}}_{g,n} \oplus \mathcal{H} \to \Omega^{0,1}(f^*TX) \quad \text{where} \quad \begin{cases} L_f(\xi) &= \nabla\xi + J\nabla\xi\, j \\ L_0(\beta) &= -2K_\beta df j \end{cases}$$

In fact, this L_f is exactly (twice) the Dolbeault derivative $\overline{\partial}$. Therefore, $\mathrm{Ker}(L_f)$ are $\mathrm{Coker}(L_f)$ are identified with the Dolbeault cohomology groups $H^0(f^*TX)$ and $H^{0,1}(f^*TX)$, respectively.

Proposition A.2 *Under either of the two equivalent conditions of Proposition A.1 there are natural identifications $H^1(f^*TX) \simeq H^{0,1}(f^*TX)$ and $H^0(X, \Omega^2) \simeq \mathcal{H}$ under which identification the map is identified with (A.3) with*

$$L_0 : \mathcal{H} \to \mathit{Coker}(L_f \oplus Jdf).$$

By Proposition A.1 this map is injective if and only if the family moduli space $\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X, A)$ is compact.

Proof. It follows by comparing the formulas for L_0 and $(A.3)$ that L_0 maps $\mathcal{H}$ into $\mathrm{Coker}(L_f)$. On the other hand, given $h \in T_j\overline{\mathcal{M}}_{g,n}$, there is a family j_t with $j_0 = j$ and $\frac{dj_t}{dt}\big|_{t=0} = h$. It follows from Proposition 1.3b and $\langle \beta, A\rangle = 0$ that

$$0 = \frac{d}{dt}\bigg|_{t=0} \int_{(\Sigma, j_t)} f^*(\beta) = \frac{d}{dt}\bigg|_{t=0} \int_{(\Sigma, j_t)} \langle df + Jdf j_t, K_\beta f_* j_t\rangle = \int_\Sigma \langle Jdf(h), K_\beta f_* j\rangle.$$

This implies that L_0 maps $\mathcal{H}$ into $\mathrm{Coker}(L_f \oplus Jdf)$. □

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