New Exponential Bounds and Approximations for the Computation of Error Probability in Fading Channels

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***Abstract*—We present new exponential bounds for the Gaussian Q function (one- and two-dimensional) and its inverse, and for M-ary phase-shift-keying (MPSK), M-ary differential phase-shift-keying (MDPSK) error probabilities over additive white Gaussian noise channels. More precisely, the new bounds are in the form of the sum of exponential functions that, in the limit, approach the exact value. Then, a quite accurate and simple approximate expression given by the sum of two exponential functions is reported. The results are applied to the general problem of evaluating the average error probability in fading channels. Some examples of applications are also presented for the computation of the pairwise error probability of space–time codes and the average error probability of MPSK and MDPSK in fading channels.**

***Index Terms*—Bounds, fading, M-ary differential phase-shift keying (MDPSK), M-ary phase-shift keying (MPSK), Q function, space–time codes (STCs).**

I. Introduction

THE GAUSSIAN Q function, or, equivalently, the error function $\mathrm{erf}(.)$ and its complement $\mathrm{erfc}(.)$ are of great importance whenever Gaussian variables occur [1], [2]. These functions are tabulated, and often available as built-in functions in mathematical software tools. However, in many cases it is useful to have closed-form bounds or approximations instead of the exact expression, to facilitate expression manipulations [3], [4]. In fact, exponential-type bounds or approximations are particularly useful in evaluating the bit-error probability in many communication theory problems, such those arising in coding, fading, and multichannel reception [5].

Here, we provide new exponential-type upper bounds on the Q function and its inverse. The two-dimensional (2-D) case has also been considered. Moreover, a quite accurate approximation is developed in the form of the sum of two exponentials. Generally, bounds or approximations are not suitable for application to average error-probability evaluation because their accuracy is not guaranteed for a wide range of values. However, we show that the accuracy of our results is preserved when used to evaluate the average error probability in fading channels. Some examples of applications are reported for the computation of the pairwise error probability (PEP) for space–time codes (STCs) and the average error probability of M-ary phase-shift keying (MPSK) and M-ary differential phase-shift keying (MDPSK).

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II. Improved Exponential-Type Bounds on the $\mathrm{ERFC}(.)$

The complementary error function is usually defined as [2]

$$\mathrm{erfc}\,(x) \triangleq \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt. \qquad (1)$$

The tail probability of a unit variance zero mean Gaussian random variable is the Q function, which is related to the $\mathrm{erfc}(.)$ by

$$Q(x) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right). \qquad (2)$$

In the following, we will focus our attention on the $\mathrm{erfc}(.)$, all results also being useful for the Q function by the relation in (2).

A few years ago, the following integral of an exponential form for the Q function appeared in [6]

$$\mathrm{erfc}(x) = \frac{2}{\pi} \int_0^{\pi/2} \exp\left(-\frac{x^2}{\sin^2\theta}\right) d\theta,\ x \ge 0. \qquad (3)$$

Although this alternative form can be obtained by trivial manipulations of the results given in Weinstein [7] and Pawula *et al.* [8], it is not explicitly stated in either paper.

In the past, some exponential-type bounds have been derived. By adopting the Chernoff–Rubin bound we have, for $x \ge 0$, the exponential-type bound [1]

$$\mathrm{erfc}(x) \le 2e^{-x^2}. \qquad (4)$$

This can be improved by a factor 1/2. In fact, it is not difficult to show that the following also holds [1]:

$$\mathrm{erfc}(x) \le e^{-x^2}. \qquad (5)$$

In [9], it was observed that the bound in (5) can be derived from (3) by replacing the integrand with its maximum that occurs at $\theta = \pi/2$ as follows:

$$\begin{aligned}\mathrm{erfc}(x) &= \frac{2}{\pi} \int_0^{\pi/2} \exp\left(-\frac{x^2}{\sin^2\theta}\right) d\theta \\ &\le \frac{2}{\pi} \int_0^{\pi/2} \exp\left(-x^2\right) d\theta = \exp\left(-x^2\right). \end{aligned} \qquad (6)$$

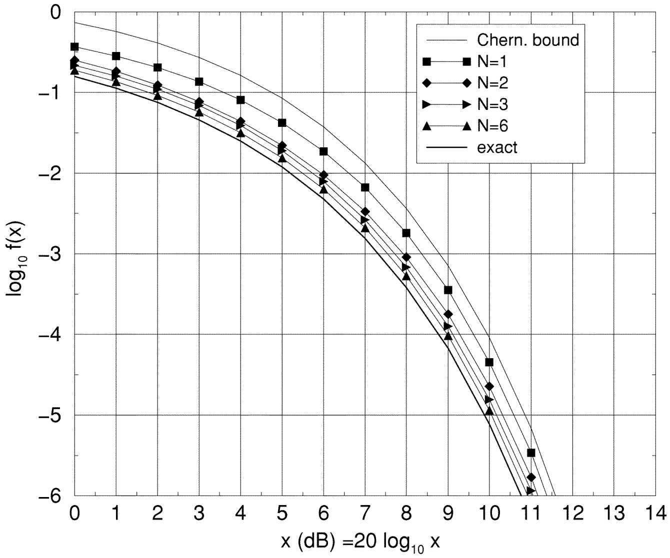

Fig. 1. Comparison among exponential bounds on the $\mathrm{erfc}(\cdot)$.

The main idea of this work is that the previous bound can be improved in a simple way. For this purpose, let us first note that since

$$\exp\left(-\frac{x^2}{\sin^2\theta}\right) \tag{7}$$

is a monotonically increasing function in θ for $0 \le \theta \le \pi/2$, then choosing arbitrarily $N+1$ values of θ such that $\theta_0, \theta_1, \ldots, \theta_N$ with $0 = \theta_0 \le \theta_1 \cdots \le \theta_N = \pi/2$, we can write the following improved exponential bound:

$$\begin{aligned}\mathrm{erfc}(x) &\le \frac{2}{\pi}\sum_{i=1}^{N}\int_{\theta_{i-1}}^{\theta_i}\exp\left(-\frac{x^2}{\sin^2\theta_i}\right)d\theta \\ &= \sum_{i=1}^{N} a_i \exp\left(-b_i x^2\right)\end{aligned} \tag{8}$$

where

$$a_i = \frac{2(\theta_i - \theta_{i-1})}{\pi}, \quad b_i = \frac{1}{\sin^2\theta_i}. \tag{9}$$

By increasing N, that is the number of θ values, the bound tends to the exact value. In fact, the right hand side of (8) corresponds to the numerical evaluation of the integral in (3) by the rectangular rule, that in this case also provides an upper bound. In other words, the integrand function is Riemann integrable. For example, from (8), by choosing $N = 1$, we have (5). By increasing N, we obtain tighter upper bounds. With $N = 2$ and $\theta_1 = \pi/4$, we have

$$\mathrm{erfc}(x) \le \tfrac{1}{2}\, e^{-2x^2} + \tfrac{1}{2}\, e^{-x^2}. \tag{10}$$

With $N = 3$ and $\theta_1 = \pi/3$, $\theta_2 = \pi/4$, we have

$$\mathrm{erfc}(x) \le \tfrac{1}{3}\, e^{-4x^2} + \tfrac{1}{6}\, e^{-2x^2} + \tfrac{1}{2}\, e^{-x^2}. \tag{11}$$

In general, the intermediate points $\theta_1, \ldots, \theta_{N-1}$ can be chosen arbitrarily, for example, trying to obtain the best bound, or simply equispaced, but satisfying $0 = \theta_0 \le \theta_1 \cdots \le \theta_N = \pi/2$. We verified that the value $\theta_1 = \pi/4$ for the case $N = 2$ is a good choice. In Fig. 1, we report the behavior of (8) with equispaced points, i.e., with $\theta_i = i\pi/(2N)$, $i = 1, \ldots, N$. Note that the case $N = 2$ is the bound in (10).

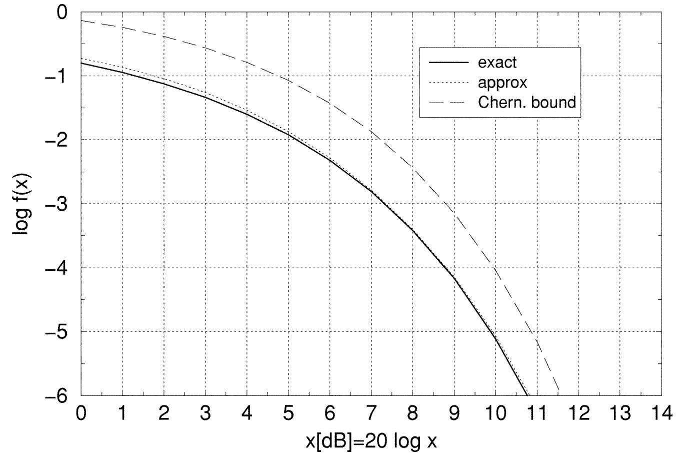

Fig. 2. Comparison between $\mathrm{erfc}(\cdot)$, the approximation (14), and the Chernoff bound.

III. A Tight and Simple Approximation for the $\mathrm{ERFC}(.)$

Starting from (3), a quite good and simple approximate expression can be obtained for $\mathrm{erfc}(.)$. In fact, by applying the numerical evaluation by trapezoidal rule in the case $N = 2$ for an arbitrary point θ, we have

$$\mathrm{erfc}(x) \simeq g(x, \theta) = \left(\frac{1}{2} - \frac{\theta}{\pi}\right) e^{-x^2} + \frac{1}{2} e^{-(x^2/\sin^2\theta)}. \tag{12}$$

Parameter θ is chosen here to minimize the integral of the relative error in the range of values of interest (the classical minimum mean square error optimization does not give good results in this case)

$$\theta_{\mathrm{opt}} = \arg\min_{\theta} \frac{1}{R}\int_0^R \frac{|g(x,\theta) - \mathrm{erfc}(x)|}{\mathrm{erfc}(x)} \cdot dx. \tag{13}$$

The optimum value in the range $0 - R = 13$ dB has been calculated numerically to be $\theta_{\mathrm{opt}} \simeq \pi/3$ leading to

$$\mathrm{erfc}(x) \simeq g(x, \theta_{\mathrm{opt}}) = \tfrac{1}{6}\, e^{-x^2} + \tfrac{1}{2}\, e^{-4x^2/3}. \tag{14}$$

Actually, it can be verified that (14) provides a tight upper bound for $x > 0.5$. This can be seen in Fig. 2 where the function $g(x, \theta_{\mathrm{opt}})$ is plotted. As can be noted, there is good agreement with $\mathrm{erfc}(x)$ for a wide range of abscissa values. Expression (14) becomes useful in those communication theory problems (e.g., coding) where an exponential-type function makes the solution or the evaluation simpler, without substantial loss of accuracy. An example of such an application is reported in Section VI.

IV. Bounds on the $\mathrm{INVERFC}(.)$

Another important function useful in dealing with statistical problems is the inverse complementary error function, defined as $\mathrm{inverfc}(y) \triangleq \{x\colon \mathrm{erfc}(x) = y\}$. It is worthwhile observing that, to our knowledge, although some approximations can be found in [10], bounds on the inverse error function have not been investigated in the literature. Here, we derive simple upper bounds on the inverse $\mathrm{erfc}(.)$ function from the exponential-type bounds on $\mathrm{erfc}(.)$ previously presented. In fact, the first (trivial) bound from (5) is

$$\mathrm{inverfc}(y) \leq \sqrt{-\ln(y)}. \tag{15}$$

An improved bound can be derived by inverting (10) as follows:

$$\mathrm{inverfc}(y) \leq \sqrt{\ln\left(\frac{1+\sqrt{1+8y}}{4y}\right)}. \tag{16}$$

Numerically, it can be verified (see, e.g., Fig. 1) that at $y = 10^{-2}$, $y = 10^{-3}$, $y = 10^{-6}$, the bound in (16) is 0.69, 0.46, and 0.22 dB closer, respectively, to the exact $\mathrm{inverfc}(y)$ value with respect to the bound in (15).

V. Bounds for the 2-D Joint Gaussian Q Function

A similar approach can be followed to bound the 2-D joint Gaussian Q function, starting from the representation presented in [17]. There, it was shown that the 2-D Gaussian Q function can be written as

$$Q(x, y; \rho) = \frac{1}{2\pi}\int_0^{\theta_x^*} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta + \frac{1}{2\pi}\int_0^{\theta_y^*} \exp\left(-\frac{y^2}{2\sin^2\theta}\right) d\theta, \quad x \geq 0,\ y \geq 0 \tag{17}$$

with ρ the correlation coefficient and

$$\theta_x^* = \arctan\left(\frac{x/y\sqrt{1-\rho^2 x/y}}{1-\rho x/y}\right)$$
$$\theta_y^* = \arctan\left(\frac{y/x\sqrt{1-\rho^2 y/x}}{1-\rho y/x}\right). \tag{18}$$

Here, the inverse tangent function principal value is taken in the interval $[0, \pi)$ so that $0 \leq \theta_y^*, \theta_x^* < \pi$. Since the integrand in each term of (17) is increasing in the interval $\theta \in [0, \pi/2]$ and decreasing for $\theta \in (\pi/2, \pi)$, two cases have to be considered. The first case is where both $1-\rho\, y/x$ and $1-\rho\, x/y$ are positive. Then, the integrands in (17) are monotonically increasing functions of θ in their respective integration intervals. Thus, the integral can be upper bounded by

$$Q(x, y; \rho) \leq \sum_{i=1}^{N} a_{x_i}\exp\left(-b_{x_i}x^2\right) + \sum_{i=1}^{N} a_{y_i}\exp\left(-b_{y_i}y^2\right), \quad y > \rho x,\ x > \rho y \tag{19}$$

where

$$a_{x_i} = \frac{\theta_{x_i}-\theta_{x_{i-1}}}{2\pi}, \quad b_{x_i} = \frac{1}{2\sin^2\theta_{x_i}}$$
$$a_{y_i} = \frac{\theta_{y_i}-\theta_{y_{i-1}}}{2\pi}, \quad b_{y_i} = \frac{1}{2\sin^2\theta_{y_i}} \tag{20}$$

and $\theta_{x_i}, \theta_{y_i}$ are $N+1$ arbitrarily chosen values satisfying $0 = \theta_{x_0} \leq \theta_{x_1} \cdots \leq \theta_{x_N} = \theta_x^*$ and $0 = \theta_{y_0} \leq \theta_{y_1} \cdots \leq \theta_{y_N} = \theta_y^*$, respectively. For example, with uniform spacing, we have $\theta_{x_i} = \theta_x^* \cdot i/N$ and $\theta_{y_i} = \theta_y^* \cdot i/N$.

For the case where $1-\rho\, x/y$ or $1-\rho\, y/x$ is negative, then in the interval between $\pi/2$ and the upper limit, the integrand is a monotonically decreasing function of θ and can be upper bounded by a downward staircase function. Hence, by further subdividing the condition where $1-\rho\ x/y$ or $1-\rho\ y/x$ is negative, we have, respectively

$$Q(x, y; \rho) \leq \sum_{i=1}^{N_1} a_{x_i}\exp\left(-b_{x_i}x^2\right) + \sum_{i=1}^{N} a_{y_i}\exp\left(-b_{y_i}y^2\right) + \sum_{i=N_1+1}^{N} a_{x_i}\exp\left(-b_{x_{i-1}}x^2\right), \quad y < \rho x \tag{21}$$

$$Q(x, y; \rho) \leq \sum_{i=1}^{N_1} a_{y_i}\exp\left(-b_{y_i}y^2\right) + \sum_{i=1}^{N} a_{x_i}\exp\left(-b_{x_i}x^2\right) + \sum_{i=N_1+1}^{N} a_{y_i}\exp\left(-b_{y_{i-1}}y^2\right), \quad x < \rho y \tag{22}$$

where $0 = \theta_{x_0} \leq \theta_{x_1} \cdots \leq \theta_{x_{N_1}} = \pi/2 \leq \theta_{x_{N_1+1}} \cdots \leq \theta_{x_N} = \theta_x^*$ in (21) and $0 = \theta_{y_0} \leq \theta_{y_1} \cdots \leq \theta_{y_{N_1}} = \pi/2 \leq \theta_{y_{N_1+1}} \cdots \leq \theta_{y_N} = \theta_y^*$ in (22). Parameters a_{x_i}, b_{x_i}, a_{y_i}, and b_{y_i} can be obtained again from (20).

These bounds can be applied, for example, to compute the outage probability for dual diversity selection combining over correlated distributed log-normal channels [18], [19].

VI. Upper and Lower Exponential Bounds on MPSK and MDPSK Symbol-Error Probability

Based on results obtained by Pawula *et al.* [8], [15], the following expressions exist for the symbol-error probability (SEP) of coherent MPSK and differential coherent MPSK (MDPSK):

$$P_s(E|\rho)|_{\mathrm{MPSK}} = \frac{1}{\pi} \times \int_0^{\pi-\pi/M} \exp\left(-\frac{\rho\sin^2(\pi/M)}{\sin^2\theta}\right) d\theta \tag{23}$$

and

$$P_s(E|\rho)|_{\mathrm{MDPSK}} = \frac{1}{\pi} \times \int_0^{\pi-\pi/M} \exp\left(-\frac{\rho\sin^2(\pi/M)}{1+\cos(\pi/M)\cos\theta}\right) d\theta \tag{24}$$

where ρ is the symbol signal-to-noise ratio (SNR). In (23), the integrand is monotonically increasing in the interval $(0, \pi/2)$ and monotonically decreasing in the interval $(\pi/2, \pi-\pi/M)$. It can also be noted that the integrand is symmetric around the value $\theta = \pi/2$. Thus, to upper and lower bound the SEP, one can first divide the integral into two integrals corresponding to the above integration intervals and then apply the approach discussed in the previous sections. Arbitrarily choosing $N+1$ monotonically increasing values of θ such that $\theta_0 = 0, \theta_1, \ldots, \theta_{N_1-1}, \theta_{N_1} = \pi/2, \theta_{N_1+1}, \ldots \theta_{N-1}, \theta_N =$

$\pi - \pi/M$, the following upper and lower sum of exponential bounds are obtained:

$$P_s(E|\rho)|_{\text{MPSK}} \leq \sum_{i=1}^{N_1} a_i \exp\left(-b_{i|_{\text{MPSK}}}\rho\right) + \sum_{i=N_1+1}^{N} a_i \exp\left(-b_{i-1|_{\text{MPSK}}}\rho\right) \quad (25)$$

$$P_s(E|\rho)|_{\text{MPSK}} \geq \sum_{i=2}^{N_1} a_i \exp\left(-b_{i-1|_{\text{MPSK}}}\rho\right) + \sum_{i=N_1+1}^{N} a_i \exp\left(-b_{i|_{\text{MPSK}}}\rho\right) \quad (26)$$

where $a_i = (\theta_i - \theta_{i-1})/\pi$ and

$$b_{i|_{\text{MPSK}}} = -\frac{\sin^2(\pi/M)}{\sin^2\theta_i}. \quad (27)$$

In (24), the integrand is monotonically decreasing over the entire integration interval $(0, \pi - \pi/M)$. Thus, arbitrarily choosing $N+1$ monotonically increasing values of θ such that $\theta_0 = 0, \theta_1, \theta_2, \ldots, \theta_{N-1}, \theta_N = \pi - \pi/M$, then the following upper and lower sum of exponential bounds are obtained:

$$P_s(E|\rho)|_{\text{MDPSK}} \leq \sum_{i=1}^{N} a_i \exp\left(-b_{i|_{\text{MDPSK}}}\rho\right) \quad (28)$$

$$P_s(E|\rho)|_{\text{MDPSK}} \geq \sum_{i=1}^{N} a_i \exp\left(-b_{i-1|_{\text{MDPSK}}}\rho\right) \quad (29)$$

where

$$b_{i|_{\text{MDPSK}}} = -\frac{\sin^2(\pi/M)}{1+\cos(\pi/M)\sin^2\theta_i}. \quad (30)$$

VII. Application to the Calculation of Average Error Probability

System performance investigation in fading channels often requires the evaluation of the average error probability $P(E)$ over the fading statistics, namely

$$P(E) = E\left[P(E|\rho)\right] \quad (31)$$

where $P(E|\rho)$ denotes the error probability conditioned on the instantaneous SNR ρ. In most cases, this task becomes difficult due to the nonlinear dependence of the error probability on ρ, the specific form of the nonlinearity depending on the particular modulation technique employed. Based on the form of $\text{erfc}(.)$ in (3), a general result is given in [11] for the evaluation of the above expectation over fading channels for the case where $P(E|\rho) = a\ \text{erfc}\left(b\sqrt{\rho}\right)$ with arbitrary constants a, b.

In this paper, we apply the exponential bounds of the previous sections to find a simple and accurate approach for the computation of (31) that does not require any numerical integration. In general, if the error probability can be upper bounded, as is done in (8), (25), and (28), then we can write (similarly for lower bounds)

$$P(E) \leq \sum_{i=1}^{N} a_i E[\exp\{-b_i\rho\}] = \sum_{i=1}^{N} a_i \Phi(-b_i) \quad (32)$$

where

$$\Phi(s) = E[\exp\{s\rho\}] \quad (33)$$

is the moment-generating function (MGF) associated with the random variable ρ [11]. Once the MGF of the instantaneous SNR is known (often in closed form), it is simple to evaluate, with the accuracy required, the average probability $P(E)$ using (32) without the need to perform any numerical integration.

A. Application to STCs

As an example of an application, consider the evaluation of the average PEP of STCs in the same scenario as in [12]–[14], [16] corresponding to a four-state quadrature phase-shift-keying (QPSK) STC operating in a Rayleigh-fading environment. The PEP is evaluated for a block length $B = 2$, for two transmitting and one receiving antenna, and

$$\mathbf{X} = \sqrt{E_s}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \hat{\mathbf{X}} = \sqrt{E_s}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (34)$$

where $\mathbf{X}$, $\hat{\mathbf{X}}$ are the two codewords considered and E_s denotes the symbol energy. In this case, the MGF of ρ for both independent and block fading channels is [11], [16]

$$\Phi(s) = (1 - sE_s/N_0)^{-2} \quad (35)$$

where N_0 is the single-sided noise power spectral density. By inserting (35) in (32), we obtain the bound on average PEP

$$P(\mathbf{X} \to \hat{\mathbf{X}}) = \tfrac{1}{2}\, E\left[\text{erfc}\left(\sqrt{\rho}\right)\right] \leq \tfrac{1}{2} \sum_{i=1}^{N} a_i (1 + b_i E_s/N_0)^{-2}. \quad (36)$$

If the approximation (14) is invoked, a simpler expression for the PEP can be found, namely

$$P(\mathbf{X} \to \hat{\mathbf{X}}) \simeq \tfrac{1}{12}\,(1 + E_s/N_0)^{-2} + \tfrac{1}{4}\,(1 + (4/3)E_s/N_0)^{-2}. \quad (37)$$

The accuracy of (36) and (37) can be verified by observing Fig. 3. For this particular scenario, an exact expression has also been obtained in [16]. The exact curve and the Chernoff bound [13], [14] are also reported for comparison. For example, at PEP $= 10^{-3}$ the improvement obtained for $N = 2$ and $N = 3$ with respect to the Chernoff bound is on the order of 3 dB. Moreover, the approximation (37) leads to a very tight result. These results show that both the bound and the approximation maintain their accuracy even if averaged over the fading. More generally, these bounds and approximations are useful in all cases where the MGF of the argument of the $\text{erfc}(.)$ function can be evaluated in a closed form (see [16] for more details about application to STCs).

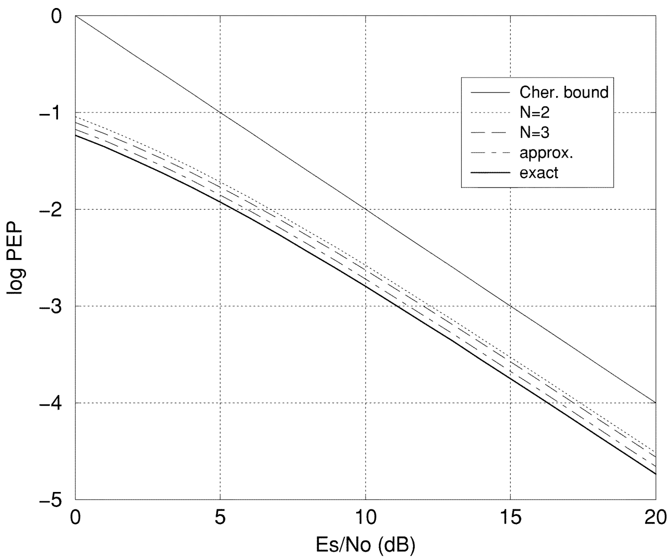

Fig. 3. PEP versus E_s/N_0 for the four-state QPSK STC considered.

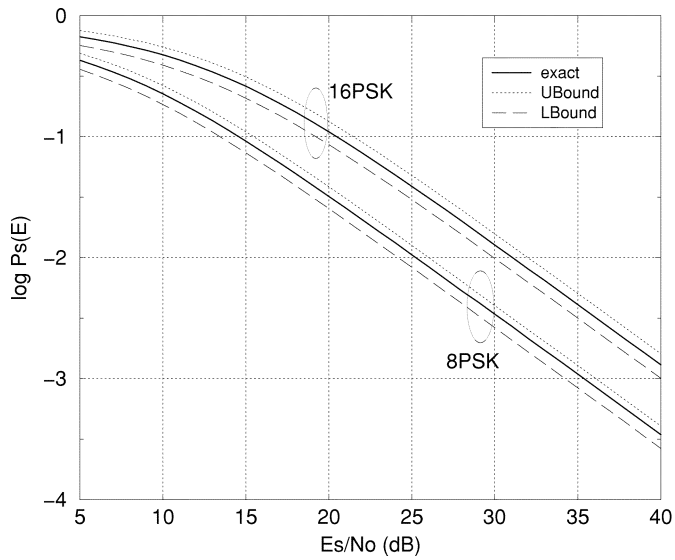

Fig. 4. $P_s(E)$ versus E_s/N_0 for 8PSK and 16PSK. $N = 8,\ N_1 = 4$.

B. Average Upper and Lower Bounds on MPSK and MDPSK SEP

The same MGF method can be exploited to find upper and lower bounds on MPSK and MDPSK SEP in fading channels. Starting from (25), (26), (28), and (29), by applying the same methodology followed in previous sections, we get the following bounds on the average SEP:

$$P_s(E)|_{\rm MPSK} \leq \sum_{i=1}^{N_1} a_i \Phi(-b_{i|_{\rm MPSK}}) + \sum_{i=N_1+1}^{N} a_i \Phi(-b_{i-1|_{\rm MPSK}}) \quad (38)$$

$$P_s(E)|_{\rm MPSK} \geq \sum_{i=2}^{N_1} a_i \Phi(-b_{i-1|_{\rm MPSK}}) + \sum_{i=N_1+1}^{N} a_i \Phi(-b_{i|_{\rm MPSK}}) \quad (39)$$

$$P_s(E)|_{\rm MDPSK} \leq \sum_{i=1}^{N} a_i \Phi(-b_{i|_{\rm MDPSK}}) \quad (40)$$

$$P_s(E)|_{\rm MDPSK} \geq \sum_{i=1}^{N} a_i \Phi(-b_{i-1|_{\rm MDPSK}}) \quad (41)$$

for MPSK and MDPSK, respectively. The MGF can be evaluated from (33) once the fading statistics are known and is reported for a large variety of channels in [11].

In Figs. 4 and 5, the accuracy of the bounds can be verified for 8PSK, 16PSK, 8DPSK, and 16DPSK schemes in Rayleigh fading. Also in this case, the accuracy, in terms of SNR required for a fixed target $P_s(E)$, is preserved for the whole range of SNR of interest.

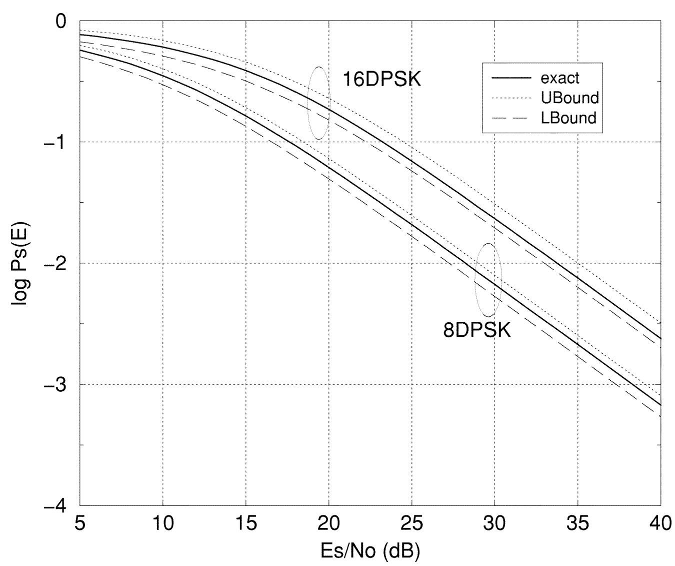

Fig. 5. $P_s(E)$ versus E_s/N_0 for 8DPSK and 16DPSK. $N = 8$.

VIII. Conclusion

In this paper, exponential bounds for one-dimensional and 2-D Gaussian Q functions have been presented. In the limit of a large number of terms, these bounds approach their corresponding exact values. Moreover, an accurate and simple approximate expression for the Q function is reported. The general problem regarding the evaluation of the average error probability in fading channels has been addressed by using these bounds and approximation in situations where other kinds of bounds (e.g., Chernoff–Rubin) fail. In particular, some examples have been given for the computation of the PEP of STCs and the average error probability of MPSK and MDPSK in fading channels.

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