Exponential stability of discrete-time switched linear positive systems with time-delay

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Abstract

This paper addresses the exponential stability problem for a class of discrete-time switched linear positive systems (DSLPSs) with time-delay. Firstly, an exponential stability criterion for a non-switched delay positive system is proposed. Then, by constructing a suitable co-positive type Lyapunov–Krasovskii functional, an exponential stability criterion for switched delay positive systems is derived via the average dwell time approach, and a stability criterion for switched positive systems with multiple delays is stated and its proof is sketched. Finally, a numerical example is given to illustrate the obtained results.

1. Introduction

A switched system is a type of hybrid dynamical system that combines discrete states and continuous states. Informally, it consists of a family of dynamical subsystems and a rule, called a switching signal, which determines the switching manner between the subsystems. Many dynamical systems can be modeled as such switched systems [1–3].

Recently, the importance of switched linear positive systems (SLPSs) has been highlighted by many researchers due to their broad applications in communication systems [4], multiagent systems [5,6], and systems theory [7,8]. A positive system implies that its states and outputs are nonnegative whenever the initial conditions and inputs are nonnegative [9–15]. A SLPS means a switched linear system, in which each subsystem itself is a positive system.

On the other hand, time-delay phenomena widely exist in dynamic systems. Although many results on time-delay systems have been reported [16–20], only recently has the linear positive systems with time delay become a topic of major interest. To list a few, a stability criterion of positive fractional continuous-time linear systems with delays was derived in [21]. A necessary and sufficient stability condition for positive systems with constant delay was obtained in [22] by means of a linear co-positive Lyapunov functional, which is a powerful tool for tackling positive systems [23,24], and a sufficient stability condition for delayed positive systems with uncertainties was given in [25]. It is worth mentioning that up to now, little effort has been devoted to switched positive linear systems in the presence of time delay, which is theoretically challenging and of fundamental importance to numerous applications [26]. The exponential stability problem of switched positive continuous-time linear systems with constant time delays was considered in [27]. However, to the best of our knowledge, the exponential stability problem of discrete-time switched linear positive systems (DSLPSs) has not been fully investigated, which motivates our present work.

In this paper, we will focus on the derivation of sufficient conditions for the exponential stability of DSLPSs with time delay. The main contributions of this paper can be summarized as follows: (1) a co-positive type Lyapunov–Krasovskii functional for a given non-switched discrete-time positive system in the presence of constant delay is firstly established; (2) by...
constructing a suitable piecewise co-positive type Lyapunov–Krasovskii functional, an exponential stability criterion for the underlying systems with average dwell time (ADT) switching is derived in terms of a set of linear matrix inequalities (LMIs); (3) a stability condition of DSLPSs with multiple delays is also developed.

The remainder of the paper is organized as follows. In Section 2, problem formulation and some necessary lemmas are given. In Section 3, the main results are developed. A numerical example is given to illustrate the effectiveness of the proposed results in Section 4. Concluding remarks are given in Section 5.

Notation: In this paper, $A > 0 (A \succeq 0)$ means that all elements of matrix $A$ are positive (non-negative). $A > B (A \succeq B)$ means that $A - B > 0 (A - B \succeq 0)$. $Q \succ 0 (<0)$ represents that matrix $Q$ is negative semi-definite (negative definite). $R^n$ is the $n$-dimensional real vector space, and $R^n_{++}$ is the set of vectors whose elements are all nonnegative. $A'$ denotes the transpose of matrix $A$. We define $I_r = \{0, \ldots, 0, 1, \ldots, 0\}$. $\omega$ is the set of nonnegative integers. The 1-norm of a vector $x \in R^n$ is defined as $\|x\| = \sum_{k=1}^{n} |x_k|$, where $x_k$ is the $k$th element of $x$.

2. Problem formulation and preliminaries

Consider the following discrete-time switched linear system (DSLS) with time-delay:

$$\begin{align*}
    \begin{cases}
        x(k+1) &= A_{\sigma(k)}x(k) + A_d x(k-h), \\
        x(k) &= \varphi(k), & k = -h, \ldots, 0,
    \end{cases}
\end{align*}$$

where $x(k) \in R^n$ is the state vector; $A$ and $A_d$ are constant matrices with appropriate dimensions; $\sigma(k) : N^+ \mapsto M = \{1, 2, \ldots, m\}$ is the switching signal with $m$ being the number of subsystems; $h > 0$ denotes the constant delay; and $\varphi(k)$ is the initial condition.

**Definition 1.** System (1) is said to be positive if, for any initial condition $\varphi(k) \succeq 0$, $k = -h, \ldots, 0$, and any switching signal $\sigma(k)$, the state trajectory $x(k) \succeq 0$ holds for all $k \in N^+$.

**Lemma 1** [28]. System (1) is positive if and only if $A_p \succeq 0$ and $A_p \varphi \succeq 0$, $p \in M$.

**Remark.** In the light of (1), it is clear that the $p$th subsystem of (1) is positive if and only if $A_p \succeq 0$ and $A_p \varphi \succeq 0$, $p \in M$.

Now, we present the following exponential stability definition of system (1).

**Definition 2** [29]. System (1) is globally uniformly exponentially stable (GUES) under switching signal $\sigma(k)$ if there exist constants $a > 0$ and $b > 0$ such that the solution of the system satisfies

$$\|x(k)\| \leq q\|x(0)\|e^{-bk}, \quad \forall k \in N^+,$$

where $\|x(0)\|_c = \sup_{h \geq 0} \|x(h)\|$.

**Definition 3** [30]. For any switching signal $\sigma(k)$ and any $T_2 > T_1 > 0$, let $N_\sigma(T_1, T_2)$ denote the number of switchings of $\sigma(k)$ over the interval $[T_1, T_2]$. For given constants $T_\sigma > 0$ and $N_0 \geq 0$, if the inequality

$$N_\sigma(T_1, T_2) \leq N_0 + \frac{T_2 - T_1}{T_\sigma}$$

holds, then the positive constant $T_\sigma$ is called the average dwell time and $N_0$ is called the chattering bound. As commonly used in the literature, we choose $N_0 = 0$ in this paper.

3. Main results

In this section, we will investigate the exponential stability of positive system (1). In order to obtain the main result, we first consider the following non-switched delay positive system

$$\begin{align*}
    \begin{cases}
        x(k+1) &= Ax(k) + A_d x(k-h), \\
        x(k) &= \varphi(k), & k = -h, \ldots, 0,
    \end{cases}
\end{align*}$$

where $A \succeq 0$ and $A_d \succeq 0$ are constant matrices, and $\varphi(k) \succeq 0$ is the initial condition.

**Lemma 2.** Given a positive constant $x$, if there exist vectors $v, u \in R^n$ such that

$$\Psi = \text{diag}\{\psi_1, \psi_2, \ldots, \psi_n; \psi_1', \psi_2', \ldots, \psi_n'\} < 0,$$

then...
where
\[ \psi_r = (a_r^T - e^{-\frac{1}{2}I}) v + u_r, \quad \psi'_r = a_r^T v - e^{-\frac{1}{2}h} u_r, \quad r \in \mathbb{N} = \{1, 2, \ldots, n\}, \]
with \( a_r(a_{rh}) \) represents the \( r \)-th column vector of matrix \( A(A_{rh}) \), \( v = [v_1, v_2, \ldots, v_n]^T \), and \( u = [u_1, u_2, \ldots, u_n]^T \), then along the trajectory of positive system (4), we have
\[ V(k) < e^{-2k}V(0). \]  
(6)

**Proof.** Choose the following co-positive type Lyapunov–Krasovskii functional candidate for positive system (4)
\[ V(k) = x^T(k) v + \sum_{s=k-h}^{k-1} e^{2(s-k+1)} x^T(s) u, \]  
(7)
where \( v, u \in \mathbb{R}_+^n \) are vectors to be determined. Along the trajectory of positive system (4), we have
\[ V(k+1) = e^{-2}V(k) = x^T(k+1) v + \sum_{s=k-h}^{k-1} e^{2(s-k+1)} x^T(s) u - e^{-2} x^T(k) v - e^{-2} \sum_{s=k-h}^{k-1} e^{2(s-k+1)} x^T(s) u \]
\[ = x^T(k) A^T v - x^T(k) v - x^T(k) (A^T - e^{-1}) v + x^T(k) (A^T - e^{-1}) v - e^{-2} x^T(k) v + e^{-2} x^T(k-h) u \]
\[ = x^T(k) (A^T v - e^{-2} v + u) + x^T(k-h) (A^T v - e^{-2} u). \]
It follows from (5) that \( V(k+1) < e^{-2}V(k) \). Therefore, one has
\[ V(k) < e^{-2}V(k-1) < \cdots < e^{-2k}V(0). \]  
□

**Remark 2.** It is noted that (6) gives a decay estimation of the Lyapunov–Krasovskii functional (7). A delay-dependent sufficient condition for the existence of the estimation (6) is presented in (5), which is formulated as an LMI with respect to the elements of vectors \( v \) and \( u \). We can directly solve the feasibility problem of the LMI (5) via LMI toolbox in MATLAB to obtain the vectors \( v \) and \( u \). In addition, a smaller \( \alpha \) will be favorable to the feasibility of the LMI (5).

Now, we are in a position to provide an exponential stability condition for positive system (1) in the following theorem.

**Theorem 1.** Consider positive system (1). Let \( \alpha > 0 \) be a given constant. If there exist vectors \( v_p \in \mathbb{R}_+^n, u_p \in \mathbb{R}_+^n, \forall p \in M, \) such that
\[ \Psi_p = \text{diag}(\psi_{p1}, \psi_{p2}, \ldots, \psi_{pm}, \psi'_{p1}, \psi'_{p2}, \ldots, \psi'_{pm}) < 0, \]  
(8)
where
\[ \psi_{pr} = (a_{pr}^T - e^{-\frac{1}{2}T}) v_p + u_{pr}, \quad \psi'_{pr} = a_{pr}^T v_p - e^{-\frac{1}{2}h} u_{pr}, \]
with \( a_{pr}(a_{rhp}) \) represents the \( r \)-th column vector of matrix \( A_p(A_{rhp}) \), and \( v_{pr}(u_{pr}) \) denotes the \( r \)-th element of vector \( v_p(u_p) \), then the system is GUES for any switching signal \( \sigma(k) \) with the average dwell time
\[ T_a > T_a = \frac{\ln \mu}{\alpha}, \]  
(9)
where \( \mu \geq 1 \) satisfies
\[ v_p \leq v_{pl}, \quad u_p \leq u_{pl}, \quad \forall p,l \in M, \ p \neq l. \]  
(10)
Moreover, the state decay of positive system (1) is given by
\[ \|x(k)\| \leq \begin{pmatrix} \bar{\varepsilon}_2 + \varepsilon_1 \\ \bar{\varepsilon}_3 \end{pmatrix}\begin{pmatrix} 1 \\ e^{-\frac{\bar{\varepsilon}2}{\varepsilon_1}} \end{pmatrix} e^{-\frac{\bar{\varepsilon}3}{\varepsilon_1}}\|x(0)\|, \]  
(11)
where
\[ \varepsilon_1 = \min_{1 \leq p \leq M} \{ \varepsilon_{pr} \}, \quad \bar{\varepsilon}_2 = \max_{1 \leq p \leq M} \{ \varepsilon_{pr} \}, \quad \bar{\varepsilon}_3 = \max_{1 \leq p \leq M} \{ \varepsilon_{pr} \}. \]

**Proof.** Choose the following Lyapunov–Krasovskii functional for positive system (1)
\[ V(k) = x^T(k) v_{\sigma(k)} + \sum_{s=k-h}^{k-1} e^{2(s-k+1)} x^T(s) u_{\sigma(k)}, \]  
(12)
where \( v_p, u_p \in \mathbb{R}_+^n, \ p \in M, \) are to be determined.
For any given integer \( k > 0 \), we let \( 0 = k_0 < k_1 < \cdots < k_q = k_{N_{\max}} \) denote the switching instants of \( \sigma(k) \) over the interval \([0, k)\). According to (10) and (12), we can easily obtain
\[ V(k_g) \leq \mu V(k_g), \quad g = 1, 2, \ldots, q. \] (13)

From Lemma 2 and (8), we have, for \( k \in [k_g, k_{q+1}) \),
\[ V(k) = e^{-\eta(k-k_0)}V(k_0). \] (14)

Combining (13) and (14) leads to
\[ V(k) = e^{-\eta(k-k_0)}V(k_0) < e^{-\eta(k-k_0)}\mu V(k_{q+1}) \]
\[ = e^{-\eta(k-k_1)}\mu V(k_{q+1}) < e^{-\eta(k-k_2)}\mu^2 V(k_{q+2}) \]
\[ < \ldots < e^{-\eta k}e^{N_2(0)}V(0). \] (15)

In view of (3) and (9), we have
\[ V(k) < e^{-\eta k}e^{N_2(0)}\ln V(0) \leq e^{-\left(\frac{\ln\mu}{\eta}\right)k}V(0). \] (16)

Denoting \( \beta_1 = \min_{(p,q) \in \mathbb{Z}^2} \{ u_{pq} \} \), \( \beta_2 = \max_{(p,q) \in \mathbb{Z}^2} \{ u_{pq} \} \), \( \beta_3 = \max_{(p,q) \in \mathbb{Z}^2} \{ u_{pq} \} \), it yields that
\[ V(k) \geq \beta_1 \| x(k) \|, \] (17)
\[ V(0) \leq (\beta_2 + \beta_3) \sup_{-h \in \mathbb{Z}^2, 0} \| x(0) \|. \] (18)

Combining (17) and (18) we obtain
\[ \| x(k) \| \leq \frac{(\beta_2 + \beta_3)}{\beta_1} e^{-\left(\frac{\ln\mu}{\eta}\right)k} \sup_{-h \in \mathbb{Z}^2, 0} \| x(0) \|. \] (19)

Thus, positive system (1) is GUES for any switching signal with average dwell time (9). □

**Remark 3.** When \( \mu = 1 \) in Theorem 1, we obtain \( \nu_p = \nu_1 \) and \( u_p = u_1 \), \( \forall p, l \in M \), which lead to \( T_\sigma = 0 \). In this case, Theorem 1 provides a sufficient condition for the existence of a common co-positive type Lyapunov–Krasovskii functional, which implies that the switching signal can be arbitrary.

**Remark 4.** In Theorem 1, a sufficient delay-dependent exponential stability condition related to an LMI problem with respect to the elements of vectors \( \nu_p \) and \( u_p \) for positive system (1) with average dwell time switching is derived, which is expected to be helpful to the field of control synthesis for DSLPSs in the presence of time delay. Also, the result can be conveniently verified via LMI toolbox in MATLAB.

**Remark 5.** It can be seen from Theorem 1 that we need to solve \( 4mn \) scalar inequalities to obtain \( 2mn \) scalar decision variables. Obviously, the computational complexity will be accordingly increased as \( m \) and \( n \) get larger.

**Corollary 1.** Given a positive constant \( x \). If there exist vectors \( \nu = \nu_p \in \mathbb{R}^n \) and \( u = u_p \in \mathbb{R}^n \), \( \forall p \in M \), such that (8) holds, then positive system (1) is GUES for any switching signal \( \sigma(k) \). Moreover, the state decay of positive system (1) is given by
\[ \| x(k) \| \leq \left( \frac{\beta_2 + \beta_3}{\beta_1} \right) e^{-\left(\frac{\ln\mu}{\eta}\right)k} \sup_{-h \in \mathbb{Z}^2, 0} \| x(0) \|. \]

where \( \beta_1, \beta_2 \) and \( \beta_3 \) are defined in Theorem 1.

In practice, it is known that multiple delays are more general. Therefore, it is of significance to extend the aforementioned result to the following DSLPS with multiple delays
\[ \begin{cases} 
  x(k + 1) = A_{e(k)}x(k) + \sum_{i=1}^{w} A_{e(k)}^i x(k - h_i), \\
  x(k) = \varphi(k), \quad k = -h, \ldots, 0,
\end{cases} \] (20)

where \( h_i, i \in \{1, 2, \ldots, w\} \), are multiple time delays, \( \bar{h} = \max_{1 \leq i \leq w} h_i \), \( A_p \geq 0 \) and \( A_p^i \geq 0 \). \( \forall i \in \{1, 2, \ldots, w\}, p \in M \), are constant matrices, and \( \varphi(k) \geq 0 \) is the initial condition.

**Theorem 2.** Consider positive system (20). Let \( \alpha > 0 \) be a given constant. If there exist vectors \( \nu_p \in \mathbb{R}^n \), \( u_p \in \mathbb{R}^n \), \( \forall p \in M \), such that
\[ \text{diag}\{\tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_{pn}, \psi_{p11}, \psi_{p12}, \ldots, \psi_{p1w}, \psi_{p21}, \ldots, \psi_{p2w}, \ldots, \psi_{pm1}, \ldots, \psi_{pmw} \} < 0, \] (21)
where \( \tilde{\psi}_r = (a_{pr}^T - e^{-\tau_k})v_p + w_{at}v_p, \quad \tilde{\psi}_i = (a_{pr}^i)^T v_p - e^{-\tau_h}u_{pr}, \quad i \in \{1, 2, \ldots, n\}, \quad r \in \{1, 2, \ldots, m\}, \) with \( a_{pr}(a_{pr}^i) \) represents the \( r \)th column vector of matrix \( A_r(A_r^i), \) and \( v_p(u_p) \) denotes the \( r \)th element of vector \( v_p(u_p), \) then the system is GUES for any switching signal with average dwell time (9), where \( \mu \geq 1 \) satisfies (10).

**Proof.** Choose the following Lyapunov–Krasovskii functional for system (20)

\[
V(k) = x^T(k)v_{g(k)} + \sum_{i=1}^{w} e^{\alpha i}e^{-\gamma (k-i)}x^T(s)u_{e(k)}.
\]

Following the proof line of Theorem 1, Theorem 2 can be obtained. The detailed proof is omitted here. \( \square \)

### 4. Numerical example

In this section, a numerical example is provided to test the obtained results. Consider positive system (1) with parameters as follows

\[
A_1 = \begin{bmatrix} 0.1 & 0.6 \\ 0.2 & 0.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.2 & 0.5 \\ 0.1 & 0.2 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

In this example, \( n = 2, \) \( m = 2, \) \( a_{11}^1 = [0.1 0.2], \) \( a_{12}^1 = [0.6 0.3], \) \( a_{11}^2 = [0.2 0.1], \) \( a_{12}^2 = [0.5 0.2], \) \( a_{d11}^1 = [0.1 0], \) \( a_{d12}^1 = [0 0], \) \( a_{d21}^2 = [0 1], \) \( a_{d22}^2 = [0 0]. \)

Let \( v_1 = [v_{11} \ v_{12}]^T, \) \( v_2 = [v_{21} \ v_{22}]^T, \) \( u_1 = [u_{11} \ u_{12}]^T \) and \( u_2 = [u_{21} \ u_{22}]^T. \) Choosing \( \alpha = 0.3 \) and \( h = 2, \) and using (8), we have

\[
v_{11} \geq 0, \quad v_{12} \geq 0, \quad v_{21} \geq 0, \quad v_{22} \geq 0,
\]

\[
u_{11} \geq 0, \quad u_{12} \geq 0, \quad u_{21} \geq 0, \quad u_{22} \geq 0,
\]

\[
\tilde{\psi}_{11} = (a_{11}^1 - e^{-2\tau_1})v_1 + u_{11} = 0.1v_{11} + 0.2v_{12} + e^{-0.3}v_{11} + u_{11} < 0,
\]

\[
\tilde{\psi}_{12} = (a_{12}^1 - e^{-2\tau_2})v_2 + u_{12} = 0.6v_{11} + 0.3v_{12} - e^{-0.3}v_{12} + u_{12} < 0,
\]

\[
\tilde{\psi}_{21} = (a_{21}^1 - e^{-2\tau_1})v_1 + u_{21} = 0.2v_{21} + 0.1v_{22} - e^{-0.3}v_{21} + u_{21} < 0,
\]

\[
\tilde{\psi}_{22} = (a_{22}^1 - e^{-2\tau_2})v_2 + u_{22} = 0.5v_{21} + 0.2v_{22} - e^{-0.3}v_{22} + u_{22} < 0,
\]

\[
\tilde{\psi}_{11} = a_{d11}^1 v_1 - e^{-\tau_h}u_{11} = 0.1v_{11} - e^{-0.6}u_{11} < 0,
\]

\[
\tilde{\psi}_{12} = a_{d12}^1 v_1 - e^{-\tau_h}u_{12} = e^{-0.6}u_{12} < 0,
\]

\[
\tilde{\psi}_{21} = a_{d21}^2 v_2 - e^{-\tau_h}u_{21} = 0.1v_{21} - e^{-0.6}u_{21} < 0,
\]

\[
\tilde{\psi}_{22} = a_{d22}^2 v_2 - e^{-\tau_h}u_{22} = e^{-0.6}u_{22} < 0.
\]

Then solving the above inequalities by LMI toolbox in MATLAB gives rise to

\[
v_{11} = 0.8605, \quad v_{12} = 1.6787, \quad v_{21} = 1.0309, \quad v_{22} = 1.7459,
\]

\[
u_{11} = 0.1672, \quad u_{12} = 0.1119, \quad u_{21} = 0.2705, \quad u_{22} = 0.2144.
\]

Then,

\[
v_1 = \begin{bmatrix} 0.8605 \\ 1.6787 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1.0309 \\ 1.7459 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 0.1672 \\ 0.1119 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0.2705 \\ 0.2144 \end{bmatrix}.
\]

Therefore, the LMI (8) is feasible with respect to the vectors \( v_1, \) \( v_2, \) \( u_1 \) and \( u_2. \) Substituting the obtained vectors \( v_1, \) \( v_2, \) \( u_1 \) and \( u_2 \) into (10), one has

\[
0.8605 - 1.0309\mu \leq 0, \quad 1.0309 - 0.8605\mu \leq 0, \quad 1.6787 - 1.7459\mu \leq 0,
\]

\[
1.7459 - 1.6787\mu \leq 0, \quad 0.1672 - 0.2705\mu \leq 0, \quad 0.2705 - 0.1672\mu \leq 0,
\]

\[
0.1119 - 0.2144\mu \leq 0, \quad 0.2144 - 0.1119\mu \leq 0.
\]
Then from the above inequalities, we can compute the minimal $\mu = 1.9160$. Furthermore, we can obtain from (9) that $T_a = 2.1673$. Thus the system is GUES for any switching signal with average dwell time $T_a > T_a = 2.1673$.

Choose $T_a = 3$, the simulation results are shown in Figs. 1 and 2, where the initial state condition is $x(0) = [5\ 8]^T$, and $x(k) = [0\ 0]^T$, $k = -2, -1$. It can be seen from Figs. 1 and 2 that the system is stable.

5. Conclusions

This paper has dealt with the stability analysis problem for a class of discrete-time switched positive systems with time-delay. We have obtained sufficient conditions under which the underlying system is exponentially stable under average dwell time switching. Moreover, a stability criterion for switched positive systems with multiple delays is proposed. Finally, a numerical example is given to illustrate the obtained results.

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