Constructive Immersion and Invariance Stabilization for a Class of Underactuated Mechanical Systems
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Abstract: In this paper a constructive approach to the stabilization of a desired equilibrium for a class of underactuated mechanical systems via the Immersion & Invariance methodology (I&I) is proposed. The design procedure shows that cases of mechanical systems with underactuation degree greater than one are included. This work generalizes the results recently reported by the authors, where an approach to obviate the solution of the corresponding PDEs for a class of nonlinear systems was proposed. Finally, our approach is successfully applied to the inertia wheel pendulum system and an interesting connection with the Interconnection and Damping Assignment Passivity-Based Control (IDA-PBC) is revealed.

Keywords: Mechanical systems; Nonlinear systems; Stabilization; Invariance; System immersion

1. INTRODUCTION

The method of I&I for stabilization of nonlinear systems was first proposed in Astolfi & Ortega [2003] and has been recently summarized in Astolfi, Karagiannis & Ortega [2007]. In the I&I approach the desired behavior of the system to be controlled is captured by the free choice of a target dynamical system. The control objective is to find a controller which guarantees that the closed-loop system asymptotically behaves like the target system achieving asymptotic model matching. The success of this methodology is witnessed by its wide range of applications. The major result of Astolfi & Ortega [2003], that constitutes the basis of the present note, is the following theorem.

Theorem 1. Consider the system

\[ \dot{x} = f(x) + g(x)u, \]  
with state \( x \in \mathbb{R}^n \) and control \( u \in \mathbb{R}^m \), with an equilibrium point \( x_* \in \mathbb{R}^n \) to be stabilized. Let \( s \leq n \) and assume we can find mappings

\[ \alpha : \mathbb{R}^s \rightarrow \mathbb{R}^s, \quad \pi : \mathbb{R}^s \rightarrow \mathbb{R}^n, \quad c : \mathbb{R}^s \rightarrow \mathbb{R}^m, \]

\[ \phi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-s}, \quad \psi : \mathbb{R}^{n \times (n-s)} \rightarrow \mathbb{R}^m, \]

such that the following hold.

1. This paper was not presented at any IFAC meeting. Corresponding author I. Sarras.

\[ \alpha(x) = \chi(x), \]

with state \( x \in \mathbb{R}^s \) has an asymptotically stable equilibrium at \( x_* = \pi(\xi) \).

(H2) (Immersion condition) For all \( \xi \in \mathbb{R}^s \)

\[ f(\xi) + g(\xi)v(\xi) = \nabla_x \pi(\xi) \alpha(\xi). \]

(H3) (Implicit manifold) The set identity

\[ \{ x \in \mathbb{R}^n \mid \phi(x) = 0 \} \]

\[ = \{ x \in \mathbb{R}^n \mid x = \pi(\xi) \text{ for some } \xi \in \mathbb{R}^s \} \]

holds.

(H4) (Manifold attractivity and trajectory boundedness) All trajectories of the system

\[ \dot{x} = \nabla_x f(x) + g(x)\psi(x, z), \]

are bounded and satisfy

\[ \lim_{t \to \infty} z(t) = 0. \]

Then \( x_* \) is an asymptotically stable equilibrium of the closed loop system

\[ \dot{x} = f(x) + g(x)\psi(x, \phi(x)). \]

Theorem 1 lends itself to the following interpretation. Given the system (1) and the target dynamical system (2) find, if possible, a manifold \( M \), described implicitly by \( \{ x \in \mathbb{R}^n \mid \phi(x) = 0 \} \), and in parameterized form by \( \{ x \in \mathbb{R}^n \mid x = \pi(\xi), \xi \in \mathbb{R}^s \} \), which can be rendered
invariant and attractive, and such that the restriction of the closed-loop system to \( \mathcal{M} \) is described by \( \dot{\xi} = \alpha(\xi) \). Notice, however, that we do not propose to apply the control \( u = c(\pi(\xi)) \) that renders the manifold invariant, instead we design a control law \( u = \psi(x, z) \) that drives to zero the coordinate \( z \) and keeps the system trajectories bounded. Notice from (5) that \( z \), called off-the-manifold coordinate, is a measure related to the distance of the system trajectories from the manifold \( \mathcal{M} \).

In standard applications of I&I the target system is a priori defined, hence condition (H1) is automatically satisfied. Given the target system, the equation (3) of condition (H2) defines a set of PDEs in the unknown \( \pi \), where \( c \) is a free parameter. Recently, Acosta, Ortega, Astolfi \cite{acosta2008} proposed a procedure to obviate the solution of the PDEs for a class of underactuated mechanical systems including the cart–pendulum system. More specifically, it was proposed to leave \( \alpha \) as a free parameter and view the PDEs (3) as algebraic equations relating \( \alpha \) with \( \pi \) (and its partial derivatives). Then suitable expressions for \( \pi \) were selected so that the desired stability properties for the target dynamics were ensured.

The present note continues in the same spirit as Acosta, Ortega, Astolfi \cite{acosta2008} by generalizing and formalizing the results for a class of underactuated mechanical systems. However, the Hamiltonian formulation is adopted here for both model and target dynamics, providing a structured framework that reveals the suitable choice of the controller.

2. I&I FOR UNDERACTUATED MECHANICAL SYSTEMS

We consider \( n \) degree of freedom mechanical systems modeled in Hamiltonian form as
\[
(\Sigma): \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{pmatrix} \nabla_q H^T \\ \nabla_p H^T \end{pmatrix} + \begin{pmatrix} 0 \\ G(q) \end{pmatrix} u, \tag{8}
\]
where \( q \in \mathbb{R}^n, p \in \mathbb{R}^n \) are the generalized positions and momenta respectively. Further, the Hamiltonian function \( H: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is the total energy of the system and is given as
\[
H(q,p) = \frac{1}{2} p^T M^{-1}(q)p + V(q) = K(q,p) + V(q), \tag{9}
\]
where \( M = M^T > 0 \) is the mass matrix and \( K, V \) the kinetic and potential energy functions respectively.

We consider the class of underactuated mechanical systems where \( G = G(q) \in \mathbb{R}^{n \times m} \) has constant rank \( m < n \). Hence a matrix \( G^\perp \in \mathbb{R}^{(n-m) \times n} \), called a left annihilator, of rank \( n-m \) exists such that
\[
G^\perp G = 0, \quad \text{rank} \left[ \begin{array}{c} G^\perp \\ G^T \end{array} \right] = n.
\]

In order to enhance readability we repeat conditions (H1)-(H4) of Theorem 1 for the class of Hamiltonian mechanical systems that admit as target system a (lower-order) Hamiltonian system.

(H1) (Target system) The system
\[
(\Sigma_t): \begin{pmatrix} \dot{\xi}_q \\ \dot{\xi}_p \end{pmatrix} = \begin{bmatrix} 0 \\ -I_n \end{bmatrix} \begin{pmatrix} \nabla_{\xi_q} H_t^T \\ \nabla_{\xi_p} H_t^T \end{pmatrix}, \tag{10}
\]
with the Hamiltonian function \( H_t: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \)
\[
H_t(\xi_q, \xi_p) = \frac{1}{2} \xi_p^T M_t^{-1}(\xi_q) \xi_p + V_t(\xi_q)
= K_t(\xi_q, \xi_p) + V_t(\xi_q), \tag{11}
\]
and states \( \xi_q, \xi_p \in \mathbb{R}^n \), has an asymptotically stable equilibrium at \( \xi_q \in \mathbb{R}^n \times \mathbb{R}^\perp \) and \( x^* = \pi(\xi_q) \), where \( \pi \triangleq \text{col}(\xi_q, \xi_p) \) and \( x^* \triangleq \text{col}(q,p) \). Moreover, \( M_t = M_t^T > 0 \) is the target mass matrix, \( K_t(\xi_q, \xi_p) \) is the target damping matrix and \( K_t, V_t \) the target kinetic and potential energy functions respectively.

(H2) (Immersion condition) For all \( \xi \in \mathbb{R}^n \),
\[
[\nabla x^T H^T + G \ c(x)]|_{x=\pi(\xi)} = \nabla \pi(\xi)(J_t - M]\nabla \pi H_t^T \tag{12}
\]
with
\[
J_t \triangleq \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \ G_t \triangleq \begin{bmatrix} 0 \\ G(q) \end{bmatrix}, \ J_r(\xi) \triangleq \begin{bmatrix} 0 \\ R_t(\xi) \end{bmatrix},
\]

(H3) (Implicit manifold) The set identity
\[
\{ x \in \mathbb{R}^n \times \mathbb{R}^n \mid \phi(x) = 0 \}
= \{ x \in \mathbb{R}^n \times \mathbb{R}^n \mid x = \pi(\xi) \text{ for some } \xi \in \mathbb{R}^n \times \mathbb{R}^n \}\tag{13}
\]
holds.

(H4) (Manifold attractivity and trajectory boundedness) All trajectories of the system
\[
\dot{z} = \nabla_x \phi(x) \left[ \nabla x^T H^T + G \ c(x) \right], \tag{14}
\]
\[
\dot{x} = \nabla x^T H^T + G \ c(x), \tag{15}
\]
are bounded and satisfy
\[
\lim_{t \to \infty} z(t) = 0. \tag{16}
\]

Now, that we have put the framework for the I&I stabilization in the Hamiltonian case we will devote the next section in showing that for the class of mechanical systems that can be rendered, by feedback or coordinate change, to have a constant block diagonal inertia matrix \( M^2 \), the set of partial differential equations (PDEs) arising from the immersion condition (12) can be brought in a convenient form that reveals a particular set of parameterized solutions. Then we will propose a general form of the control law that achieves stabilization of the desired equilibrium \( x^* = (q_*, 0) \).

3. CONSTRUCTIVE PROCEDURE

In order to show how the proposed I&I approach becomes constructive let’s consider first the following partitions
\[
q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \ p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \ G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \ M^{-1} = \begin{bmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{bmatrix}
\]
with constant \( M_{11} \in \mathbb{R}^{s \times s}, M_{22} \in \mathbb{R}^{(n-s) \times (n-s)} \) and assume that \( V = V(q_1) \) that yields the Hamiltonian equations of motion as
\[
(\Sigma): \begin{cases} \dot{q}_1 &= M_{11} p_1 \\ \dot{q}_2 &= M_{22} p_1 \\ \dot{p}_1 &= -\nabla q_1^T V + G_1 u \\ \dot{p}_2 &= G_2 u, \end{cases} \tag{17}
\]
\[
2 \text{ In Sarras (2010) we provide sufficient conditions for a mechanical system to keep its mechanical structure after applying partial feedback linearization. This class of mechanical systems can be also treated by the proposed approach under some conditions.}
\]
with $G_1 \in \mathbb{R}^{s \times m}$ and $G_2 \in \mathbb{R}^{(n-s) \times m}$. Throughout
we make the following dimensionality and invertibility
assumptions 3:

**Assumption A.1.** $s = \frac{n}{2}$ and $m = s$.

**Assumption A.2.** $G_i = G_i(q_1)$

$$\det(G_i(q_1, l)) \neq 0, \quad i = 1, 2.$$  

**Remark 1.** The first part of Assumption A.1 simply says that we restrict our attention to mechanical systems with an underactuation degree equal to half of the dimension of the configuration space which is even dimensional. The second part indicates that we should choose the target system’s dimension of the configuration space to equal the number of available actuators.

Moreover, let’s consider the following choice of the immersion $\pi$ as

$$x = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} = \pi(\xi) = \begin{pmatrix} \pi_1(\xi) \\ \pi_2(\xi) \end{pmatrix} = \begin{pmatrix} \xi_q \\ \xi_p \end{pmatrix},$$

with $\pi_{1, 2}, \pi_{22} : \mathbb{R}^s \to \mathbb{R}^s$. Now, from the PDEs of (12) we have that

$$\begin{align*}
M_{11} \xi_p &= M_{11}^{-1}(\xi_q) \xi_p, \\
M_{22} \pi_{22}(\xi) &= \nabla_{\xi_2} \pi_{12} 1 M_{11} \xi_p - \nabla_{\xi_2} \pi_{12} \nabla_{\xi_2} V^T_t + R_t(\xi) M_{11} \xi_p, \\
G(\pi_1(\xi)) c(\pi(\xi)) &= \nabla_{\xi_2} V^T_t \bigg|_{x=\pi(\xi)} - \nabla_{\xi_2} \pi_2 \nabla_{\xi_2} V^T_t + \nabla_{\xi_2} \pi_2 R_t(\xi) M_{11} \xi_p. 
\end{align*}$$

From the first one we have that $M_{22}^{-1} = M_{11}$, while from the second one we have the expression of $\pi_{22}$ as a function of $\pi_{12}$. The third multiplication by $G^1(q_1) = [I_s - G_1 G_2^1]$ and evaluated on the manifold, yields

$$G^1(q_1) \left\{ \nabla_{\xi_2} V^T_t \bigg|_{x=\pi(\xi)} - \nabla_{\xi_2} \pi_2 \nabla_{\xi_2} V^T_t \right\} = G^1(q_1) \left\{ \nabla_{\xi_2} \pi_2 R_t(\xi) - \nabla_{\xi_2} \pi_2 \right\} M_{11} \xi_p.$$  

Then, we define the mapping $\Delta : \mathbb{R}^s \to \mathbb{R}^{s \times s}$

$$\Delta(\xi) \triangleq G^1(q_1) \nabla_{\xi_2} \pi_2 - I_s - G_1 G_2^{-1} \nabla_{\xi_2} \pi_2,$$  

and make the following assumption to ensure the inverse around the equilibrium $\xi_*$ of the matrix $\Delta(\xi)$

**Assumption A.3.** $\det(\Delta(\xi_*)) \neq 0$.

Under this assumption we have that one particular set of solutions of the PDEs is given when we take

$$\begin{align*}
G^1(q_1) \left\{ \nabla_{\xi_2} V^T_t \bigg|_{x=\pi(\xi)} - \nabla_{\xi_2} \pi_2 \nabla_{\xi_2} V^T_t \right\} &= 0, \\
G^1(q_1) \left\{ \nabla_{\xi_2} \pi_2 R_t(\xi) - \nabla_{\xi_2} \pi_2 \right\} &= 0, 
\end{align*}$$

and it follows that the parameterized solutions are given by

$$\begin{align*}
R_t(\xi) &= -\Delta^{-1}(\xi) G_1 G_2^{-1}(\pi_1(\xi)) \nabla_{\xi_2} \pi_{22} \\
\nabla_{\xi_2} V^T_t &= -\Delta^{-1}(\xi) \nabla_{\xi_2} V^T_t \bigg|_{x=\pi(\xi)}. 
\end{align*}$$

**Remark 2.** In the case where the inertia matrices $M, M_t$ are not constant, then we would have to take into account the kinetic energies that would lead to the following parametrized solution for $K_t$:

$$\nabla_{\xi_2} K^T_t = -\Delta^{-1}(\xi) G_1 G_2^{-1}(\pi_1(\xi)) \nabla_{\xi_2} V^T_t \bigg|_{x=\pi(\xi)}$$

This, of course, imposes a restriction on the possible choices of $\pi$’s.

In order to ensure the asymptotic stability of the equilibrium $\xi_* = (\xi_{q*}, 0)$ for the target system we impose the following assumptions:

**Assumption A.4.**

(i) The potential energy function $V_t(\xi_*)$, satisfies $V_t'(\xi_{q*}) = 0$ and $V_t''(\xi_{q*}) > 0$.

(ii) The damping function $R_t(\xi_*, \xi_*)$ is such that $R_t(\xi_{q*}, 0) > 0$.

Thus, an appropriate choice of the mapping $\pi_{12}$, since $\pi_{22}$ is given in (19) as a function of $\pi_{12}$, will guarantee the desired stability requirements. Moreover, we observe that since $V_t$ should be a function only of $\xi_q$ so should $\Delta$ which from (21) imposes that

$$\pi_{22}(\xi_2) = \phi_1(\xi_q) \xi_p + \phi_2(\xi_q),$$

with $\phi_1 : \mathbb{R}^s \to \mathbb{R}^{s \times s}$ and $\phi_2 : \mathbb{R}^s \to \mathbb{R}^s$. Moreover, from (19) we have also that

$$\pi_{12}(\xi) = \phi_3(\xi_q) \xi_p + \phi(\xi_q),$$

with $\phi_3 : \mathbb{R}^s \to \mathbb{R}^{s \times s}$ and $\phi : \mathbb{R}^s \to \mathbb{R}^s$. The new mappings defined, $\phi_i$, $i = 1, ..., 4$, have to satisfy on the one hand, the equilibrium condition of the target dynamics, stated in (H1) of Theorem 1, yielding $\phi_3(\xi_q) = \phi_4(\xi_q) = 0$; and on the other hand, the relations

$$\begin{align*}
\phi_1(\xi_q) - \nabla_{\xi_2} \phi_4 &= -\phi_3 R_t(\xi_q), \\
\phi_2(\xi_q) - \nabla_{\xi_2} \phi_3 &= 0,
\end{align*}$$

where we have introduced a splitting of the free matrix $R_t(\xi_q)$ similar to the splitting of the $J_2$-term in Gómez-Estern & Van der Schaft [2004] as

$$R_t(\xi_q, \xi_p) = R_t^1(\xi_q) + R_t^2(\xi_q, \xi_p),$$

and considered for simplicity that $M_{11} = M_{22} = I_s$.

Recapitulating, we have that

$$\Delta(\xi) = I_s - G_1 G_2^{-1} \phi_1,$$

$$\nabla_{\xi_2} V^T_t = -\Delta^{-1}(\xi) \nabla_{\xi_2} V^T_t \bigg|_{x=\pi(\xi)} R_t(\xi_q, \xi_p) = -\Delta^{-1}(\xi) G_1 G_2^{-1} \nabla_{\xi_2} \phi_2 - \Delta^{-1}(\xi) G_1 G_2^{-1} \nabla_{\xi_2} \phi_3(\xi_q).$$

Now, in order to ensure attractivity of the manifold we define the off-the-manifold coordinates as

$$\begin{align*}
\tilde{z}_1 &= p_2 - \nabla_{\xi_2} \phi_2 p_2 - \nabla_{\xi_2} \phi_3 \nabla_{\xi_2} V^T + \nabla_{\xi_2} \phi_3 G_2 \psi, \\
\tilde{z}_2 &= -\nabla_{\xi_2} \phi_2 p_2 + \nabla_{\xi_2} \phi_3 \nabla_{\xi_2} V^T + (G_2 - \nabla_{\xi_2} \phi_3 G_2) \psi,
\end{align*}$$

and calculating the $z$-dynamics yields

$$\begin{align*}
\tilde{z}_1(t) &= p_2 - \nabla_{\xi_2} \phi_2 p_2 + \nabla_{\xi_2} \phi_3 \nabla_{\xi_2} V^T + \nabla_{\xi_2} \phi_3 G_2 \psi, \\
\tilde{z}_2(t) &= -\nabla_{\xi_2} \phi_2 p_2 + \nabla_{\xi_2} \phi_3 \nabla_{\xi_2} V^T + (G_2 - \nabla_{\xi_2} \phi_3 G_2) \psi
\end{align*}$$

Using the expressions given in (23), (19) and (22) leads to

$$\begin{align*}
\tilde{z}_1 &= p_2 - \nabla_{\xi_2} \phi_2 p_1 + \nabla_{\xi_2} \phi_3 \nabla_{\xi_2} V^T + \nabla_{\xi_2} \phi_3 G_2 \psi, \\
\tilde{z}_2 &= -\nabla_{\xi_2} \phi_2 p_1 + \nabla_{\xi_2} \phi_3 \nabla_{\xi_2} V^T + (G_2 - \nabla_{\xi_2} \phi_3 G_2) \psi.
\end{align*}$$

(29)
and
\[
\dot{z}_2 = -(\nabla_{q_1}(\phi_3 p_1) + \nabla_{q_1} \phi_2) p_1 + \phi_1 \nabla_{q_1} V + (G_2 - \phi_1 G_1) \psi = G_2 G_1^{-1} \Delta R_{p1} p_1 + \phi_1 \Delta \nabla_{q_1} V_1 + (G_2 - \phi_1 G_1) \psi.
\]

(30)

Now, we select the control law
\[
\psi(x, z) = -G_1^{-1} \{ G_1 G_2^{-1} \phi_1 \nabla_{q_1} V_1 + R_{p1} p_1 - \phi_3^{-1}(A z_1 + B z_2) \},
\]
which is well-defined under the assumption

**Assumption A.5.** The matrix \(\phi_3(q_1)\) is bounded and, invertible i.e. \(\text{det}(\phi_3(q_1)) \neq 0\).

Additionally, the free constant matrices \(A, B \in \mathbb{R}^{s \times s}\) satisfy

**Assumption A.6.**

- \(A \geq \epsilon_1 I_s\) for some constant \(\epsilon_1 > 0\)
- \(\Delta B \leq -\epsilon_2 I_s\) for some constant \(\epsilon_2 > 0\).

Thus, the closed loop z-system dynamics become

\[
\begin{align*}
\dot{z}_1 &= -A z_1 - (B - I_s) z_2 \\
\dot{z}_2 &= \Delta(A z_1 + B z_2),
\end{align*}
\]

(32)

where we defined \(\Delta(q_1) \triangleq G_2 G_1 \Delta \phi_3^{-1}\) which is bounded. The following proposition states our main result.

**Proposition 1.** For any functions \(\phi_i, i = 1, \ldots, 4\), satisfying (25) while verifying Assumptions A.3, A.5, A.6 and such that Assumption A.4 holds for any functions \(V_i, R_i\) given in (26) and for appropriate positive constants \(\epsilon_1, \gamma_1, \gamma_2, k_1, k_2, k_3\), the equilibrium \(x^* = (q^*, 0)\) of the mechanical system (17) with the I&I controller (31) is (locally) asymptotically stable.

**Proof.** Consider the Lyapunov function candidate \(V_\psi = \frac{1}{2} \gamma_1 |z_1|^2 + \frac{1}{2} \gamma_2 |z_2|^2\), with constants \(\gamma_1, \gamma_2 > 0\). Taking its derivative along trajectories of (32) yields

\[
\begin{align*}
\dot{V}_\psi &= -\gamma_1 z_1^T A z_1 + \gamma_2 z_2^T \Delta B z_2 \\
&\quad + z_1^T \left( \gamma_1 (I_s - B) + \gamma_2 A^T \Delta^T \right) z_2 \\
&\quad \leq -\epsilon_1 \gamma_1 - \epsilon_3 |P|^2 \frac{1}{2} |z_1|^2 - (\epsilon_2 \gamma_2 - \frac{1}{\epsilon_3}) |z_2|^2 \\
&\quad \leq -\epsilon_4 V_\psi,
\end{align*}
\]

(33)

for some positive constants \(\gamma_1, \gamma_2, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\) and where we used Young’s inequality for the cross term in the first equality as well as Assumptions A.5-A.6. Thus, we conclude exponential convergence to the manifold \(x = \pi(\xi)\). In order to conclude asymptotic stability of the zero equilibrium for the extended system we examine the subsystem

\[
\begin{align*}
\dot{q}_1 &= p_1 \\
\dot{p}_1 &= -\nabla_{q_1} V_\psi - R_{q_1}(q_1, p_1) p_1 + \phi_3^{-1}(q_1) (A z_1 + B z_2) \\
\dot{z}_1 &= -A z_1 - (B - I_s) z_2 \\
\dot{z}_2 &= \Delta(A z_1 + B z_2).
\end{align*}
\]

(34)

Taking the derivative of the candidate Lyapunov function \(W = V_\psi + \frac{1}{2} |p_1|^2 + V_1(q_1)\), along trajectories of (34) yields

\[
\dot{W} = -p_1^T R_{p1} p_1 + p_1^T \phi_3^{-1} A z_1 + p_1^T \phi_3^{-1} B z_2 - \gamma_2 z_1^T A z_1 \\
- \gamma_3 z_1^T (B - I_s) z_2 + \gamma_2 z_2^T \Delta A z_1 + \gamma_2 z_2^T \Delta B z_2 \\
\leq -(k_1 - \frac{k_2}{2}) |A^T \phi_3^{-T}|^2 - \frac{k_3}{2} |B^T \phi_3^{-T}|^2 |p_1|^2 \\
- (\epsilon_1 \gamma_1 - \epsilon_3 |P|^2 - \frac{1}{2 k_2} |z_1|^2 \\
- (\epsilon_2 \gamma_2 - \frac{1}{2 k_3} |z_2|^2
\]

(35)

for appropriate choices of the positive constants \(k_1, k_2, k_3, \gamma_1, \gamma_2\) and invoking (ii) of Assumption A.4. Note that \(A, B\) can be chosen such that \(k_2 |A^T \phi_3^{-T}|^2 + k_3 |B^T \phi_3^{-T}|^2 < 2 k_1\). Also, by picking \(\gamma_1, \gamma_2\) large enough ensures the resulting inequality. Finally, by (27) we have that

\[
\begin{align*}
q_2 &= z_1 + \phi_2(q_1) p_1 + \phi_4(q_1) \\
p_2 &= z_2 + \phi_5(q_1) p_1 + \phi_6(q_1).
\end{align*}
\]

(36)

Under Assumption A.5 and since \(z_1, z_2\) converge exponentially to zero while \(q_1, p_1\) converge asymptotically to \((q_1, 0)\), we conclude asymptotic stability of the equilibrium \(x^*\).

**Remark 3.** The unique restriction on underactuation degree is \(m = \frac{s}{2}\).

4. PHYSICAL EXAMPLE. INERTIA WHEEL PENDULUM

![Inertia wheel pendulum](image_url)

Fig. 1. Inertia wheel pendulum.

The example that we consider is the inertia wheel pendulum. The dynamic equations of the system, after a simple change of coordinates and scaling Ortega, Spong, Gómez-Erran & Blankenstein [2002], can be written as (8) with

\[
M = I_s, V(q_1) = m_4 (\cos q_1 - 1), G = \cos(-1, 1),
\]

\[
G^+ = [1 \ 1], m_3 = m_g L,
\]

where \(q_1 = \theta_1, q_2 = \theta_1 + \theta_2\) with \(\theta_1, \theta_2\) the angles of the pendulum and the disk, \(m\) the pendulum mass and \(L\) its length.

We want to immerse the one-dimensional pendulum – whose potential energy and damping functions are to be designed – into the two-dimensional inertia wheel pendulum. Thus, from (22) we have that

\[
\begin{align*}
R_t(\xi) &= \underbrace{\nabla_{\xi} \phi_2}_{R_t(\xi)} + \underbrace{\nabla_{\xi} \phi_4}_{\Delta(\xi)} \\
\nabla_{\xi} V_\psi^T &= -m_4 \sin \xi \underbrace{\Delta(\xi)}_{\Delta(\xi)} \\
\Delta(\xi) &= 1 + \phi_1(\xi).
\end{align*}
\]
From (25) we finally get that
\[ \phi_2(\xi_q) = \phi_4 \frac{m_3 \sin \xi_q}{\Delta(q)} \]  
(38)
\[ \phi_1(\xi_q) - \nabla_{\xi_q} \phi_4 = -\phi_4 \frac{\nabla_{\xi_q} \phi_2}{\Delta(q)} \]  
(39)
\[ \nabla_{\xi_q} \phi_3 = \phi_1 \frac{\nabla_{\xi_q} \phi_3}{\Delta(q)} \]  
(40)

By using (38) to eliminate \( \phi_2 \) in (39) we finally get the following ODEs to be satisfied:
\[ \phi_1(\xi_q) - \nabla_{\xi_q} \phi_4 = -\phi_4 \frac{m_3 \sin \xi_q}{\Delta(q)} \]  
(41)
\[ \nabla_{\xi_q} \phi_3 = \phi_1 \frac{\nabla_{\xi_q} \phi_3}{\Delta(q)} \]  
(42)

Leaving \( \phi_1 \) free, we get the following form for \( \phi_2, \phi_3 \) and \( \phi_4 \):
\[ \phi_3(\xi_q) = c_2 m_3 \sin \xi_q \]  
(43)
\[ \phi_4(\xi_q) = c_2(1 + \phi_1(\xi_q)) = c_2 \Delta(\xi_q) \]  
(44)
\[ \phi_4(\xi_q) = \int_{\xi_q}^{\xi_q'} (\phi_1(\mu) + c_2 \cos \mu, d\mu + c_1) \]  
(45)

with \( c_1, c_2 \) free constants to be chosen appropriately. It is immediate that the control law (31) of Proposition 1 with \( A = -\lambda_1 \) and \( B = \lambda_2 - \lambda_1 c_2 \) with \( \lambda_1, \lambda_2 \) such that the characteristic polynomial
\[ \Gamma(\sigma) = \sigma^2 + \lambda_1 \sigma + \lambda_1 \]
has all roots with negative real part, drives exponentially all trajectories of the \( z \)-subsystem to zero. Upon evaluation on the manifold, the controller is explicitly defined as
\[ \psi(x, \phi(x)) = -\phi_1(\xi_q) V'_1(\xi_q) + R_1(\xi_q, p_1) \]  
(46)
\[ -\frac{\Delta(\xi_q) [q_2 - (\phi_3(\xi_q) p_1 + \phi_4(\xi_q))]}{\Delta(\xi_q)} \]
\[ -\frac{(\lambda_2 - \lambda_1 c_2)}{\Delta(\xi_q)} [p_2 - (\phi_1(\xi_q) p_1 + \phi_2(\xi_q))] \]

The design is completed by proposing functions \( \phi(q_1) \) verifying Assumptions A.3 - A.4. To this end we will first investigate the properties of the functions \( \phi_1, \phi_2 \) to be satisfied. From (37) we have that
\[ V'_1(0) = 0. \]
Now, we calculate the second derivative of the target potential energy with respect to \( \xi \),
\[ V''_1(\xi_q) = \frac{m_3(\cos \xi_q(1 + \phi_1(\xi_q)) + m_3 \sin \xi_q \phi'_1(\xi_q)}{\Delta'(\xi_q)} \]
and evaluate it at the equilibrium, that is
\[ V''_1(0) = -\frac{m_3}{\Delta(0)} \]
where \( \Delta(0) = 1 + \phi_1(0) \). Moreover, we evaluate the damping function at the equilibrium and we get
\[ R_1(0) = \frac{\phi_1'(0)}{\Delta'(0)} = \frac{c_2 m_3}{\Delta(0)} \]
It follows directly that the conditions to be satisfied for the stability of the equilibrium are the following:
\[ \phi_1(0) < -1 \iff \Delta(0) < 0, \quad c_2 < 0. \]  
(47)

One trivial choice consists of taking \( \phi_1 = -c \) for \( c > 0 \). For simplicity we will consider \( c = 2 \). Moreover, we can state the following.

**Proposition 2.** The I&I controller (46), with the choice \( \phi_1 = -2 \), ensures that the zero equilibrium of the closed-loop Inertia Wheel Pendulum system is almost globally asymptotically stable.

**Proof.** First, we observe that the \( z \)-dynamics take the form
\[ \dot{z} = Az, \]
with \( A \) a Hurwitz matrix, and thus, we have that \( z = 0 \) is a globally exponentially stable (GES) equilibrium. Since from Proposition 1 we have established local asymptotic stability, it only remains to examine the (target) dynamics on the manifold in order to determine the domain of attraction of the zero equilibrium. To this end, we have that for the above choice of \( \phi_1 \) the explicit expressions for \( V' \) and \( R \) are given as:
\[ V_1 = m_3(1 - \cos \xi_q) \]
\[ R_1 = -c_2 m_3 \cos \xi_q \]

Hence, the target dynamics takes the following form:
\[ (\Sigma_r) : \left\{ \begin{array}{l} \dot{\xi}_q = \xi_p \\ \dot{\xi}_p = -m_3 \sin \xi_q + c_2 m_3 \cos(\xi_q) \xi_p \end{array} \right. \]  
(48)

It can be easily observed that the target dynamics resembles the dynamics of a simple pendulum with friction\(^4\). However, the damping coefficient is not a constant but in fact changes sign periodically. Despite this we will be able to conclude almost global asymptotic stability of the zero equilibrium. To this end consider the candidate Lyapunov function
\[ W(\xi_q, \xi_p) = \frac{1}{2}(\xi_p - c_2 m_3 \sin \xi_q)^2 + m_3(1 - \cos \xi_q) \]
which has a unique global minimum at zero. By taking the derivative along the trajectories of (\( \Sigma_r \)) we get
\[ \dot{W} = c_2 m_3^2 \sin^2 \xi_q \leq 0, \]

since \( c_2 < 0 \). Now, we define the set
\[ \Omega = \{ \xi_q \in \mathbb{S}^1, \xi_p \in \mathbb{R} | \dot{W} = 0 \} = \{ \xi_q = \pm k \pi, \xi_p = 0 \}, \]
with \( k = 0, 1 \), whose largest invariant set corresponds to the equilibria of (\( \Sigma_r \)). By looking at the linearized system for each of these equilibria we have that the only stable equilibrium is the zero equilibrium. However, the system at the unstable equilibria has one eigenvalue with strictly positive real part and one eigenvalue with strictly negative real part. Associated to the latter there is a stable manifold and trajectories starting in this manifold will converge to the downward position. However, it is known that an \( k \)-dimensional invariant manifold of an \( n \)-dimensional system has Lebesgue measure zero if \( l < n \). Hence, the set of initial conditions that converges to the downward position has zero measure. By LaSalle’s invariance principle we conclude almost global asymptotic stability of the zero equilibrium.

The other equilibria that can be approached by the trajectories correspond to the number of full swings a trajectory would take before it settles at the upright position. In Fig. 2,3 we show the smooth closed-loop response for two different sets of initial conditions.

**Remark 4.** (Comparison with IDA-PBC controller)

The control law derived in Ortega, Spong, Gómez-Estern

\(^4\) Corresponds also to the dynamics of the phase-locked loop (PLL) system, see Rantzer [2001].
Some directions of future research are:

- Physical damping to be included in the design.
- Explore the connection between IDA–PBC and I&I.

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