Strong and Weak Convergence Theorems for Common Fixed Points of Two Nonself Asymptotically Nonexpansive Mappings in Banach Spaces

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Abstract

Suppose that \(K\) is a nonempty closed convex subset of a uniformly convex and smooth Banach space \(E\) with \(P\) as a sunny nonexpansive retraction. Let \(T_1, T_2 : K \to E\) be two weakly inward nonself asymptotically nonexpansive mappings with respect to \(P\) with two sequences \(\{k_n^{(i)}\} \subset [1, \infty)\) satisfying \(\sum_{n=1}^{\infty} (k_n^{(i)} - 1) < \infty\) \((i = 1, 2)\) and \(F(T_1) \cap F(T_2) = \{x \in K : T_1x = T_2x = x\} \neq \emptyset\), respectively. For any given \(x_1 \in K\), suppose that \(\{x_n\}\) is a sequence generated iteratively by

\[
\begin{aligned}
x_{n+1} &= \alpha_n x_n + \beta_n (PT_1)^n y_n + \gamma_n (PT_2)^n y_n, \\
y_n &= \alpha_n x_n + \beta_n (PT_1)^n z_n + \gamma_n (PT_2)^n z_n, \\
z_n &= \alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n,
\end{aligned}
\]

where \(\{\alpha_n\}, \{\beta_n\}, \) and \(\{\gamma_n\}\) \((i = 1, 2, 3)\) are sequences in \([a, 1-a]\) for some \(a \in (0, 1)\), satisfying \(\alpha_n + \beta_n + \gamma_n = 1\) \((i = 1, 2, 3)\). Under some suitable conditions, the strong and weak convergence theorems of \(\{x_n\}\) to a common fixed point of \(T_1\) and \(T_2\) are obtained.

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1 Introduction

Let $K$ be a nonempty closed convex subset of a real uniformly convex Banach space $E$. A self-mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. A self-mapping $T : K \to K$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [1, \infty)$ satisfying $k_n \to 1$ as $n \to \infty$ such that

$$\|T^nx - T^ny\| \leq k_n \|x - y\|, \quad \forall x, y \in K, n \geq 1. \quad (1.1)$$

A self-mapping $T : K \to K$ is said to be uniformly $L$-Lipschitzian if there exists a constant $L \geq 0$ such that

$$\|T^nx - T^ny\| \leq L \|x - y\|, \quad \forall x, y \in K, n \geq 1. \quad (1.2)$$

As a generalization of the class of nonexpansive maps, the class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] in 1972, who proved that if $K$ is a nonempty bounded closed convex subset of a real uniformly convex Banach space and $T$ is an asymptotically nonexpansive self-mapping of $K$, then $T$ has a fixed point.

Recently, Chidume et al. [2] further generalized the class of asymptotically nonexpansive mappings introduced by Goebel and Kirk [1], and proposed the concept of nonself asymptotically nonexpansive mapping defined as follows:

**Definition 1.1.** [2] Let $K$ be a nonempty subset of real normed linear space $E$. Let $P : E \to K$ be the nonexpansive retraction of $E$ onto $K$.

(1) A nonself mapping $T : K \to E$ is called *asymptotically nonexpansive* if there exist sequences $\{k_n\} \in [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^nx - T^ny\| \leq k_n \|x - y\|, \quad \forall x, y \in K, n \geq 1. \quad (1.3)$$

(2) A nonself mapping $T : K \to E$ is said to be *uniformly $L$-Lipschitzian* if there exists a constant $L \geq 0$ such that

$$\|T^nx - T^ny\| \leq L \|x - y\|, \quad \forall x, y \in K, n \geq 1. \quad (1.4)$$

By using the following iterative algorithm:

$$x_1 \in K, \quad x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \quad \forall n \geq 1. \quad (1.5)$$

Chidume et al. [2] established demiclosed principle, strong and weak convergence theorems for such mappings in uniformly convex Banach spaces. Since then, some authors [3-6] have studied the strong and weak convergence theorem for such mappings.
As a matter of fact, if $T$ is a self-mapping, then $P$ is a identity mapping. Thus (1.3) and (1.4) reduce to (1.1) and (1.2) as $T$ is a self-mapping, respectively. In addition, if $T : K \to E$ is asymptotically nonexpansive in light of (1.3) and $P : E \to K$ is a nonexpansive retraction, then $PT : K \to K$ is asymptotically nonexpansive in light of (1.1). Indeed, for all $x, y \in K$ and $n \geq 1$, by (1.3), it follows that
\[
\| (PT)^nx - (PT)^ny \| = \| PT(PT)^n^{-1}x - PT(PT)^n^{-1}y \|
\leq \| T(PT)^n^{-1}x - T(PT)^n^{-1}y \|
\leq k_n \| x - y \|. \tag{1.6}
\]
Conversely, it may not be true. Therefore, Zhou et al.[7] introduced the following generalized definition recently.

**Definition 1.2.**[7] Let $K$ be a nonempty subset of real normed linear space $E$. Let $P : E \to K$ be a nonexpansive retraction of $E$ onto $K$.

1. A nonself mapping $T : K \to E$ is called asymptotically nonexpansive with respect to $P$ if there exist sequences $\{k_n\} \in [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that
\[
\| (PT)^nx - (PT)^ny \| \leq k_n \| x - y \|, \forall x, y \in K, n \geq 1. \tag{1.7}
\]

2. A nonself mapping $T : K \to E$ is said to be uniformly $L$-Lipschitzian with respect to $P$ if there exists a constant $L \geq 0$ such that
\[
\| (PT)^nx - (PT)^ny \| \leq L \| x - y \|, \forall x, y \in K, n \geq 1. \tag{1.8}
\]

Furthermore, by studying the following iterative process:
\[
x_{n+1} = \alpha_n x_n + \beta_n (PT_1)^nx_n + \gamma_n (PT_2)^nx_n, \forall x_1 \in K, n \geq 1, \tag{1.9}
\]
where $\{\alpha_n\}, \{\beta_n\},$ and $\{\gamma_n\}$ are three sequences in $[a, 1 - a]$ for some $a \in (0, 1)$, satisfying $\alpha_n + \beta_n + \gamma_n = 1$, Zhou et al.[7] obtained some strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings with respect to $P$ in uniformly convex Banach spaces. As a consequence, the main results of Chidume et al. [2] are deduced.

Inspired and motivated by those work mentioned above and three step iteration method proposed by Noor[8], in this paper, we construct a three step iteration scheme for approximating common fixed points of two nonself asymptotically nonexpansive mappings with respect to $P$ and to prove some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces.
2 Preliminaries

Let $K$ be a nonempty closed convex subset of a real uniformly convex Banach space $E$ with retraction $P$. Let $T_1, T_2 : K \to E$ be two nonself asymptotically nonexpansive mappings with respect to $P$. For approximating common fixed points of such mappings, we further generalize the iteration scheme (1.9) as follows:

\[
\begin{align*}
  x_{n+1} & = \alpha_n x_n + \beta_n (PT_1)^n y_n + \gamma_n (PT_2)^n z_n, \\
  y_n & = \alpha_n x_n + \beta_n (PT_1)^n z_n + \gamma_n (PT_2)^n z_n, \\
  z_n & = \alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n,
\end{align*}
\]

(2.1)

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}(i = 1, 2, 3)$ are sequences in $[0, 1)$, satisfying $\alpha_n + \beta_n + \gamma_n = 1(\forall i = 1, 2, 3)$.

For the sake of convenience, we restate the following concepts and results:

Let $E$ be a Banach space with dimension $E \geq 2$. The modulus of $E$ is the function $\delta_E(\epsilon) : (0, 2] \to [0, 1]$ defined by

\[
\delta_E(\epsilon) = \inf \left\{ 1 - \frac{1}{2} \| x + y \| : \| x \| = 1, \| y \| = 1, \epsilon = \| x - y \| \right\}.
\]

A Banach space $E$ is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

Let $E$ be a Banach space and $S(E) = \{ x \in E : x = 1 \}$. The space $E$ is said to be smooth if

\[
\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t}
\]

exists for all $x, y \in S(E)$.

A subset $K$ of $E$ is said to be retract if there exists continuous mapping $P : E \to K$ such that $Px = x$ for all $x \in K$. A mapping $P : E \to E$ is said to be a retraction if $P^2 = P$. Let $C$ and $K$ be subsets of a Banach space $E$. A mapping $P$ from $C$ into $K$ is called sunny if $P(Px + t(x - Px)) = Px$ for $x \in C$ with $Px + t(x - Px) \in C$ and $t \geq 0$.

Note that, if mapping $P$ is a retraction, then $Pz = z$ for every $z \in R(P)$ (the range of $P$). It is well-known that every closed convex subset of a uniformly convex Banach space is a retract.

For any $x \in K$, the inward set $I_K(x)$ is defined as follows:

\[
I_K(x) = \{ y \in E : y = x + \lambda(z - x), z \in K, \lambda \geq 0 \}.
\]

A mapping $T : K \to E$ is said to satisfy the inward condition if $Tx \in I_K(x)$ for all $x \in K$. $T$ is said to satisfy the weakly inward condition if, for each $x \in K$, $Tx \in \text{cl}I_K(x)$ ($\text{cl}I_K(x)$ is the closure of $I_K(x)$).
A Banach space $E$ is said to satisfy Opial’s condition if, for any sequence \( \{x_n\} \) in $E$, $x_n \rightharpoonup x$ implies that
\[
\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|
\]
for all $y \in E$ with $y \neq x$, where $x_n \rightharpoonup x$ denotes that \( \{x_n\} \) converges weakly to $x$.

Let $K$ be a nonempty closed subset of a real Banach space $E$. $T : K \to E$ is said to be demicompact if, for any sequence \( \{x_n\} \subset K \) with $\|x_n - Tx_n\| \to 0$ ($n \to \infty$), there exists a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) such that \( \{x_{n_j}\} \) converges strongly to $x^* \in K$.

A mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is said to be demi-closed at $p$ if whenever \( \{x_n\} \) is a sequence in $D(T)$ such that \( \{x_n\} \) converges weakly to $x^* \in D(T)$ and \( \{Tx_n\} \) converges strongly to $p$, then $Tx^* = p$.

We need the following lemmas for our main results.

**Lemma 2.1.** [9] Let \( \{a_n\}, \{\delta_n\}, \{b_n\} \) be sequences of nonnegative real numbers satisfying
\[
a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \forall n \geq 1,
\]
if $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \to \infty} a_n$ exists.

**Lemma 2.2.** [10] Let $E$ be a real uniformly convex Banach space and let $B_r(0)$ be the closed ball of $E$ with center at the origin and radius $r \geq 0$. Then, there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that
\[
\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \gamma \|z\|^2 - \lambda \mu g(\|x - y\|)
\]
for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

**Lemma 2.3.** [11] Let $E$ be a real smooth Banach space, let $K$ be a nonempty closed convex subset of $E$ with $P$ as a sunny nonexpansive retraction, and let $T : K \to E$ be a mapping satisfying weakly inward condition. Then $F(PT) = F(T)$.

**Lemma 2.4.** [12] Let $E$ be a uniformly convex Banach space and $K$ a nonempty closed convex subset of $E$. Let $T : K \to K$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $\{k_n\} \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed at zero, that is, for each sequence \( \{x_n\} \) in $K$, if the sequence \( \{x_n\} \) converges weakly to $q \in K$ and \( \{(I - T)x_n\} \) converges strongly to 0, then $(I - T)q = 0$. 

3 Main Results

Lemma 3.1. Let $K$ be a nonempty closed convex subset of a real uniformly convex Banach space $E$. Let $T_1, T_2 : K \to E$ be two nonself asymptotically non-expansive mappings with respect to $P$ with two sequences $\{k_n^{(1)}\}, \{k_n^{(2)}\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty} (k_n^{(i)} - 1) < \infty (i = 1, 2)$, respectively. Suppose that $\{x_n\}$ is defined by (2.1), where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\} (i = 1, 2, 3)$ are sequences in $[a, 1 - a]$ for some $a \in (0, 1)$. If $F := F(T_1) \cap F(T_2) \neq \emptyset$, then

(1) $\lim_{n \to \infty} \|x_n - q\|$ exists, $\forall q \in F$;

(2) $\lim_{n \to \infty} d(x_n, F)$ exists, where $d(x_n, F) = \inf_{q \in F} \|x_n - q\|$;

(3) $\lim_{n \to \infty} \|x_n - (PT_i)x_n\| = 0 (i = 1, 2)$.

Proof. (1) Setting $k_n = \max\{k_n^{(1)}, k_n^{(2)}\}$. Since $\sum_{n=1}^{\infty} (k_n^{(i)} - 1) < \infty (i = 1, 2)$, so $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. For any $q \in F$, by (2.1), we have

$$
\|z_n - q\| = \|\alpha_n x_n - q\| + \beta_n ((PT_1)^n x_n - q) + \gamma_n ((PT_2)^n x_n - q)\|
\leq \alpha_n \|x_n - q\| + \beta_n k_n \|x_n - q\| + \gamma_n k_n \|x_n - q\|
= k_n \|x_n - q\|. \quad (3.1)
$$

By (2.1) and (3.1), we have

$$
\|y_n - q\| = \|\alpha_n x_n - q\| + \beta_n ((PT_1)^n z_n - q) + \gamma_n ((PT_2)^n z_n - q)\|
\leq \alpha_n \|x_n - q\| + \beta_n k_n^2 \|x_n - q\| + \gamma_n k_n \|x_n - q\|
\leq k_n^2 \|x_n - q\|. \quad (3.2)
$$

and hence, it follows from (2.1) and (3.2) that

$$
\|x_{n+1} - q\| = \|\alpha_n x_n - q\| + \beta_n ((PT_1)^n y_n - q) + \gamma_n ((PT_2)^n y_n - q)\|
\leq \alpha_n \|x_n - q\| + \beta_n k_n^3 \|x_n - q\| + \gamma_n k_n \|x_n - q\|
\leq k_n \|x_n - q\| = (1 + \delta_n) \|x_n - q\|. \quad (3.3)
$$

where $\delta_n = k_n^3 - 1$ satisfying $\sum_{n=1}^{\infty} \delta_n < \infty$, since $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ is equivalent to $\sum_{n=1}^{\infty} (k_n^3 - 1) < \infty$. Thus, by (3.3) and Lemma 2.1, we obtain that $\lim_{n \to \infty} \|x_n - q\|$ exists for each $q \in F$.

(2) This conclusion can be easily shown by taking infimum in (3.3) for all $q \in F$.

(3) Assume, by the conclusion of (1), $\lim_{n \to \infty} \|x_n - q\| = d$, and from (2.1), (3.2), and Lemma 2.2, we have

$$
\|x_{n+1} - q\|^2 = \|\alpha_n x_n - q\|^2 + \beta_n ((PT_1)^n y_n - q) + \gamma_n ((PT_2)^n y_n - q)\|^2
$$
\begin{equation}
\leq \alpha_{n1}\|x_n - q\|^2 + \beta_{n1}(PT_1)^n y_n - q\|^2 + \gamma_{n1}(PT_2)^n y_n - q\|^2
\end{equation}

\begin{equation}
-\alpha_{n1}\beta_{n1}g_1(\|x_n - (PT_1)^n y_n\|)
\end{equation}

\begin{equation}
\leq \alpha_{n1}\|x_n - q\|^2 + (\beta_{n1} + \gamma_{n1})k_n^2\|y_n - q\|^2 - a^2g_1(\|x_n - (PT_1)^n y_n\|)
\end{equation}

\begin{equation}
\leq (\alpha_{n1} + (\beta_{n1} + \gamma_{n1})k_n^4)\|x_n - q\|^2 - a^2g_1(\|x_n - (PT_1)^n y_n\|)
\end{equation}

\begin{equation}
\leq k_n^4\|x_n - q\|^2 - a^2g_1(\|x_n - (PT_1)^n y_n\|),
\end{equation}

which implies that \(g_1(\|x_n - (PT_1)^n y_n\|) \to 0\) as \(n \to \infty\). Since \(g_1 : [0, \infty) \to [0, \infty)\) with \(g_1(0) = 0\) is a continuous strictly increasing convex function, it follows that

\begin{equation}
\lim_{n \to \infty} \|x_n - (PT_1)^n y_n\| = 0. \quad (3.4)
\end{equation}

Similarly, we have

\begin{equation}
\lim_{n \to \infty} \|x_n - (PT_2)^n y_n\| = 0. \quad (3.5)
\end{equation}

Noting that

\begin{equation}
\|x_n - q\| \leq \|x_n - (PT_1)^n y_n\| + (PT_1)^n y_n - q\|
\end{equation}

\begin{equation}
\leq \|x_n - (PT_1)^n y_n\| + k_n\|y_n - q\|
\end{equation}

we obtain from (3.4) that, by taking \(\liminf\) on both sides in the inequality above,

\begin{equation}
d = \liminf_{n \to \infty} \|x_n - q\| \leq \liminf_{n \to \infty} k_n\|y_n - q\| = \liminf_{n \to \infty} \|y_n - q\|.
\end{equation}

In addition, it follows from (3.2) that \(\limsup_{n \to \infty} \|y_n - q\| \leq d\), thus

\begin{equation}
\lim_{n \to \infty} \|y_n - q\| = d. \quad (3.6)
\end{equation}

Hence, by (2.1), (3.1), (3.6) and Lemma 2.2, we have

\begin{equation}
\|y_n - q\|^2 = \|\alpha_{n2}(x_n - q) + \beta_{n2}(PT_1)^n z_n - q\|^2 + \gamma_{n2}(PT_2)^n z_n - q\|^2
\end{equation}

\begin{equation}
-\alpha_{n2}\beta_{n2}g_2(\|x_n - (PT_1)^n z_n\|)
\end{equation}

\begin{equation}
\leq \alpha_{n2}\|x_n - q\|^2 + \beta_{n2}(PT_1)^n z_n - q\|^2 + \gamma_{n2}(PT_2)^n z_n - q\|^2
\end{equation}

\begin{equation}
-\alpha_{n2}\beta_{n2}g_2(\|x_n - (PT_1)^n z_n\|)
\end{equation}

\begin{equation}
\leq \alpha_{n2}\|x_n - q\|^2 + (\beta_{n2} + \gamma_{n2})k_n^2\|z_n - q\|^2 - a^2g_2(\|x_n - (PT_1)^n z_n\|)
\end{equation}

\begin{equation}
\leq (\alpha_{n2} + (\beta_{n2} + \gamma_{n2})k_n^3)\|x_n - q\|^2 - a^2g_2(\|x_n - (PT_1)^n z_n\|)
\end{equation}

\begin{equation}
\leq k_n^3\|x_n - q\|^2 - a^2g_2(\|x_n - (PT_1)^n z_n\|),
\end{equation}

which implies that \(g_2(\|x_n - (PT_1)^n z_n\|) \to 0\) as \(n \to \infty\), where \(g_2 : [0, \infty) \to [0, \infty)\) with \(g_2(0) = 0\) is a continuous strictly increasing convex function. Consequently,

\begin{equation}
\lim_{n \to \infty} \|x_n - (PT_1)^n z_n\| = 0. \quad (3.7)
\end{equation}
Similarly, we have
\[ \lim_{n \to \infty} \| x_n - (PT_2)^n z_n \| = 0. \] (3.8)

Then, since
\[ \| x_n - q \| \leq \| x_n - (PT_1)^n z_n \| + \| (PT_1)^n z_n - q \| \]
\[ \leq \| x_n - (PT_1)^n z_n \| + k_n \| z_n - q \| \]
we have from (3.7) that, by taking \( \liminf \) on both sides in the inequality above,
\[ d = \liminf_{n \to \infty} \| x_n - q \| \leq \liminf_{n \to \infty} k_n \| z_n - q \| = \liminf_{n \to \infty} \| z_n - q \|. \]

In addition, it follows from (3.1) that \( \limsup_{n \to \infty} \| z_n - q \| \leq d \), thus
\[ \lim_{n \to \infty} \| z_n - q \| = d. \] (3.9)

Next, it follows from (2.1), (3.9) and Lemma 2.2 that
\[ \| z_n - q \|^2 = \| \alpha_3(x_n - q) + \beta_3((PT_1)^n x_n - q) + \gamma_3((PT_2)^n x_n - q) \|^2 \]
\[ \leq \alpha_3 \| x_n - q \|^2 + \beta_3 \| (PT_1)^n x_n - q \|^2 + \gamma_3 \| (PT_2)^n x_n - q \|^2 \]
\[ - \alpha_3 \beta_3 g_3(\| x_n - (PT_1)^n x_n \|) \]
\[ \leq (\alpha_3 + (\beta_3 + \gamma_3)k_n^2) \| x_n - q \|^2 - a^2 g_3(\| x_n - (PT_1)^n x_n \|) \]
\[ \leq k_n^2 \| x_n - q \|^2 - a^2 g_3(\| x_n - (PT_1)^n x_n \|). \]

Thus, \( g_3(\| x_n - (PT_1)^n x_n \|) \to 0 \) as \( n \to \infty \), where \( g_3 : [0, \infty) \to [0, \infty) \) with \( g_1(0) = 0 \) is a continuous strictly increasing convex function. This implies that
\[ \lim_{n \to \infty} \| x_n - (PT_1)^n x_n \| = 0. \] (3.10)

Similarly, it can be proved that
\[ \lim_{n \to \infty} \| x_n - (PT_2)^n x_n \| = 0. \] (3.11)

Furthermore, we claim that \( \| x_{n+1} - x_n \| \to 0 \) as \( n \to \infty \). In fact, by (2.1), we have
\[ \| x_{n+1} - x_n \| = \| \beta_1((PT_1)^n y_n - x_n) + \gamma_1((PT_2)^n y_n - x_n) \|
\[ \leq \beta_1 \| (PT_1)^n y_n - x_n \| + \gamma_1 \| (PT_2)^n y_n - x_n \|. \]

Hence, it follows from (3.4) and (3.5) that
\[ \lim_{n \to \infty} \| x_{n+1} - x_n \| = 0. \] (3.12)
Since any asymptotically nonexpansive mapping with respect to $P$ must be uniformly $L$-Lipschitzian with respect to $P$, where $L = \sup_{n \geq 1} \{k_n\} \geq 1$, we have

$$
\|x_{n+1} - (PT_i)x_{n+1}\| \leq \|x_{n+1} - (PT_i)^{n+1}x_{n+1}\| + \|(PT_i)x_{n+1} - (PT_i)^{n+1}x_{n+1}\|
$$

$$
\leq \|x_{n+1} - (PT_i)^{n+1}x_{n+1}\| + L\|x_{n+1} - (PT_i)^{n}x_{n+1}\|
$$

$$
\leq \|x_{n+1} - (PT_i)^{n+1}x_{n+1}\| + L\|x_{n} - (PT_i)^{n}x_{n}\|
$$

$$
+ L(L+1)\|x_{n+1} - x_{n}\|.
$$

Consequently, by (3.10), (3.11), and (3.12), it can be obtained that

$$
\lim_{n \to \infty} \|x_{n+1} - (PT_i)x_{n+1}\| = 0 \quad (i = 1, 2).
$$

(3.13)

This completes the proof.

**Theorem 3.2.** Let $K$ be a nonempty closed convex subset of a real uniformly convex and smooth Banach space $E$ with $P$ as a sunny nonexpansive retraction. Let $T_1, T_2 : K \to E$ be two weakly inward nonself asymptotically nonexpansive mappings with respect to $P$ with two sequences $\{k_n^{(1)}\}, \{k_n^{(2)}\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty} (k_n^{(i)} - 1) < \infty (i = 1, 2)$, respectively. Suppose that $\{x_n\}$ is defined by (2.1), where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ $(i = 1, 2, 3)$ are sequences in $[a, 1-a]$ for some $a \in (0, 1)$. If $PT_1$ and $PT_2$ satisfy Condition (B) with respect to the sequence $\{x_n\}$, i.e., there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $f(d(x_n, F_1)) \leq \max_{1 \leq i \leq 2} \{\|x_n - (PT_i)x_n\|\}$ and $F_1 := F(PT_1) \cap F(PT_2) = \{x \in K : PT_1x = PT_2x = x\} \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of $T_1$ and $T_2$.

**Proof.** It follows from Lemma 2.3 that $F_1 = F$, where $F$ is the common fixed point set of $T_1$ and $T_2$. Since $PT_1$ and $PT_2$ satisfy Condition (B) with respect to the sequence $\{x_n\}$, that is to say

$$
f(d(x_n, F)) \leq \max_{1 \leq i \leq 2} \{\|x_n - (PT_i)x_n\|\}.
$$

Taking $\limsup$ as $n \to \infty$ on both sides in the inequality above, we get

$$
\lim_{n \to \infty} f(d(x_n, F)) = 0,
$$

which implies $\lim_{n \to \infty} d(x_n, F) = 0$ by the definition of the function $f$.

Now we show that $\{x_n\}$ is a Cauchy sequence. By (3.3), we may assume that $\sum_{n=1}^{\infty} \delta_n = M \geq 0$. Since $\lim_{n \to \infty} d(x_n, F) = 0$, then for any $\epsilon > 0$, there exists a positive integer $N$ such that $d(x_n, F) < \epsilon/2e^M$ for all $n \geq N$. On the
other hand, there exists a $p \in F$ such that $\|x_N - p\| = d(x_N, F) < \epsilon/2e^M$, because $d(x_N, F) = \inf_{q \in F} \|x_N - q\|$ and $F$ is closed. Thus, for any $n > N$, it follows from (3.3) that

$$
\|x_n - p\| \leq (1 + \delta_n)\|x_n - p\| \leq \prod_{i=N}^{n} (1 + \delta_i)\|x_N - p\|
$$

$$
\leq e^{\sum_{i=N}^{n}(1+\delta_i)}\|x_N - p\| \leq e^M\|x_N - p\|.
$$

Hence, for any $n, m > N$,

$$
\|x_n - x_m\| \leq \|x_n - p\| + \|x_m - p\| \leq 2e^M\|x_N - p\| < \epsilon.
$$

This implies that $\{x_n\}$ is a Cauchy sequence. Thus, there exists a $x \in K$ such that $x_n \to x$ as $n \to \infty$, since $E$ is complete. Then, $\lim_{n \to \infty} d(x_n, F) = 0$ yields that $d(x, F) = 0$. Further, it follows from the closedness of $F$ that $x \in F$. This completes the proof.

**Theorem 3.3.** Let $K$ be a nonempty closed convex subset of a uniformly convex and smooth Banach space $E$ satisfying Opial’s condition with $P$ as a sunny nonexpansive retraction. Let $T_1, T_2 : K \to E$ be two weakly inward nonself asymptotically nonexpansive mappings with respect to $P$ with two sequences $\{k^{(1)}_n\}, \{k^{(2)}_n\} \subset [1, \infty)$ satisfying $\sum_{i=1}^{\infty} (k^{(i)}_n - 1) < \infty (i = 1, 2)$, respectively. Suppose that $\{x_n\}$ is defined by (2.1), where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\} (i = 1, 2, 3)$ are sequences in $[a, 1-a]$ for some $a \in (0, 1)$. If $F := F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ converges weakly to some common fixed point of $T_1$ and $T_2$.

**Proof.** For any $q \in F$, by Lemma 3.1, we know that $\lim_{n \to \infty} \|x_n - q\|$ exists. We now prove that $\{x_n\}$ has a unique weakly subsequential limit in $F$. First of all, since $PT_1$ and $PT_2$ are self-mappings from $K$ into itself, therefore, Lemmas 2.3, 2.4, and 3.1 guarantee that each weakly subsequential limit of $\{x_n\}$ is a common fixed point of $T_1$ and $T_2$. Secondly, Opial’s condition guarantees that the weakly subsequential limit of $\{x_n\}$ is unique. Consequently, $\{x_n\}$ converges weakly to a common fixed point of $T_1$ and $T_2$. This completes the proof.

**References**


