Extending Similarity Measures of Interval Type-2 Fuzzy Sets to General Type-2 Fuzzy Sets

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Similarity Measures

One of the most common tools of fuzzy logic is similarity measures. A similarity measure between fuzzy sets indicates the degree to which the fuzzy sets are similar as a value between 0 and 1.

Although some methods have been developed for interval type-2 fuzzy sets, fewer methods exist for general type-2 fuzzy sets.

This presentation presents a general method of extending similarity measures on interval type-2 fuzzy sets to similarity measures on general type-2 fuzzy sets.
Similarity Properties

Four properties of similarity measures for fuzzy sets that are commonly used in the literature are:

**Reflexivity**: \( s(\tilde{A}, \tilde{B}) = 1 \iff \tilde{A} = \tilde{B} \)

**Symmetry**: \( s(\tilde{A}, \tilde{B}) = s(\tilde{B}, \tilde{A}) \)

**Transitivity**: If \( \tilde{A} \leq \tilde{B} \leq \tilde{C} \), then \( s(\tilde{A}, \tilde{B}) \geq s(\tilde{A}, \tilde{C}) \)

**Overlapping**: If \( \tilde{A} \cap \tilde{B} \neq \emptyset \), then \( s(\tilde{A}, \tilde{B}) > 0 \); otherwise, \( s(\tilde{A}, \tilde{B}) = 0 \)

Note that it is not necessary for a similarity measures to have all of these properties as the application of the measure may not depend on all of them.
The following four existing interval type-2 similarity were reviewed and tested against general type-2 fuzzy sets.

- Zeng & Li
- Jaccard
- Gorzałczany
- Bustince

Additional similarity measures exist in the literature, but for conciseness only these four are currently considered.
A general type-2 fuzzy set can be represented by slicing the third dimension \((z)\) at level \(z_i\) to create a zSlices-based general type-2 fuzzy set.

The resulting set will consist of zSlices which are interval type-2 fuzzy sets with a secondary membership grade of \(z_i\).
Similarity Measures for General Type-2 Fuzzy Sets

By using zSlices-based general type-2 fuzzy sets, a measure of similarity on interval type-2 fuzzy sets can be applied to each zSlice, and the results for each zSlice can be combined as follows:

\[ S_{ZS}(\tilde{A}, \tilde{B}) = \frac{\sum_{i \in L} z_i S_{\lambda}(\tilde{A}_{z_i}, \tilde{B}_{z_i})}{\sum_{i \in L} z_i} \]

where \( S_{\lambda}(\tilde{A}_{z_i}, \tilde{B}_{z_i}) \) is any similarity measure for interval type-2 fuzzy sets. Sets \( \tilde{A}_{z_i} \) and \( \tilde{B}_{z_i} \) are zSlices from sets \( \tilde{A} \) and \( \tilde{B} \) at zLevel \( z_i \), and \( L \) is the set of zLevels used by \( \tilde{A} \) and \( \tilde{B} \). For example, if \( \tilde{A} \) and \( \tilde{B} \) have three zLevels where \( z_1 = 0.33 \), \( z_2 = 0.66 \) and \( z_3 = 1 \) then \( L = \{0.33, 0.66, 1\} \).
zSlices-based General Type-2 Fuzzy Sets with different numbers of zLevels

Diagram showing examples of zSlices-based General Type-2 Fuzzy Sets with different numbers of zLevels.
zSlices-based General Type-2 Fuzzy Sets with different numbers of zLevels
zSlices-based General Type-2 Fuzzy Sets with different numbers of zLevels

The similarity of sets with different numbers of zLevels is calculated using the union of their zLevels as follows:

\[ L = \bigcup_{m=1}^{M} z_m \cup \bigcup_{n=1}^{N} z_n \]

Where \(M\) and \(N\) are the number of zLevels used by each respective fuzzy set. In this example, \(M = 4\) and \(N = 3\), so

\[ L = \{0.25, 0.5, 0.75, 1.0\} \cup \{0.33, 0.66, 1.0\} = \{0.25, 0.33, 0.5, 0.66, 0.75, 1.0\}. \]
zSlices-based General Type-2 Fuzzy Sets with different numbers of zLevels
Reviewing Interval Type-2 Similarity Measures

<table>
<thead>
<tr>
<th></th>
<th>$S(\tilde{A}, \tilde{A})$</th>
<th>$S(\tilde{A}, \tilde{B})$</th>
<th>$S(\tilde{A}, \tilde{C})$</th>
<th>$S(\tilde{A}, \tilde{D})$</th>
<th>$S(\tilde{A}, \tilde{E})$</th>
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<tr>
<td>Zeng &amp; Li</td>
<td>1.0</td>
<td>0.538</td>
<td>0.345</td>
<td>0.371</td>
<td>0.461</td>
</tr>
<tr>
<td>Jaccard</td>
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<td>0.071</td>
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<td>(1.0, 1.0)</td>
<td>(0.0, 1.0)</td>
<td>(0.0, 0.0)</td>
<td>(0.0, 0.0)</td>
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<tr>
<td>Bustince</td>
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<td>(0.0, 0.15)</td>
<td>(0.0, 0.15)</td>
<td>(0.0, 0.15)</td>
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Converting Interval Type-2 Fuzzy Sets into General Type-2 Fuzzy Sets
Interval Type-2 Similarity Measures on zSlices-based General Type-2 Fuzzy Sets

\[
S(\tilde{A}, \tilde{A}) \quad S(\tilde{A}, \tilde{B}) \quad S(\tilde{A}, \tilde{C}) \quad S(\tilde{A}, \tilde{D}) \quad S(\tilde{A}, \tilde{E})
\]

<table>
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<tr>
<th>Method</th>
<th>$S(\tilde{A}, \tilde{A})$</th>
<th>$S(\tilde{A}, \tilde{B})$</th>
<th>$S(\tilde{A}, \tilde{C})$</th>
<th>$S(\tilde{A}, \tilde{D})$</th>
<th>$S(\tilde{A}, \tilde{E})$</th>
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</thead>
<tbody>
<tr>
<td>Zeng &amp; Li</td>
<td>1.0</td>
<td>0.496</td>
<td>0.267</td>
<td>0.345</td>
<td>0.443</td>
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<tr>
<td>Jaccard</td>
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<td>0.335</td>
<td>0.041</td>
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<td>Gorzałczany</td>
<td>(1.0, 1.0)</td>
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<td>(0.33, 0.66)</td>
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<tr>
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<td>(0.05, 0.1)</td>
<td>(0.05, 0.1)</td>
<td>(0.05, 0.1)</td>
<td>(0.05, 0.1)</td>
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Diagram showing the interval type-2 fuzzy sets and their similarity measures.
Summary

We have introduced a general method of extending existing similarity measures on interval type-2 fuzzy sets to similarity measures on general type-2 fuzzy sets through the use of the zSlices based general type-2 fuzzy set representation.

The extension preserves all the common initial properties for similarity measures of the interval type-2 case, namely transitivity, symmetry, transitivity and overlapping.
Questions?