An Introduction to Chaos Theory

Paul Tobin

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"Deviner avant de démontrer! Ai-je besoin de rappeler que c'est ainsi que se sont faites toutes les découvertes importantes. Guessing before proving! Need I remind you that it is so that all important discoveries have been made? The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living". Jules Henri Poincaré

1 Introduction to Chaos

Ok, I agree, this quote seems a bit pretentious but it captures my feelings about chaos theory which is one of the most exciting topics I've come across in my career! In this paper we investigate chaos theory which will support my first blog on the role of PSpice simulation. Chaos theory, a subset of 'complexity theory'. The science of complexity involves the principle of *Self-Organizing Criticality*,of which the human brain is a great example, where large neurons organize themselves to form an extremely complex connective network which can solve complex problems with a rapidity still not matched by present day computers. It has been postulated by Walter Freeman III how chaos plays an important role in brain functioning and attempts to explain how it operates as fast as it does! This short introduction to chaos theory will outline how we can use a chaotic source to produce unlimited amounts of 'cryptographic keys' for secure saving of data in Cloud computing (CC). The word chaos is from the Greek verb to gape open and normally used to mean total disarray such as seen in the aftermath of an explosion. However, we will see that chaos theory is anything but total disarray! A little history to outline how chaos theory developed by a relatively few people: The Greek philosopher, Anaxagoras [1], the great Scottish scientist, James Clerk Maxwell (1876), Hadamard (1898) [2], Poincaré in 1890 [3], [4] and Andronov (1929) [5]. The exploration of chaos (it wasn't called chaos theory yet) continued in a haphazard fashion and in 1975, Tien-Yien Li and James A. Yorke, introduced the term 'Chaos' in their paper "Period Three Implies Chaos" [6].

Chaos theory is the science of nonlinear topics and has many potential applications in engineering, physics, medicine, biology, and economics, etc. Engineers have focused on the linear aspects of engineering and largely ignored the interesting nonlinear world. Indeed, the world of mathematics and science has, until recently, ignored nonlinear system which are "unsolvable" in the traditional sense. However, to get over this we can make nonlinear systems approximately linear for small perturbations about points of equilibrium called fixed points (FP). We will use FPs to design a suitable threshold point using chaos source signals later. Of course, nonlinear design techniques were used when designing signal generators, class C power amplifiers, modulators, etc., but this was in a limited sense only and didn't consider any chaotic aspects.

Nature, by and large, is not linear and straight lines are rarely, if ever, observed in the natural World. In essence, electrical engineers have dismissed nature's evolutionary design techniques and mainly considered linear designs. Nature, through evolutionary steps, created some of the most complex structures in the known universe, of which the human brain is its finest example. Nonlinearity is a necessary condition for equations to be chaotic, such equations which cannot really be solved analytically and requires the power of personal computers to provide a solution.

King Oscar II of Sweden and Norway (1872-1907), announced a competition for a prize of 2500 kroner to celebrate his 60^{th} birthday to anyone who could prove that the Solar system was stable and would not fly apart at some future date. Jules Henri Poincaré (1854-1912), who was a brilliant mathematician and physicist [é], took on the challenge and won the prize for his paper 'sur le problème à trois corps' '*On the problem of three bodies and the equations of equilibrium*". Nonlinear system dynamics was first investigated by this brilliant French polymath who recognised chaos as an almost impossible topic to analyse mathematically.

Another French mathematician, Edvard Phragmen, pointed out serious mistakes in Poincare's orbital stability calculations and resulted in Poincaré withdrawing his book from circulation. Nevertheless, Poincaré's flawed solution created new areas of maths such as topology and the study of dynamical systems. He is considered by many to be the 'Father of Chaos Theory'. On a historical note, the three-body problem was eventually solved by the Finnish Mathematician, Karl Sundman in 1912, but the solution converged so slowly that it wasn't useful!

Poincaré also invented a new ways of thinking using pictures to visualize problems rather than rigorous mathematical analysis. He added to the concept of phase space by visualizing the trajectory of a complex dynamical system as it evolves. His work on two-dimensional transverse slices from complex three-dimensional flows is called a Poincaré section and will be examined later. This revolutionized continuous, hard-to-visualize 3-D systems into much simpler 2-D digital maps. Henri Poincaré also introduced the important concept called bifurcation in a mathematics paper in 1885 [7]. However, chaos theory was not investigated for a long time after Poincaré except for a few mathematicians such as George David Birkhoff (1884-1944) and some European and Russian mathematicians.

In the mid-sixties Edward Lorenz, a meteorologist/mathematician (a student of Birkhoff), discovered quite by accident, one of the hallmarks of chaos theory whilst modelling weather patterns,- *Sensitivity to Initial Conditions* (SIC) [8]. Weather patterns, dripping taps, etc. are examples of nonlinear dynamical systems which exhibit chaotic behaviour and are highly sensitive to initial conditions (IC). The trajectory, starting from some IC, of these chaotic systems will proceed along a certain path, however, setting the IC to a tiny, but different starting value, will produce a very different trajectory/path. Thus, systems that are sensitivity to initial conditions, means that the final output will be vastly different for very small changes in the initial conditions. This is often referred to as the "*Butterfly effect"*, a term mentioned in a lecture by Edward Lorenz in 1972 -"Predictability: Does the flap of a butterfly's wings in Brazil set off a Tornado in Texas"[9].

Chaotic systems are deterministic over a short period, hence we can make predictions in weather patterns up to about ten about days. According to Strogatz's chaos videos "Does the butterfly effect imply that chaotic systems are unpredictable? –No, they are predictable in the short term because of their deterministic character, but they become unpredictable after a certain amount of time, called the *horizon of predictability* [10] and is the time required for tiny errors to double in size. The Solar system trajectory is accurate to about 5 million years." After that, the path will deviate significantly. Another person who has been ignored in chaos literature, but who contributed to chaos theory, is *Mary Lucy Cartwright* [1900-1998] [11].

1.1 Nonlinear Dynamical Chaotic Systems

Chaotic systems are classified as *conservative* or *dissipative*. Astronomical systems are examples of conservative systems and sometimes referred to as 'Physicist's chaos', where the phase space volume remains unchanged but changes its overall shape in time. Dissipative systems, or 'Engineer's chaos', are considered in this blog and have a phase space volume that shrinks with time and contain 'Strange Attractors'. Dissipative systems are further classified further as *continuous* and *discrete*; continuous system-analogue systems operate over all time, whereas discrete or digital systems, exist at discrete periods of time.

Another classification for nonlinear chaotic oscillatory systems, autonomous and non-autonomous. Systems that have an external exciting input signal are called *non-autonomous*, meaning time is involved. The equations on the left hand side of these system normally show a periodic forcing input signal. When no external input is applied to the chaotic system and which do not contain time-dependent parameters, they are said to be *autonomous*. An example of an autonomous system is the Lorenz equation system examined in the first blog and which models atmospheric convection. On the other hand, a non-autonomous chaotic example is the chaotic Duffing nonlinear oscillator which has an external cosine forcing input signal, $\cos\omega t$.

1.2 Chaos applications

There are many potential applications for chaos but in this blog we look at a cryptographic application for securing data in the Cloud. A dynamic chaotic system equation contains state variables and constants to represent the 'state' of the system in *phase space* with time. The trajectory (a flow or orbit), traces the history of the system as time evolves. For example, this could be the speed of a planet in space or a particle in space, with measurable states such as velocity, acceleration. However, the dynamical systems in this treatment are concerned with state changes of voltage and current in electronic circuits starting from some initial condition and produce chaotic signals.

1.3 Phase Space

Phase Space is a mathematical representation of the state of a system in two or three dimensions and there is much confusion over who introduced the concept of phase space. A fine paper by David Nolte explains the history of this concept for those of you interested. The concept of phase space was mistakenly attributed to Liouville and Gibbs (1902) but it was Boltzmann who introduced the concept. He gave credit to Liouville thus doing himself out of the honour of being its creator. Henri Poincaré and the Irish mathematician, Sir William Edwin Hamilton (1806-65), along with Carl Jacobi, also contributed to this concept. In a 1971 paper, 'On the Nature of Turbulence' by David Ruelle and Floris Takens, they described how variables in phase space trace out two and three dimensional shapes which they called '*strange attractors*. The number of variables describing the attractor is its *dimensionality*.

Figure 1 show a 3-D picture of Lorenz phase space with a Poincaré section through the phase space.

In discrete time systems, the flow is called a *map* or *orbit* and produce time-voltage or time-current, vectors. Jules Henri Poincaré explained the impossibility of getting meaningful results from these time series and introduced the *Poincare section* - a beautifully-simple mathematical technique for reducing the dimensionality of the 3-D phase/state space by one and thus simplified its complexity. We need at least three state variables for an autonomous continuous chaotic system - a requirement for the *Poincaré-Bendixson theorem*. For a 2-D sampled system, the third dimension comes from sampled time. Differential equations solve the trajectory of continuous systems and difference equations solve the trajectory of discrete chaotic maps.

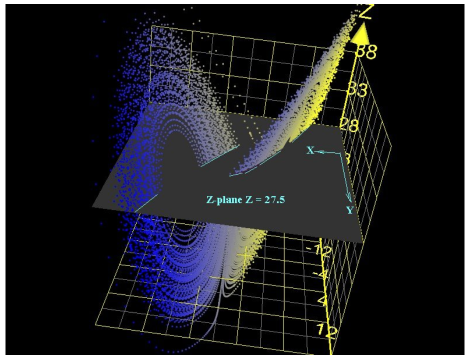

Figure 1. Lorenz 30D phase space showing a Poincaré section

A question that could be asked "What is the difference between a random (noisy or stochastic) signal and a chaotic deterministic signal"? To answer this question let's consider the simple schematic in Figure 3 which shows two generators; one is a PSPice noise generator introduced in the present version, and the other is chaotic signal from a Lorenz generator, both are attached to an analogue delay achieved using a correctly terminated and buffered ***T*** part. The delay is necessary to try and correlate past and present values. For a noise signal there should be no correlation between past and present values. A chaotic time series is a stochastic signal that occurring in a deterministic way, whereas a random noisy signal is completely stochastic. Individual trajectories in a chaotic time series are totally unpredictable, but the overall behaviour is not.

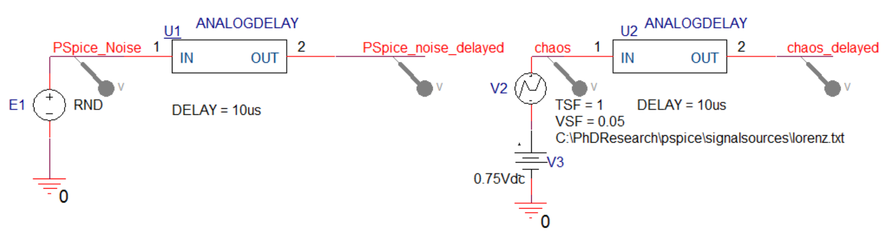

Figure 2. Noise and chaos signals

The outputs from these generators produce two time series that look similar. However, if we delay each signal by a suitable amount, and plot the delayed versus the undelayed signal, we get results shown in Figure 3.

We see for the random noise signal, there is no correlation between the delayed and undelayed signals and the attractor shows this very well.

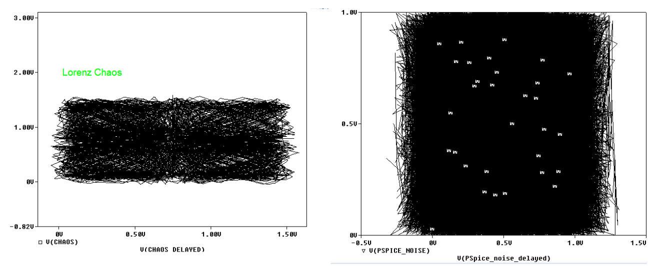

Figure 3. Chaotic and noise attractors showing no correlation between the signal and a delayed version of it.

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