Envelopes of generic tangential families and their stability *

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Abstract

We study tangential families, i.e. 1-parameter families of rays emanating tangentially from a curve. We give normal forms of generic tangential families under Left-Right equivalence and we show that their envelopes are smooth or have a second order self-tangency. We prove the stability of these envelopes under small tangential deformations, that is, perturbations of the family among tangential families. We give examples where these families and deformations naturally arise, proving for instance that envelopes of tangent geodesics to a curve in a Riemannian surface are generically stable under small deformations of the metric.

Keywords : Envelope theory, Left-Right equivalence, Tangential families.
2000 MSC : 14B05, 14H15, 58K25, 58K40, 58K50.

Introduction

Envelope Theory is a classical subject in Geometry, deeply related to the Theories of Caustics and Wave Fronts, that naturally arises in Geometric Optics and Singularity Theory.

The (geometric) envelope of a family of plane curves is a new curve, tangent at each point to a curve of the family. For example, the envelope of all the normal lines to a non circular ellipse is a closed curve, affine equivalent to the astroid. This curve is the singularity locus of the ellipse equidistant fronts (see figure 1).

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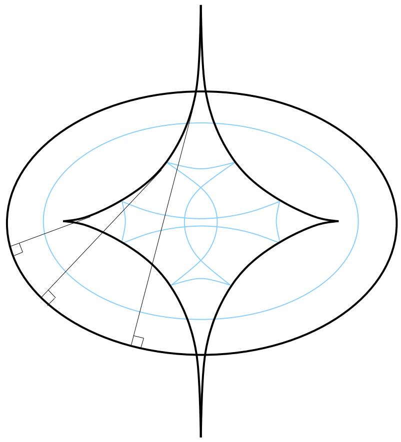

Figure 1: Envelope of normal lines to the ellipse.

Envelope Theory has a long history, which beginning may be considered Huygens' investigation of caustics of rays of light, in the second half of the XVII Century.

However, it is only with René Thom's paper [10], published in 1963, that Envelope Theory has been cleared up by Singularity Theory. Thom showed, in particular, that the only generic singularities of envelopes of 1-parameter families of plane curves are semicubic cusps and transversal self-intersections.

Normal forms for generic families of plane curves have been found by V.I. Arnold. He proved that a generic family can be reduced, near a regular point of its envelope, to one of three normal forms (listed in section 1.2 below). These normal forms are discussed in [3], Ch. I, §3. Some of them first appeared in [1]; see also [2].

In this paper we study a special class of families of curves, that we call tangential families. This case is not covered by Thom's nor Arnold's theories. A family of tangent curves to a given curve (the support of the family) is a tangential family if it can be parametrized by the tangency point of its curves with the support, as for instance a system of rays emanating tangentially from the support curve. For a first investigation of envelopes of tangential families (in the case of singular support) see [7] and [8].

Tangential families naturally arise in Geometry of Caustics (see [4]) and in Differential Geometry. For example, any curve in the Euclidean plane or in a Riemannian surface defines the tangential family of its tangent lines or tangent geodesics. In these two examples a perturbation of the Euclidean or Riemannian structure induces a deformation of the family such that deformed families are tangential. More generally, we call tangential deformation of a tangential family every deformation among tangential families.

The aim of this paper is to study generic singularities of envelopes of tangential families and to prove their stability under small tangential deformations.

This paper is divided into three parts. In the first one we present our results. Namely, we prove that there are two classes of generic tangential families, not equivalent under coordinate changes, and that their envelopes are smooth or have a second order self-tangency (see figure 2). Our main theorem states the stability of envelopes of generic tangential families under small tangential deformations.

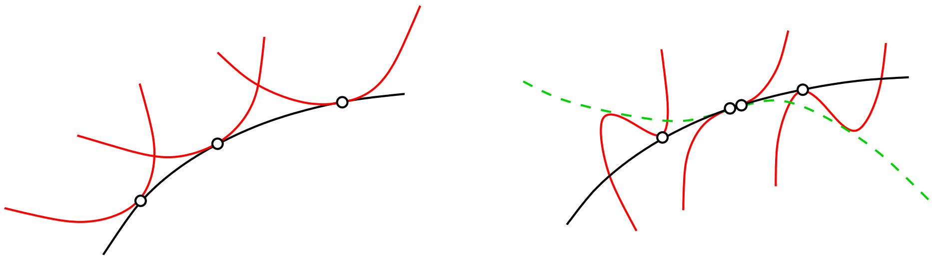

Figure 2: Two generic non equivalent tangential families.

The second part is devoted to some applications of this theory. We generalize the basic example of tangent lines to a curve replacing usual lines with "straight" lines defined by a local projective or Riemannian structure. As a consequence of our envelope stability theorem, we prove that the envelopes of these tangential families are generically stable under small perturbations of the structure.

In the last part we prove the results of the first part, using standard techniques of Singularity Theory of smooth maps, as for instance the Preparation Theorem of Mather-Malgrange-Weierstrass and the Finite Determinacy Theorem.

Acknowledgements. I wish to express my deep gratitude to V.I. Arnold for suggesting me to study this problem and for his careful reading of this paper. I also like to thank M. Garay for useful discussions and comments.

1 Presentation of results

Unless otherwise specified, all the objects considered below are supposed smooth, that is of class $\mathscr{C}^\infty$. For instance, a curve is a dimension 1 smooth submanifold of $\mathbb{R}^2$.

1.1 Tangential families and their envelopes

A map $f : \mathbb{R}^2 \to \mathbb{R}^2$ of the variables ξ, t defines the 1-parameter family of plane curves parameterized by t (indicating the point on the curve) and indexed by ξ (indicating the curve in the family); the map is a *(regular) parameterization* of the family if $\partial_t f$ does not vanish. We set $f_\xi(t) := f(\xi, t)$.

Definition. A 1-parameter family of plane curves is a *tangential family* if it admits a parameterization which partial derivatives at $(\xi, t = 0)$ are non zero parallel vectors. The curve parameterized by $\xi \mapsto f(\xi, 0)$ is the *support* of the family.

In other terms, a tangential family of support γ is a 1-parameter family of tangent curves to γ that can be parameterized by the tangency point. We denote such a family by $\{\Gamma_q : q \in \gamma\}$, where Γ_q is the tangent curve to the support at the point q.

Definition. The *graph* of a tangential family $\{\Gamma_q : q \in \gamma\}$ is the smooth surface

$$\Phi := \{(q, p) : q \in \gamma, p \in \Gamma_q\} \subset \gamma \times \mathbb{R}^2 \ .$$

Let us consider the two natural projections of the graph on γ and $\mathbb{R}^2$, $\pi_1 : (q, p) \mapsto q$ and $\pi_2 : (q, p) \mapsto p$.

Remark. The first projection π_1 is a fibration. The images by π_2 of its fibers are the curves of the family.

Definition. The *envelope in the source* of a tangential family is the critical set of the projection π_2 of its graph to the plane. The *envelope* of the family is the apparent contour of its graph in the plane (that is, the set of critical values of π_2).

Remark. The envelope of a tangential family is the critical value set of any of its parameterizations. In particular, the support of a tangential family is a branch of its envelope.

Let us end this section with some examples.

Example 1. The family of tangent lines to the parabola $y = x^2$ is parameterized by

$$f(\xi, t) := (\xi + t, \xi^2 + 2\xi t) \ .$$

The envelope in the source is $\{t = 0\}$. Therefore, the envelope of the tangent lines coincides with the support parabola.

Example 2. Let us consider in the plane $\{x, y\}$ the tangential family of support $y = 0$, defined by the graphs $y = P_\xi(x)$ of the polynomials

$$P_\xi(x) := (x - \xi)^2(x - 2\xi) \ ,$$

showed in figure 3. Its envelope in the source has two branches, $x = \xi$ and $2x = 3\xi$. The envelope of the family is the union of the x-axis and the curve of equation $x^3 + 27y = 0$.

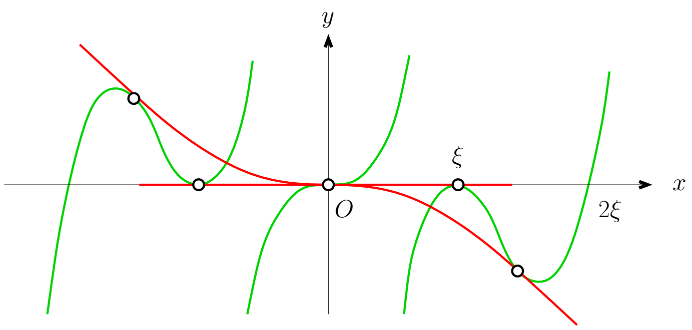

Figure 3: The tangential family of Example 2.

Given a family of plane curves, a branch of its envelope is said to be *geometric* if it is tangent at any point to a curve of the family (different from the branch itself). The envelopes of the tangential families considered in the preceding examples are all geometric.

Example 3. The envelope of the tangent lines to the cubic parabola $y = x^3$ is $\{y = x^3\} \cup \{y = 0\}$. The line $\{y = 0\}$ belongs to the family and no other line in the family is tangent to it. Hence, this branch of the envelope is not geometric.

1.2 Equivalence of tangential families

Our study of tangential families being local, we will consider their parameterizations as map germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, that is as elements of $(\mathfrak{m}_2)^2$, where $\mathfrak{m}_2$ is the space of function germs in two variables vanishing at the origin.

Definition. We will denote by X_0 the subset of $(\mathfrak{m}_2)^2$ formed by all the map germs parameterizing a tangential family: $f \in (\mathfrak{m}_2)^2$ belongs to X_0 if and only if $\partial_\xi \bar{f}(\xi, 0)$ and $\partial_t \bar{f}(\xi, 0)$ are non zero parallel vectors for every map $\bar{f} : \mathbb{R}^2 \to \mathbb{R}^2$ which germ at $(0, 0)$ is f, provided that ξ is small enough.

In this definition, the two parameters ξ and t play different roles: ξ is the parameter on the base of the family and t is the parameter on each fiber of the family. However, to study envelopes of tangential families we do not need to distinguish the parameters. In this case we will use the notation (s, t) instead of (ξ, t) for the parameters, keeping the letter ξ for the variable on the support.

Denote by $\mathrm{Diff}(\mathbb{R}^2, 0)$ the group of diffeomorphism germs of the plane keeping fixed the origin, and by $\mathscr{A}$ the direct product $\mathrm{Diff}(\mathbb{R}^2, 0) \times \mathrm{Diff}(\mathbb{R}^2, 0)$. Consider the subgroup $\mathscr{A}_0$ of $\mathscr{A}$, formed by all the diffeomorphism germs $(\varphi, \psi) \in \mathscr{A}$ such that φ is fibered with respect to the first variable, that is of the form $(\xi, t) \mapsto (\eta(\xi), u(\xi, t))$. Then $\mathscr{A}_0$ and $\mathscr{A}$ act on $(\mathfrak{m}_2)^2$ by the rule

$$(\varphi, \psi) \cdot f := \psi \circ f \circ \varphi^{-1}$$

(this means changing the coordinate systems in the source and in the target of the map germ).

Definition. Two map germs $f, g \in (\mathfrak{m}_2)^2$ are *Left-Right equivalent*, or $\mathscr{A}$-*equivalent*, if they belong to the same $\mathscr{A}$-orbit, that is if there exists a commutative diagram

$$\begin{array}{ccc} (\mathbb{R}^2, 0) & \xrightarrow{f} & (\mathbb{R}^2, 0) \\ \downarrow & & \downarrow \\ (\mathbb{R}^2, 0) & \xrightarrow{g} & (\mathbb{R}^2, 0) \end{array}$$

in which the vertical arrows are diffeomorphism germs. Similarly, f and g are $\mathscr{A}_0$-*equivalent* if g belongs to the $\mathscr{A}_0$-orbit $\mathscr{A}_0 \cdot f$ of f.

Remark. X_0 is $\mathscr{A}_0$-stable but not $\mathscr{A}$-stable: $\mathscr{A}_0 \cdot X_0 = X_0$, $\mathscr{A} \cdot X_0 \neq X_0$. Indeed, for every $f \in X_0$, the orbit $\mathscr{A} \cdot f$ in X contains map germs not defining tangential families: $\mathscr{A} \cdot f \not\subset X_0$.

Example 4. The map germ parameterizing the family of tangent lines to the parabola, considered in Example 1, is $\mathscr{A}$-equivalent to (ξ, t^2). This germ does not parameterize a tangential family.

This is the reason why in order to classify tangential families it is natural to consider the $\mathscr{A}_0$-equivalence instead of $\mathscr{A}$-equivalence.

Remark. V.I. Arnold showed (see [1], [2] and [3]) that every generic 1-parameter family of plane curves, near a regular point of its envelope, is $\mathscr{A}_0$-equivalent to one of the following three normal forms:

$$\text{(a)} \quad \big(\xi + t, t^2\big) \ , \quad \text{(b)} \quad (\xi + \xi t + t^3, t^2) \ , \quad \text{(c)} \quad \big(t, (\xi + t^2)^2\big) \ .$$

The families parameterized by these normal forms in the plane $\{x, y\}$ are depicted in figure 4. The envelopes of these families are the line $y = 0$. Only $\mathscr{A}_0$-equivalent families to the normal form (a) are tangential families. The converse is discussed at the end of the next section.

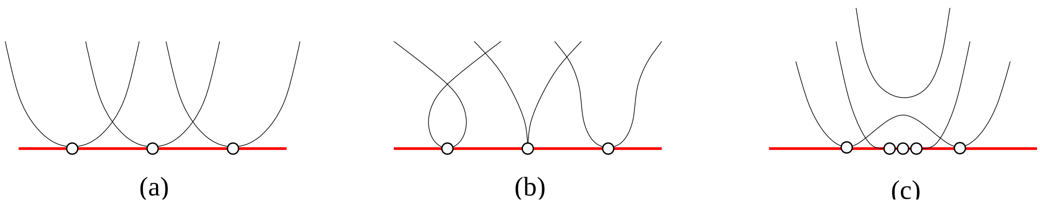

Figure 4: Generic families of curves near a regular point of the envelope.

In order to classify tangential families with respect to their envelope, we can consider $\mathscr{A}$-equivalence instead of $\mathscr{A}_0$-equivalence, because the envelope of a tangential family is diffeomorphic to the critical value set of every map germ $\mathscr{A}$-equivalent to any of its parameterizations.

Definition. We denote by $X \subset (\mathfrak{m}_2)^2$ the set of all the map germs $\mathscr{A}$-equivalent to (parameterizations of) tangential families:

$$X := \bigcup_{f \in X_0} \mathscr{A} \cdot f \ .$$

Remark. The set X is a stratified submanifold of $(\mathfrak{m}_2)^2$, stable under the action of $\mathscr{A}$: $\mathscr{A} \cdot X = X$.

The smoothness of X has to be understood as the smoothness of its restrictions to every jet space $J^N(\mathbb{R}^2, \mathbb{R}^2)$, endowed with the standard $\mathscr{C}^N$ topology. The action of $\mathscr{A}$ on $(\mathfrak{m}_2)^2$ gives rise to the quotient action of $\mathscr{A}^N$ on $J^N(\mathbb{R}^2, \mathbb{R}^2)$, where $\mathscr{A}^N$ is the group of N-jets at $(0, 0)$ of pairs of local diffeomorphisms. Since $\mathscr{A}^N$ is a Lie group et J^N is a finite dimensional space, $\mathscr{A}^N$-orbits in $J^N(\mathbb{R}^2, \mathbb{R}^2)$ are smooth.

1.3 Envelopes of generic tangential families

The action of the group $\mathscr{A}$ on the submanifold X allow us to reduce "almost any" tangential family to some simple normal form.

Definition. Given a tangential family, the fiber $\pi_2^{-1}(O)$ defines a unique characteristic direction in the tangent plane to the graph of the family at the origin, that we call the *vertical* direction.

Remark. The branch of the envelope in the source of a tangential family, which projection in the plane is the support of the family, is not vertical (see figure 5).

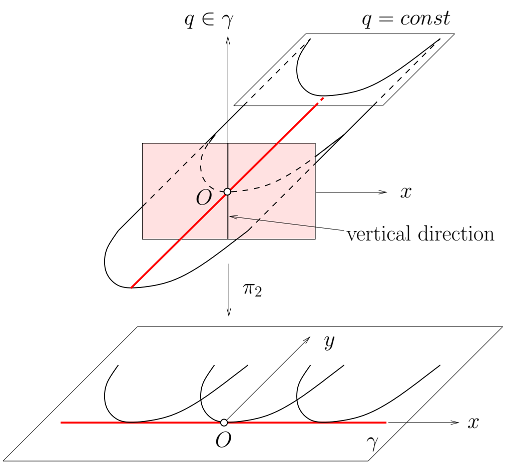

Figure 5: The vertical direction in the graph of a tangential family of support $y = 0$.

Definition. A tangential family is said to be *generic* if its envelope in the source has not more than two branches crossing at the origin and these branches are transversal and non vertical.

We point out that in the preceding definition one has to count the envelope branches with their algebraic multiplicity.

Example 5. The tangential family parameterized by the map germ

$$(\xi, t) \mapsto (\xi + t, t^4 + 4t^2\xi^2)$$

is not generic. Indeed, its envelope in the source has three branches, namely, the support $y = 0$ and the origin $(0, 0)$ counted twice, corresponding to the two complex branches of equations $t = (1 \pm i)\xi$.

The genericity conditions stated above can be applied to the elements of X replacing "envelope in the source" by "critical set". Notice that this property is stable under $\mathscr{A}$-equivalence: if $f \in X_0$ is generic, then every element of $\mathscr{A} \cdot f$ is generic.

Definition. A generic tangential family is said to be of *first type* if its envelope in the source has one branch, of *second type* otherwise.

The generic tangential family depicted in figure 5 is of first type.

Example 6. The family of tangent lines to $y = x^2$ is of first type. The family of tangent lines to $y = x^3$ and the tangential family of Example 2 are of second type.

The next statement is a geometric characterization of generic tangential families. The proof is in section 3.1.

Proposition 1. *A tangential family is of first type if and only if each curve of the family has a first order tangency with its support. If a tangential family is of second type, then the curve of the family tangent to the support at the origin has tangency order greater than* 1 *(the tangency order of the other curves being* 1*).*

Let $G \subset X$ be the set of $\mathscr{A}$-equivalent map germs to generic tangential families. This set split off into the two components G_1 and G_2 of $\mathscr{A}$-equivalent map germs to generic tangential families of first and second type respectively.

Theorem 1. *The set $X \setminus G$ is a codimension 1 stratified submanifold of X; each set G_i is an orbit under the action of $\mathscr{A}$ over X. In particular, every element of G_i is $\mathscr{A}$-equivalent to the normal form f_i, where*

$$f_1(s,t) := (s+t, t^2) \ , \quad f_2(s,t) := (s+t, st^2) \ .$$

The proof is in section 3.2. Notice that the first assertion justifies the term "generic" for the elements of G.

Theorem 1 has the following consequences.

Corollary 1. *All the tangential families of fixed type are $\mathscr{A}$-equivalent.*

Corollary 2. *Envelopes of tangential families of first type are smooth, while envelopes of tangential families of second type have a second order self-tangency.*

Let us end this section discussing the action of $\mathscr{A}_0$ on $X_0 \subset X$. Theorem 1 does not hold if we consider $\mathscr{A}_0$-equivalence instead of $\mathscr{A}$-equivalence.

Example 7. Tangential families considered in Examples 2 and 3 are $\mathscr{A}$-equivalent but not $\mathscr{A}_0$-equivalent.

However, the two normal forms in Theorem 1 are parameterizations of tangential families, taking $s = \xi$ as distinguished parameter. In particular, f_1 is one of Arnold's $\mathscr{A}_0$-normal forms discussed in section 1.2 (family (a) in figure 4). The tangential family parameterized by the normal form f_2 is showed in figure 6.

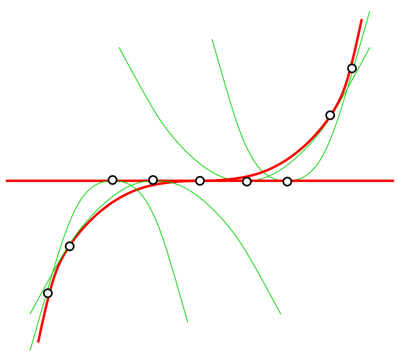

Figure 6: The tangential family parameterized by the normal form f_2.

Therefore, a 1-parameter family of plane curves is $\mathscr{A}_0$-equivalent to the normal form f_1 if and only if it is a tangential family of first type. In other words, the set $G_1 \cap X_0$ is an unique orbit for the action of $\mathscr{A}_0$ on X_0.

On the other hand, the $\mathscr{A}_0$-normal form of second type tangential families has an infinite number of moduli; actually, the set $G_2 \cap X_0$ contains an infinity of $\mathscr{A}_0$-orbits, each one being of infinite codimension.

1.4 Tangential deformations and envelope stability

We have seen in the preceding section that envelopes of generic tangential families may be smooth or singular, according to the type of the family.

In the first case the envelope is *stable*: it changes diffeomorphically under small deformations of the family.

The envelope of a second type family has a second order self-tangency. According to Thom, this envelope singularity is not stable. However, the normal form f_2 has no moduli; the singularity is therefore *simple*.

The instability of the envelope singularity means that it is possible to destroy the self-tangency by an arbitrary small perturbation of the germ parameterizing the family.

In some situations, as for instance in the case of geodesic tangential families under small perturbations of the metric, it may be permitted to perturb a tangential family only among tangential families.

Definition. A *tangential deformation* of a tangential family parameterized near the origin by $f \in X_0$ is a p-family of tangential families, parameterized by a p-family of maps

$$\{F_\lambda : \mathbb{R}^2 \to \mathbb{R}^2, \lambda \in \mathbb{R}^p\} ,$$

such that the germ at the origin of F_0 is f.

For example, the translation of the origin is a tangential deformation.

Remarks. (1) A tangential deformation induces a smooth deformation on the support of the family as branch of the envelope.
(2) The vector space of all the infinitesimal tangential deformations of a map germ of G is canonically identified to the (extended) tangent space to G in $(\mathfrak{m}_2)^2$ at this point.

We may state now our main result.

Theorem 2. *Envelopes of generic tangential families are stable under small tangential deformations.*

The stability of the envelope second order self-tangency must be understood as follows. Let $\{F_\lambda : \lambda \in \mathbb{R}^p\}$ be a tangential deformation of $f \in G_2 \cap X_0$ and let $\mathcal{U}$ be an arbitrary small neighborhood of $(0,0)$ in $\mathbb{R}^2$. Then the envelopes of the tangential families parameterized by F_λ have a second order self-tangency in $\mathcal{U}$, provided that λ is small enough.

In section 3.2 we will prove the next result.

Theorem 3. *The map* $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, *defined by*

$$F(\lambda; s, t) := (s + t, \lambda s + st^2) \ ,$$

is an $\mathscr{A}$*-miniversal deformation of the normal form* f_2 *in* $(\mathfrak{m}_2)^2$.

Theorem 2 follows from Theorem 3. Indeed, the well known stability of f_1 (which singularity is a fold) implies that any infinitesimal deformation of a map germ in G_1 is tangential.

On the other hand, the direction $(0, s)$ is transversal to G_2 at f_2. To see this, denote by F_λ the map obtained from F fixing the parameter value λ. For $\lambda < 0$ the critical set of F_λ has two smooth branches, while for $\lambda > 0$ it has two branches, having each one a semicubic cusp (see figure 7). Such a perestroika is called "bec à bec" in Thom's notation. Hence, F is not a tangential deformation. This proves the announced transversality and completes the proof of Theorem 2 modulo Theorem 3.

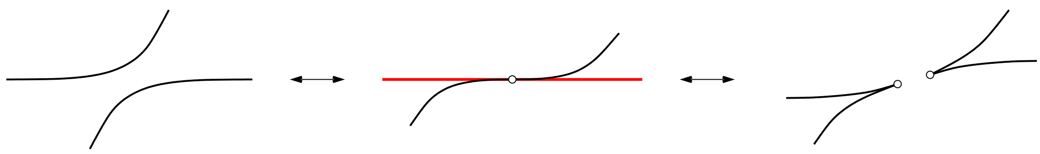

Figure 7: "Bec à bec" perestroika.

Remark. The critical sets of F_λ form a surface in the three-dimensional space $\{x, y, \lambda\}$, diffeomorphic to the (semicubic) cuspidal edge –this holds for any $\mathscr{A}$-miniversal deformation of any element of G_2. In suitable coordinates $\{u, v, w\}$, the bec à bec perestroika can be described as the metamorphose of the level set of the function $w^2 - u$ on the cuspidal edge $u^3 = v^2$ near the critical value 0 (figure 8).

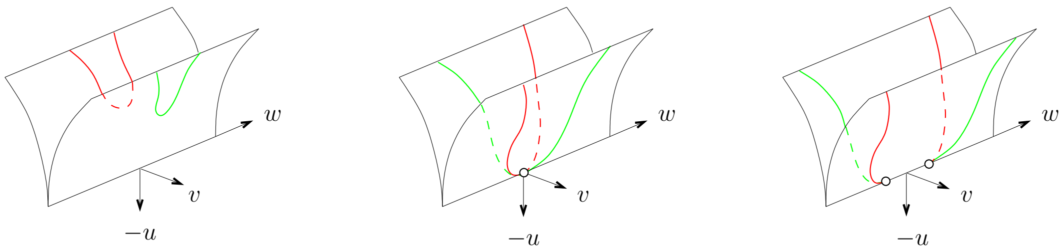

Figure 8: Level sets $w^2 - u = c$ on the cuspidal edge $u^3 = v^2$ ($c < 0$ on the left, $c = 0$ on the center and $c > 0$ on the right).

2 Examples and applications

The simplest example of tangential family is given by tangent lines to a curve. We generalize it by replacing Euclidean structure (defining usual lines) by a local projective structure or a Riemannian structure.

To apply results of section 1, we will prove that the families considered below satisfy the regularity conditions. For this, we will use the Regularity Theorem for solutions of second order differential equations with respect to initial conditions, and its variants on parameters and systems of equations. We will refer to all these statements as Regularity Theorem.

2.1 Projective tangential families

A *(local) projective structure* is the structure given, at any point of the plane near the origin, by a second order differential equation (see [3]). The graphs of their solutions are, by definition, *straight lines* for this structure. For instance, the standard projective structure in the Euclidean plane is defined by the equation $y'' = 0$. A projective structure is said to be *flat* if it is possible to reduce it to the standard projective structure by a change of local coordinates.

Remark. Let us fix a flat projective structure. The envelope of straight tangent lines to a curve having an inflection point is the union of the support curve and the straight line tangent at the inflection point. Moreover, the envelope is not geometric.

It follows from the Regularity Theorem that, given any projective structure, all the straight tangent lines to a curve define a tangential family, which support is this curve.

Corollary 3. *Given a projective structure, consider the family of straight tangent lines to a curve γ. If the support has not inflection points, then the envelope of the family is γ. If the support has an inflection point, then the envelope has two branches (one of which is γ); these branches have generically a second order tangency at the inflection point.*

Remark. This Corollary provides a characterization of non flat projective structure. Indeed, suppose that there exists a curve, having an inflection point, such that the envelope of its tangent straight lines is geometric. Then the projective structure can not to be flat.

Let us consider a family of straight tangent lines to a curve. Deformations of the projective structure induce deformations of the family. By the Regularity Theorem, these deformations are tangential deformations.

Corollary 4. *Envelopes of straight tangent lines to a plane curve for a given projective structure are generically stable under small perturbations of the structure.*

2.2 Geodesic tangential families

Let γ be a curve in a Riemannian surface. Near a point of γ, the tangent geodesics to γ form a tangential family, that we will call *geodesic tangential family*. Indeed, regularity conditions are satisfied, as follows from the Regularity Theorem.

The following is an immediately consequence of Theorem 1.

Corollary 5. *If γ has not inflection points, then the envelope of its geodesic tangential family is γ. If γ has an inflection point, then the envelope has two branches (one of which is γ), that generically have a second order tangency at the inflection point.*

Perturbations of the metric induce tangential deformations of geodesic tangential families in the surface (again by the Regularity Theorem).

Thus we get the following two consequences of Theorem 2.

Corollary 6. *Envelopes of generic geodesic tangential families in a Riemannian surface are stable under small deformation of the metric.*

Corollary 7. *Envelopes of generic geodesic tangential families on a surface in the Euclidean 3-space are stable under small perturbations of the surface.*

3 Proof of Theorems 1 and 3

3.1 Prenormal forms of tangential families

Let us start reducing every tangential family to a map germ, depending on some parameters. In order to do this, let us define as follows three real coefficients k_0, k_1 and α, depending on the family. Fix a coordinate system $\{x, y\}$ in the plane, centered at the origin, such that the support equation is $y = 0$ near the origin. Consider the curve of the family which is tangent to the support at $(\xi, 0)$. Then define k_0 and k_1 by the expansion at this point $k_0 + k_1\xi + o(\xi)$ of its curvature divided by 2. The last coefficient α is similarly defined by the expansion $k_0t^2 + \alpha t^3 + o(t^3)$ of the function which graph is the tangent curve of the family to the support at the origin.

Theorem 4. *Every tangential family is $\mathscr{A}$-equivalent to a map germ of the form*

$$(s, t) \mapsto \big(t, k_0s^2 + (\alpha - k_1)s^3 + k_1s^2t + o(3)\big) \ ,$$

where k_0, k_1 and α are the coefficients, depending on the family, defined above and $o(3)$ is a formal series in s, t of order greater than 3.

The map germ in Theorem 4 will be called a *prenormal form* of the family.

Proof. For any given tangential family, let us choose an orthogonal coordinate system as in the above definition of coefficients k_0, k_1 and α. The map $\xi \mapsto (\xi, 0)$ parameterizes the support. For any small enough value of ξ, consider the tangent curve to γ at $(\xi, 0)$. Near this point, the curve can be parameterized by its projection $\xi + s$ on the x-axis. In this manner we get a parameterization of the family, that may be written as

$$(\xi, s) \mapsto \big(s + \xi, k_0s^2 + k_1s^2\xi + \alpha s^3 + o(3)\big) \ .$$

Taking in this map s and $t := s + \xi$ as new parameters, we obtain the map germ in the statement. Theorem 4 is therefore proved. □

Genericity conditions for tangential families can be formulated in terms of coefficients k_0, k_1 and α of their prenormal forms.

Proposition 2. *A tangential family is of first type if and only if $k_0 \neq 0$; it is of second type if and only if $k_0 = 0$ and $k_1 \neq 0, \alpha$.*

Proof. Consider the prenormal form of a given tangential family. We can compute explicitly approximate solutions of its critical point equation. Some easy computations show that genericity conditions for the graphs of these solutions provide the above conditions on the coefficients. □

Taking into account the geometrical meaning of the coefficients, Proposition 1 follows from Proposition 2.

3.2 Normal forms of generic tangential families

The proof of Theorems 1 and 3 is based on the reduction of the prenormal form to the normal form and the computation of the corresponding $\mathscr{A}$-miniversal deformation. In order to do this, first we recall some facts about Singularity Theory of smooth maps. For a complete presentation of this theory, we refer the reader to [9], [6] or [5].

As before, let us denote respectively by s, t and x, y the coordinates in the source and in the target space of a map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. Such a germ defines, by the formula $f^*g := g \circ f$, a homomorphism from the ring $\mathscr{E}_{x,y}$ of function germs in the target to the ring $\mathscr{E}_{s,t}$ of function germs in the source. Hence we can consider every $\mathscr{E}_{s,t}$-module as an $\mathscr{E}_{x,y}$-module via this homomorphism.

Let $< f > \subset \mathscr{E}_{s,t}$ be the ideal generated by the components of f (for the structure of $\mathscr{E}_{s,t}$-module). Let us recall the well known Preparation Theorem of Mather-Malgrange-Weierstrass, in the case we deal with: *the module $\mathscr{E}^2_{s,t}$ is finitely generated as $\mathscr{E}_{x,y}$-module if and only if the quotient space $\mathscr{E}^2_{s,t}/\left(< f > \cdot \mathscr{E}^2_{s,t}\right)$ is a real vector space of finite dimension; moreover, a basis of the vector space provides a generator system of the module.*

The *(extended) tangent space* at f in $(\mathfrak{m}_{s,t})^2$ is the subspace of $\mathscr{E}^2_{s,t}$ defined by

$$T\mathscr{A}(f) := \mathscr{E}_{s,t} \cdot J(f) + f^*(\mathscr{E}_{x,y}) \cdot \mathbb{R}^2 \ ,$$

where $J(f)$ is the real vector space spanned by the partial derivatives of f (from now on, we will utilize the notation $\mathfrak{m}_{s,t}$ instead of $\mathfrak{m}_2$ to point out that the variables of the considered germs are s and t). The *(extended) codimension* of a map germ f is the dimension of the quotient space $\mathscr{E}^2_{s,t}/T\mathscr{A}(f)$ as real vector space. It is well known that *map germs of finite codimension are finitely determined, i.e. $\mathscr{A}$-equivalent to their Taylor's polynomial of some finite order* (Finite Determinacy Theorem).

We can now start the proof of Theorem 1. We have to show that any map germ belonging to G_i is $\mathscr{A}$-equivalent to the normal form f_i $(i = 1, 2)$. The proof is, in both cases, divided into two steps. The first one is the formal $\mathscr{A}$-equivalence between f_i and any germ in G_i (two map germs are formally $\mathscr{A}$-equivalent if they are $\mathscr{A}$-equivalent modulo arbitrary large order terms). The second step is to verify that the normal form has finite codimension. In this case formal $\mathscr{A}$-equivalence imply $\mathscr{A}$-equivalence.

Let us begin reducing map germs in G_1 to the map germ $h_1(s,t) := (t, s^2)$, which is $\mathscr{A}$-equivalent to the normal form

$$f_1(s,t) = (s + t, t^2) \ .$$

Proposition 3. *Every element of G_1 is $\mathscr{A}$-equivalent to h_1.*

Proof. Prenormal forms of first type tangential families are $\mathscr{A}$-equivalent to map germs of the form $(t, s^2 + o(2))$. Consider now such a germ $\tilde{h}_1$ that can be written as $\big(t, s^2 + P_r(s,t) + o(r)\big)$, where P_r is a homogeneous polynomial $\sum_{j=0}^{r} b_j s^j t^{r-j}$ of order $r > 2$, and define

$$\varphi(s,t) := \left(s - \frac{1}{2} \sum_{j=1}^{r} b_j s^{j-1} t^{r-j}, t \right) \ , \quad \psi(x,y) := (x, y - b_0 x^r) \ .$$

Then $\big(\psi \circ \tilde{h}_1 \circ \varphi\big)(s,t) = \big(t, s^2 + o(r)\big)$. Iterating this argument, we obtain that the initial prenormal forms are $\mathscr{A}$-equivalent to $(t, s^2 + o(r))$ for r arbitrary large. Finally, the formal $\mathscr{A}$-equivalence implies the $\mathscr{A}$-equivalence, due to the stability of the fold h_1. □

Let us now prove that all the map germs in G_2 are $\mathscr{A}$-equivalent to the map germ $h_2(s,t) := (t, s^2(s + t))$, and then to the normal form

$$f_2(s,t) = (s + t, st^2) \ .$$

Proposition 4. *Every element of G_2 is formally $\mathscr{A}$-equivalent to h_2.*

Proof. By Theorem 4 and Proposition 2, every element of G_2 is $\mathscr{A}$-equivalent to a map germ of the form

$$(t, (\alpha - k_1)s^3 + k_1 s^2 t + o(3)\big) \ , \quad k_1 \neq 0, \alpha \ .$$

which is $\mathscr{A}$-equivalent, by rescaling, to a map germ $\tilde{h}_2$ of the form

$$\big(t, s^2(s + t) + P_r(s,t) + o(r)\big) \ ,$$

where P_r is a homogeneous polynomial of order $r > 3$.

We shall now prove that every map germ of this form is $\mathscr{A}$-equivalent to a map germ $\big(t, s^2(s+t) + o(r)\big)$. This equivalence provides the formal $\mathscr{A}$-equivalence between the initial prenormal form and h_2, proving the Proposition.

We can kill the term on $s^j t^{r-j}$ in P_r for any fixed $j \in \{3, ...r\}$, changing only the term on $s^{j-1}t^{r-j+1}$ and higher order terms in the second component of $\tilde{h}_2$. This is made by the coordinate change $(s,t) \mapsto (s + cs^{j-2}t^{r-j}, t)$, for a suitable $c \in \mathbb{R}$. In a similar way, we can eliminate the term on st^{r-1}, modifying only the term on s^2t^{r-2} in the homogeneous polynomial and higher order terms. Therefore, we have reduced $\tilde{h}_2$ to a map germ $\hat{h}_2$ verifying

$$\hat{h}_2(s,t) = \big(t, s^2(s+t) + c_1 s^2 t^{r-2} + c_2 t^r + o(r)\big) \ .$$

Setting $\varphi(s,t) := (s, t - c_1 t^{r-2})$ and $\psi(x,y) := (x, y - c_2 x^r)$ we have

$$\left(\psi \circ \hat{h}_2 \circ \varphi\right)(s,t) = \big(t - c_1 t^{r-2}, s^2(s+t) + o(r)\big) \ .$$

Each germ of this type is formally $\mathscr{A}$-equivalent to a germ of the same type with $c_1 = 0$. □

Proposition 5. *The codimension of the map germ h_2 is* 1*.*

To prove this Proposition we need the following Lemma, which proof is a straight-forward computation.

Lemma 1. *All the monomials $(s^pt^q, 0)$ for $p+q \geq 0$ and $(0, s^pt^q)$ for $p+q \geq 2$ belong to the tangent space $T\mathscr{A}(h_2)$.*

Proof of Proposition 5. By the Preparation Theorem, $\mathscr{E}^2_{s,t}$ is generated, as $\mathscr{E}_{x,y}$-module, by the map germs $(1,0)$, $(0,1)$, $(s,0)$, $(0,s)$, $(s^2,0)$ and $(0,s^2)$. By Lemma 1, between these germs only $(0,s)$ does not belong to the tangent space $T\mathscr{A}(h_2)$. Now, $h_2^*(\mathfrak{m}_{x,y}) \cdot (0,s)$ is generated by $(0, st)$ and $(0, s^3(s+t))$, so it is contained in the tangent space. Therefore we obtain the equality

$$\mathscr{E}^2_{s,t} = T\mathscr{A}(h_2) \oplus \mathbb{R} \cdot \begin{pmatrix} 0 \\ s \end{pmatrix} \ ,$$

which proves the Proposition. □

The proof of Theorem 1 is completed.

The last equality implies also that $(t, \mu s + s^2(s+t))$ is an $\mathscr{A}$-miniversal deformation of the map germ h_2. Thus, the map germ

$$(\lambda; s, t) \mapsto (s+t, \lambda s + st^2)$$

is an $\mathscr{A}$-miniversal deformation of the normal form f_2. Theorem 3 is now proved.

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