Some results on polyadic algebras

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Abstract . While every polyadic algebra (**PA**) of dimension 2 is representable, we show that not every atomic polyadic algebra of dimension two is completely representable; though the class is elementary. Using higly involved constructions of Hirsch and Hodkinson we show that it is not elementary for higher dimensions a result that, to the best of our knowledge, though easily destilled from the literature, was never published. We give a uniform flexible way of constructing weak atom structures that are not strong, and we discuss the possibility of extending such result to infinite dimensions. Finally we show that for any finite $n > 1$, there are two n dimensional polyadic atom structures At_1 and At_2 that are $L_{\infty,\omega}$ equivalent, and there exist atomic $\mathfrak{A}, \mathfrak{B} \in \mathbf{PA}_n$, such that $\mathsf{At}\mathfrak{A} = \mathsf{At}_1$ and $\mathsf{At}\mathfrak{B} = \mathsf{At}_2$, $\mathfrak{A} \in \mathfrak{Nr}_n\mathbf{PA}_\omega$ and $\mathfrak{B} \notin \mathfrak{Nr}_n\mathbf{PA}_{n+1}$. This can also be done for infinite dimensions (but we omit the proof).

1 Introduction

There are two main algebraisations of first order, cylindric algebras due to Tarski and polyadic algebras due to Halmos. In the infinite dimensional, case they are significantly distinct, and it commonly accepted that they actually belong to two different universes or paradigms. One blatant difference is that polyadic algebras have continuum many opeartions, whie cylindric algebras have only countably many.

It is hard to give a rigouous mathematical definition of such a dichotomy, but they more often than not manifest contradictory behaviour. A plathora of results, existing in the literature, see [10], point out to the fact that there is some kind of dichotomy which does not need further rigorous evidence.

For finite dimensions, they are pretty close, but there are differences that are delicate and quite subtle. In the context of presence of equality for example, as is the case with cylindric algebras, when we are dealing with polyadic algebras endowed with diagonal elements, the substitution operations corresponding to transpositions, are not even finitely axiomatizable over their diagonal free case, this happens for all dimensions > 2, which suggest beyond doubt, that the substitution operations, a basic operation in polyadic algebras, that are not definable in cylindric algebras, adds a lot, and consequently one can expect that there are also non-trivial differences between the finite dimensional cylindric algebras and polyadic ones (Polyadic algebras occur in the literature under the names of quasi polyadic or finitary polyadic algebras; all three are essentially the same.)

The standard way of obtaining cylindric algebras, known as set algebras, is from a model for first order logic wih equality. Polyadic algebras, are rather obtained from models of first order logic *without* equality, and from, at least such a perspective, they are different, in so much as first order logic without is different that without equality.

On the other hand, because for some reason or another , maybe historic or aesthetic, cylindric algebras got the bigger share of research, during the lase decated, yielding plenty sophisticated deep results, using graph theory and finite combinatorics. And there is this feeling in the air that many results obatined for cylindric algebras, carry over to polyadic algebras without much ado, but no one has actually bothered to make sure that this is the case indeed, possibly under the conviction that is a systematic boring task. In some case, it definitely is, but in other, as it turns out this is not true at all, like for example the polyadic algebras constructed by Sayed Ahmed and Robin hirsch to confirm the analogue of the famous neat embeding pproblem, already proved for cylindric algebras, to polyadic algebras and other reducts thereof. Another is that, contrary to cylindric algebras, it is not known where there is a universal axiomatization of the class of representable polyadic algebras, of dimension > 2. Not only that, but such feelings often based on false intuition can be downright wrong, as we show in a minute.

2 A false impression about polyadic algebras

We give an example of cylindrfier free reducts of polyadic algebras. This is taken from [AGNS]. However, in the latter reference the example worked out by Andreka et all, addresses polyadic algebras, and it works only for dimension two. Here it works for all dimensions. Besides it answers a question of Hodkinson's, for Pinter's algebras.

Theorem 2.1. *For any ordinal $\alpha > 2$, and any infinite cardinal κ, there is an atomic algebra $\mathfrak{A} \in SA_\alpha$, with $|A| = \kappa$, that is not completely representable.In particular, $\mathfrak{A}$ can be countable.*

Proof. It suffices to show that there is an algebra $\mathfrak{A}$, and a set $S \subseteq A$, such that s_0^1 does not preserves $\sum S$. For if $\mathfrak{A}$ had a representation as stated in the theorem, this would mean that s_0^1 is completely additive in $\mathfrak{A}$.

For the latter statement, it clearly suffices to show that if $X \subseteq A$, and $\sum X = 1$, and there exists an injection $f : \mathfrak{A} \to \wp(V)$, such that $\bigcup_{x \in X} f(x) = V$, then for any $\tau \in {}^n n$, we have $\sum s_\tau X = 1$. So fix $\tau \in V$ and assume that this does not happen. Then there is a $y \in \mathfrak{A}$, $y < 1$, and $s_\tau x \leq y$ for all $x \in X$. (Notice that we are not imposing any conditions on cardinality of $\mathfrak{A}$ in this part of the proof). Now

$$1 = s_\tau(\bigcup_{x \in X} f(x)) = \bigcup_{x \in X} s_\tau f(x) = \bigcup_{x \in X} f(s_\tau x).$$

(Here we are using that s_τ distributes over union.) Let $z \in X$, then $s_\tau z \leq y < 1$, and so $f(s_\tau z) \leq f(y) < 1$, since f is injective, it cannot be the case that $f(y) = 1$. Hence, we have

$$1 = \bigcup_{x \in X} f(s_\tau x) \leq f(y) < 1$$

which is a contradiction, and we are done. Now we turn to constructing the required counterexample, which is an easy adaptation of a construction due to Andréka et all in [AGMNS] to our present situation. We give the detailed construction. One reason is for the reader's conveniance.

The other, which is more important, is that there are two major differences between our constrcustion and the forementioned one by Andrek et all. One is that our constructed algebra can have any infinite cardinality, this is not to much of a change. It has to do with enlarging an indexing set of a partition of the large enough base.

The second this, is that our construction works for *all* dimensions, and not just 2, because we are fortunate enough not to have cylindrifiers.

Now we start implementing our example. Let α be the given ordinal. Let $|U| = \mu$ be an infinite set and $|I| = \kappa$ be a cardinal such that Q_n, $n \in \kappa$, is a family of α-ary relations that form a partition of $V = {}^\alpha U^{(p)}$, for some fixed sequence $p \in {}^\alpha U$. Let $i \in I$, and let $J = I \sim \{i\}$. Then of course $|I| = |J|$. Assume that $Q_i = D_{01} = \{s \in V : s_0 = s_1\}$, and that each Q_n is symmetric; that is for any $i, j \in n$, $S_{ij} Q_n = Q_n$. It is straightforward to show that such partitions exist.

Now fix F a non-principal ultrafilter on J, that is $F \subseteq \mathcal{P}(J)$. For each $X \subseteq J$, define

$$R_X = \begin{cases} \bigcup\{Q_k : k \in X\} & \text{if } X \notin F, \\ \bigcup\{Q_k : k \in X \cup \{i\}\} & \text{if } X \in F \end{cases}$$

Let

$$\mathfrak{A} = \{R_X : X \subseteq I \sim \{i\}\}.$$

Notice that $|\mathfrak{A}| \geq \kappa$. Also $\mathfrak{A}$ is an atomic set algebra with unit R_J, and its atoms are $R_{\{k\}} = Q_k$ for $k \in J$. (Since F is non-principal, so $\{k\} \notin F$ for every k). We check that $\mathfrak{A}$ is indeed closed under the operations. Let X, Y be subsets of J. If either X or Y is in F, then so is $X \cup Y$, because F is a filter. Hence

$$R_X \cup R_Y = \bigcup\{Q_k : k \in X\} \cup \bigcup\{Q_k : k \in Y\} \cup Q_0 = R_{X\cup Y}$$

If neither X nor Y is in F, then $X \cup Y$ is not in F, because F is an ultrafilter.

$$R_X \cup R_Y = \bigcup\{Q_k : k \in X\} \cup \bigcup\{Q_k : k \in Y\} = R_{X\cup Y}$$

Thus A is closed under finite unions. Now suppose that X is the complement of Y in J. Since F is an ultrafilter exactly one of them, say X is in F. Hence,

$$\sim R_X = \sim \bigcup\{Q_k : k \in X \cup \{0\}\} = \bigcup\{Q_k : k \in Y\} = R_Y$$

so that $\mathfrak{A}$ is closed under complementation (w.r.t R_J). We check substitutions. Transpositions are clear, so we check only replacements. It is not too hard to show that

$$S_0^1(R_X) = \begin{cases} \emptyset & \text{if } X \notin F, \\ R_{\mathbb{Z}^+} & \text{if } X \in F \end{cases}$$

Now

$$\sum\{S_0^1(R_k) : k \in J\} = \emptyset.$$

and

$$S_0^1(R_J) = R_J$$

$$\sum\{R_{\{k\}} : k \in J\} = R_J = \bigcup\{Q_k : k \in J\}.$$

Thus

$$S_0^1(\sum\{R_{\{k\}} : k \in J\}) \neq \sum\{S_0^1(R_{\{k\}}) : k \in J\}.$$

The algebra required is that generated by the κ many atoms. Finally, this algebra cannot posses a complete represenation, for any such representation implies the complete additivity of the substitution operations as indicated above.

□

The following answers a question of Hodkinson's.

Corollary 2.2. *By discarding replacements, we obtain that Pinter's atomic algebras may not be completely representable*

There is a wide spread belief, almost permenantly established that like cylindric algebras, any atomic *poyadic algebras of dimension* 2 is completely representable. This is wrong. The above example, indeed shows that it is not the case, because the set algebras consrtucted above , if we impose the additional condition that each Q_n has U as its domain and range, then the algebra in question becomes closed under the first two cylindrfiers, and by the same reasoning as above, it *cannot* be completely representable.

Theorem 2.3. *The class of atomic polyadic algebras of dimension* 2 *is elementary. schema*

Proof. Let $\mathsf{At}(x)$ be the first order formula asserting that x is an atom, namely, $\mathsf{At}(x)$ is the formula $x \neq 0 \wedge (\forall y)(y \leq x \to y = 0 \vee y = x)$. We first assume that $n > 1$ is finite, the other cases degenerate to the Boolean case. For distinct $i, j < 2$ let $\psi_{i,j}$ be the formula: $y \neq 0 \to \exists x(\mathsf{At}(x) \wedge s_i^j x \neq 0 \wedge s_i^j x \leq y)$. Let Σ be obtained from the axiomatization PA_n by adding $\psi_{i,j}$ for every distinct $i, j \in 2$. Then $CRPA_2 = \mathbf{Mod}(\Sigma)$. □

3 Weakly representable atom structures that are not strongly representable

For a fixed graph $\mathbf{G}$, we define a family of labelled graphs $\mathcal{F}$ such that every edge of each graph $\Gamma \in \mathcal{F}$, is labelled by a unque label from $\mathbf{G} \cup \{\rho\}$, $\rho \notin \mathbf{G}$. Then one forms a labelled graph M which can be viewed as model of a natural signiture, namely, the one with relation symbols (a, i), for each $a \in \mathbf{G} \cup \{\rho\}$ and $i < n$. This M can be constructed as a limit of finite structures, in the spirit of Fraisse constructions. Then one takes a subset $W \subseteq {}^nM$, by roughly dropping assignments that do not satify (ρ, l) for every $l < n$. Formally, $W = \{\bar{a} \in {}^nM : M \models (\bigwedge_{i<j<n, l<n} \neg(\rho, l)(x_i, x_j))(\bar{a})\}$. All this can be done with an arbirary graph.

Now for particular choices of $\mathbf{G}$; the algebra relativized set algebras based on M, but taking only sequences in W in L_n is an atomic representable algebra. This algebra has universe $\{\phi^M : \phi \in L_n\}$ where $\phi^M = \{s \in W : M \models \phi[s]\}$. Its completion is the relatvized sets algebras consisting of ϕ^M, $\phi \in L_{\infty,\infty}$, which turns out not representable. (All logics are taken in the above signature). In fact, we will show that for certian choices of $\mathbf{G}$, it will not be even in $\mathbf{S}\mathfrak{Nr}_n CA_{n+2}$. Let us get more technical.

Example 3.1. (1) A *labelled graph* is an undirected graph Γ such that every edge (*unordered* pair of distinct nodes) of Γ is labelled by a unique label from $(\mathbf{G} \cup \{\rho\}) \times n$, where $\rho \notin \mathbf{G}$ is a new element. The colour of (ρ, i) is defined to be i. The *colour* of (a, i) for $a \in \mathbf{G}$ is i.

Now we define a class $\mathfrak{GG}$ of certain labelled graphs.The class $\mathfrak{GG}$ consists of all complete labelled graphs Γ (possibly the empty graph) such that for all distinct $x, y, z \in \Gamma$, writing $(a, i) = \Gamma(y, x)$, $(b, j) = \Gamma(y, z)$, $(c, l) = \Gamma(x, z)$, we have:

(1) $|\{i, j, l\} > 1$, or

(2) $a, b, c \in \mathbf{G}$ and $\{a, b, c\}$ has at least one edge of $\mathbf{G}$, or

(3) exactly one of a, b, c – say, a – is ρ, and bc is an edge of $\mathbf{G}$, or

(4) two or more of a, b, c are ρ.

(5) There is a countable labelled graph $M \in \mathfrak{GG}$ with the following property:

If $\triangle \subseteq \triangle' \in \mathfrak{GG}$, $|\triangle'| \leq n$, and $\theta : \triangle \to M$ is an embedding, then θ extends to an embedding $\theta' : \triangle' \to M$.

Let L^+ be the signature consisting of the binary relation symbols (a, i), for each $a \in \mathbf{G} \cup \{\rho\}$ and $i < n$. Let $L = L^+ \setminus \{(\rho, i) : i < n\}$. From now on, the logics $L^n, L^n_{\infty\omega}$ are taken in this signature. We may regard any non-empty labelled graph equally as an L^+-structure, in the obvious way.

(6) Let $W = \{\bar{a} \in {}^nM : M \models (\bigwedge_{i<j<n, l<n} \neg(\rho, l)(x_i, x_j))(\bar{a})\}$. For an $L^n_{\infty\omega}$-formula φ, we define φ^W to be the set $\{\bar{a} \in W : M \models_W \varphi(\bar{a})\}$, an we let $\mathfrak{A}$ to be the relativised set algebra with domain

$$\{\varphi^W : \varphi \text{ a first-order } L^n - \text{formula}\}$$

and unit W, endowed with the algebraic operations $\mathsf{d}_{ij}, \mathsf{c}_i$, ect., in the standard way . Fix finite $N \geq n(n-1)/2$.

$\mathbf{G}$ can be any graph that contains infinitely countably many cliques (complete subgraphs) each of size N. For example it can be $\mathbf{G} = (\mathcal{N}, E)$ with nodes $\mathcal{N}$ and i, l is an edge i.e $(i, l) \in E$ if $0 < |i - l| < N$, or a countable union of cliques, denote by $N \times \omega$.

Now let G be an infinite countable graph that contains infinitely many N cliques. Then $\mathfrak{A}$ is a representable (countable) atomic polyadic algebra but $\mathfrak{Rd}_{ca}\mathcal{C} \notin S\mathfrak{Nr}_n\mathbf{CA}_{n+2}$, its complex lagebra is isomorphic to the algebra consisting of formula in L_∞ is not representable. Further, $\mathfrak{A}$ is acually isomorphic to the term algebra over its atom structure.

The above example can be easily transferred to relation algebras as follows:

Example 3.2. We define a relation algebra atom structure $\alpha(\mathbf{G})$ of the form $(\{1'\} \cup (\mathbf{G} \times n), R_{1'}, \breve{R}, R_;)$. The only identity atom is $1'$. All atoms are self converse, so $\breve{R} = \{(a, a) : a \text{ an atom }\}$. The colour of an atom $(a, i) \in \mathbf{G} \times n$ is i. The identity $1'$ has no colour. A triple (a, b, c) of atoms in $\alpha(\mathbf{G})$ is consistent if $R; (a, b, c)$ holds. Then the consistent triples are (a, b, c) where

- one of a, b, c is $1'$ and the other two are equal, or
- none of a, b, c is $1'$ and they do not all have the same colour, or
- $a = (a', i), b = (b', i)$ and $c = (c', i)$ for some $i < n$ and $a', b', c' \in \mathbf{G}$, and there exists at least one graph edge of G in $\{a', b', c'\}$.

$\alpha(\mathbf{G})$ can be checked to be a relation atom structure. The atom structure of $\mathfrak{Rd}_{ca}\mathfrak{A}$ is isomorphic (as a cylindric algebra atom structure) to the atom structure $\mathcal{M}_n$ of all n-dimensional basic matrices over the relation algebra atom structure $\alpha(\mathbf{G})$. Indeed, for each $m \in \mathcal{M}_n$, let $\alpha_m = \bigwedge_{i,j<n} \alpha_{ij}$. Here α_{ij} is $x_i = x_j$ if $m_{ij} =$ 1' and $R(x_i, x_j)$ otherwise, where $R = m_{ij} \in L$. Then the map $(m \mapsto \alpha_m^W)_{m \in \mathcal{M}_n}$ is a well - defined isomorphism of n-dimensional cylindric algebra atom structures. We can show that thae $\mathfrak{Cm}\alpha(\mathbf{G})$ is not representable like exactly [weak] using Ramseys theore. Here we show something stronger.

Theorem 3.3. *We have $\mathfrak{Cm}\alpha(\mathbf{G})$ is not in $S\mathfrak{Ra}\mathbf{CA}_{n+2}$.*

Proof. The idea is to use relativized representations. Such algebras are localy representable, but the epresentation is global enough so that Ramseys theorem applies. Hence the full complex cylindric algebra over the set of n by n basic matrices - which is isomorphic to $\mathcal{C}$ is not in $S\mathfrak{Nr}_n\mathbf{CA}_{n+2}$ for we have a relation algebra embedding of $\mathfrak{Cm}\alpha(\mathbf{G})$ onto $\mathfrak{Ra}\mathfrak{Cm}\mathcal{M}_n$. Assume for contradiction that $\mathfrak{Cm}\alpha(\mathbf{G}) \in S\mathfrak{Ra}\mathbf{CA}_{n+2}$. Then $\mathfrak{Cm}\alpha(\mathbf{G})$ has an n-flat representation M [11] 13.46, which is n square [11] 13.10. In particular, there is a set M, $V \subseteq M \times M$ and $h : \mathfrak{Cm}\alpha(\mathbf{G}) \to \wp(V)$ such that $h(a)$ $(a \in \mathfrak{Cm}\alpha(\mathbf{G}))$ is a binary relation on M, and h respects the relation algebra operations. Here $V = \{(x, y) \in M \times M : M \models 1(x, y)\}$, where 1 is the greatest element of $\mathfrak{Cm}\alpha(\mathbf{G})$. A clique C of M is a subset of the domain M such that for $x, y \in C$ we have $M \models 1(x, y)$, equivalently $(x, y) \in V$. Since M is $n + 2$ square, then for all cliques C of M with $|C| < n+2$, all $x, y \in C$ and $a, b \in \mathfrak{Cm}\alpha(\mathbf{G})$, $M \models (a; b)(x, y)$ there exists $z \in M$ such that $C \cup \{z\}$ is a clique and $M \models a(x, z) \wedge b(z, y)$. For $Y \subseteq \mathcal{N}$ and $s < n$, set
$$[Y, s] = \{(l, s) : l \in Y\}.$$
For $r \in \{0, \ldots N - 1\}$, $N\mathcal{N} + r$ denotes the set $\{Nq + r : q \in \mathcal{N}\}$. Let
$$J = \{1', [N\mathcal{N} + r, s] : r < N, s < n\}.$$

Then $\sum J = 1$ in $\mathfrak{Cm}\alpha(\mathbf{G})$. As J is finite, we have for any $x, y \in M$ there is a $P \in J$ with $(x, y) \in h(P)$. Since $\mathfrak{Cm}\alpha(\mathbf{G})$ is infinite then M is infinite. By Ramsey's Theorem, there are distinct $x_i \in X$ $(i < \omega)$, $J \subseteq \omega \times \omega$ infinite and $P \in J$ such that $(x_i, x_j) \in h(P)$ for $(i, j) \in J$, $i \neq j$. Then $P \neq 1'$. Also $(P; P) \cdot P \neq 0$. This follows from $n+2$ squareness and that if $x, y, z \in M$, $a, b, c \in \mathfrak{Cm}\alpha(\mathbf{G})$, $(x, y) \in h(a)$, $(y, z) \in h(b)$, and $(x, z) \in h(c)$, then $(a; b) \cdot c \neq 0$. A non -zero element a of $\mathfrak{Cm}\alpha(\mathbf{G})$ is monochromatic, if $a \leq 1'$, or $a \leq [\mathcal{N}, s]$ for some $s < n$. Now P is monochromatic, it follows from the definition of α that $(P; P) \cdot P = 0$. This contradiction shows that $\mathfrak{Cm}\alpha(\mathbf{G})$ is not in $S\mathfrak{Ra}\mathbf{CA}_{n+2}$. Hence $\mathfrak{Cm}\mathcal{M}_n \notin S\mathfrak{Nr}_n\mathbf{CA}_{n+2}$.

□

We have not seen a publication of ths result, though its proof can be easily destilled from known rather involved proofs.

Lemma 3.4. *Let $\mathfrak{D}$ be a polyadic equality algebra of dimension $n \geq 3$, that is generated by the set $\{x \in D : \Delta x \neq n\}$. Then if $\mathfrak{Rd}_{qa}\mathfrak{D}$ is completely representable, then so is $\mathfrak{D}$.*

Proof. First suppose that $\mathfrak{D}$ is simple, and let $h : \mathfrak{D} \to \wp(V)$ be a complete representation, where $V = \prod_{i<n} U_i$ for sets U_i. We can assume that $U_i = U_j$ for all $i, j < n$, and if $s \in V$, $i, j < n$ and $a_i = a_j$ then $a \in h(d_{ij}$. Indeed, let $\delta = \prod d_{ij} \in \mathfrak{D}$. As C is a cyilndric algebra, we have $c_{(n)}\delta = 1$, so for each $u \in U_i$ there is an $s \in h(\delta)$ with $a_i = u$. So there exists a function $s_i : U_i \to h(\delta)$ such that $(s_i(u))_i = u$ for each $u \in U_i$.

Let U be the disjoint union of the U_is. Let $t_i : U \to U_i$ be the surjection defined by $t_i(u) = (s_j(u))_i$. Let $g : \mathfrak{D} \to \wp({}^nU)$ be defined via

$$d \mapsto \{s \in {}^nU : (t_0(a_0), \ldots, t_{n-1}(a_{n-1})) \in h(d)\}.$$

Then g is a complete representation of $\mathfrak{D}$. Now suppose $s \in {}^nU$, satisfies $s_i = s_j$ with $a_i \in U_k$, say, where $k < n$. Let $\bar{b} = s_k(a_i) = s_k(a_j) \in h(\delta)$. Then $t_i(a_i) = b_i$ and $t_j(a_j) = b_j$, so $(t_i(a_i) : i < n)$ agrees with $\bar{b}$ on coordinates i, j. Since $\bar{b} \in h(\delta)$ and $\Delta d_{ij} = \{i, j\}$, then $(t_i(a_i) : i < n) \in h(d_{ij}$ and so $s \in g(d_{ij})$, as required.

Now define $\sim_{ij} = \{(a_i, a_j) : \bar{a} \in h(d_{ij})$. Then it easy to check that $\sim_{01} = \sim_{i,j}$ is an equivalence relation on U. For $s, t \in {}^nU$, define $s \sim t$, if $s_i \sim t_i$ for each $i < n$, then $\sim$ is an equivalence relation on ${}^nU$. Let

$$E = \{d \in D : h(d) \text{ is a union of } \sim \text{ classes }\}.$$

Then

$$\{d \in D : \Delta d \neq n\} \subseteq E.$$

Furthermore, E is the domain of a complete subalgebra of $\mathfrak{C}$. Let us check this. We have $\{0, 1, d_{ij} : i, j < n\} \subseteq E$, since $\Delta 0 = \Delta 1 = \emptyset$ and $\Delta d_{ij} = \{i, j\} \neq$

n (as $n \geq 3$). If $h(d)$ is a union of $\sim$ classes, then so is ${}^nU \setminus h(d) = h(-d)$. If $S \subseteq E$ and $\sum S$ exists in $\mathfrak{D}$, then because h is complete representation we have $h(\sum^{\mathfrak{D}} S) = \bigcup h[S]$, a union of $\sim$ classes so $\sum S \in E$. Hence $E = C$. Now define $V = U/\sim_{01}$, and define $g : \mathfrak{C} \to \wp({}^nV)$ via

$$c \mapsto \{(\bar{a}/\sim_{01}) : \bar{a} \in h(c).$$

Then g is a complete representation.

Now we drop the assumption that $\mathfrak{D}$ is simple. Suppose that $h : \mathfrak{D} \to \prod_{k \in K} Q_k$ is a complete representation. Fix $k \in K$, let $\pi_k : Q \to Q_k$ be the canonical projection, and let $\mathfrak{D}_k = rng(\pi_k \circ h)$. We define diagonal elements in $\mathfrak{D}_k$ by $d_{ij} = \pi_k(h^{\mathfrak{C}}(d_{ij}))$. This expands $\mathfrak{D}_k$ to a cylindric-type algebra $\mathfrak{C}_k$ that is a homomorphic image of $\mathfrak{C}$, and hence is a cylindric algebra with diagonal free reduct $\mathfrak{D}_k$. Then the inclusion map $i_k : \mathfrak{D}_k \to Q_k$ is a complete representation of D_k. Since obviously

$$\pi_k[h[\{c \in C : \Delta c \neq n\}] \subseteq \{c \in C_k : \Delta c \neq n\}$$

and π_k, h preserve arbitrary sums, then C_k is completely generated by $\{c \in C_k : \Delta c \neq n\}$. Now $c_{(n)}x$ is a discriminator term in Q_k, so D_k is simple. So by the above $\mathfrak{C}_k$ has complete represenation $g_k : \mathfrak{C}_k \to Q'_k$. Define $g : \mathfrak{C} \to \prod_{k \in K} Q'_k$ via

$$g(c)_k = g_k(\pi_k(h(c))).$$

Then g defines a complete representation. □

Example 3.5. In definition 3.6.3 [11] a cylindric atom structure is defined from a family K of L structures, closed under forming subalgebra. This class is formulated in a language L of relation symbols $< n$. Call this atom structure $\rho(K)$. The atom structure, can be turned easily into a polyadic equality atom structure by defining accesibility relations correponding to the substituton $s_{i,j}$ by: $R_{ij} = \{[f], [g] : f, g \in \mathcal{F} : f = g \circ [i, j]\}$.

Two examples are given of such clases, what concerns us is the second (rainbow) class defined in 3.6. 9, referred to as classes based on on graph. Fix a graph Γ. The rainbow polyadic equality algebra based on this graph is denoted by $R(\Gamma)$ is the complex algebra of $\rho(K(\Gamma))$, namely $\mathfrak{C}m\rho(K(\Gamma))$. It is proved that If Γ is a countable graph, then the cylindric algebr $R(\Gamma)$ is completely reprsentable if and only if Γ contains a reflexive node or an infinite clique, This proof can be checked to work for polyadic equality algebras, and by our previous lemma, it also works for polyadic algebras.

Define K_k and Γ as in corollary 3.7.1 in [11]. Then $R(\Gamma)$ is s completely representable. But Γ has arbirary large cliques, hence it is elementay equivalent to a countable graph Δ with an infinite clique. Then $R(\Delta) \equiv R(\Gamma)$, and by the above chracteization the latter is completely representable, the former is not. Notice that $\Delta \equiv \Gamma$ as first order structures.

3.1 The infinite dimensional case

Let us try to extend the result concerning existence of weaky representable atom structures that are not strongly so. If we insist on using graphs and model theory we wil have to change our base logic, to allows infnitary formulas. For simplicity we consider the arity of formulas to be at most ω. L^ω is a quantifier logic that allows infinitary predicates of arbitrary rank, and otherwise is like first order logic, in particular quantification can be taken only on finitely many variables. L^ω_∞ is the logic obtained from L^ω by adding infinite conjunctions. Let $\mathbf{G}$ be a graph.

(1) A *labelled graph* is an undirected graph Γ such that every edge (*unordered* pair of distinct nodes) of Γ is labelled by a unique label from $(\mathbf{G} \cup \{\rho\}) \times \omega$, where $\rho \notin \mathbf{G}$ is a new element. The colour of (ρ, i) is defined to be i. The *colour* of (a, i) for $a \in \mathbf{G}$ is i. Now we define a class $\mathfrak{G}\mathfrak{G}$ of certain labelled graphs.The class $\mathfrak{G}\mathfrak{G}$ consists of all complete labelled graphs Γ (possibly the empty graph) such that for all distinct $x, y, z \in \Gamma$, writing $(a, i) = \Gamma(y, x)$, $(b, j) = \Gamma(y, z)$, $(c, l) = \Gamma(x, z)$, we have:

   (1) $|\{i, j, l\} > 1$, or

   (2) $a, b, c \in \mathbf{G}$ and $\{a, b, c\}$ has at least one edge of $\mathbf{G}$, or

   (3) exactly one of a, b, c – say, a – is ρ, and bc is an edge of $\mathbf{G}$, or

   (4) two or more of a, b, c are ρ.

(5) There is a countable labelled graph $M \in \mathfrak{G}\mathfrak{G}$ with the following property:

If $\triangle \subseteq \triangle' \in \mathfrak{G}\mathfrak{G}$, $|\triangle'| \leq n$, and $\theta : \triangle \to M$ is an embedding, then θ extends to an embedding $\theta' : \triangle' \to M$.

Let L^+ be the signature consisting of the binary relation symbols (a, i), for each $a \in \mathbf{G} \cup \{\rho\}$ and $i < \omega$. Let $L = L^+ \setminus \{(\rho, i) : i < \omega\}$. From now on, the logics $L^\omega, L^\omega_\infty$ are taken in this signature. Fix $p \in {}^\omega M$, and let $V = {}^\omega M^{(p)}$. For $\bar{a} \in V$ and $\phi \in L^\omega_\infty$ satifiability is defined the usual Tarskian way. For a formula ϕ, we write $\phi^{\mathfrak{M}}$ for all asignments in V that satisfy M.

(6) Let $W = \{\bar{a} \in V : M \models (\bigwedge_{i<j<\omega, l<\omega} \neg(\rho, l)(x_i, x_j))(\bar{a})\}$. For an L^ω_∞ formula φ, we define φ^W to be the set $\{\bar{a} \in W : M \models_W \varphi(\bar{a})\}$, an we let $\mathfrak{A}$ to be the relativised set algebra with domain

$$\{\varphi^W : \varphi \text{ an } L^\omega - \text{formula}\}$$

and unit W, endowed with the algebraic operations $\mathsf{d}_{ij}, \mathsf{c}_i$, ect., in the standard way . Let $\mathfrak{C}$ be the algebra with base φ^W in L^ω_∞ and operations as above.

For $\mathfrak{A}$ to a a representable (countable) atomic polyadic algebra, we need a graph with arbitrary large cliques. For $\mathfrak{C}$ its completion to be non-representable, we need a finite chromatic number to apply Ramseys theorem. These two conditions are incompatible. However, it might be possible in this context, to use the Erdos-Rado theorem, extending Ramseys theorem to the uncountable case, by noting that a representation of the complex algebra must have an uncountable base.

4 Two polyadic atom structures equivalent in $L_{\infty,\omega}$, generating (in their complex algebra) two polyadic algebras one in $\mathfrak{Nr}_n\mathbf{PA}_\omega$ and the other not in $\mathfrak{Nr}_n\mathbf{PA}_{n+1}$.

A class closely related to the class of completely representable algebras is that of neat reducts; the completely representable algebras are those tha have a strong neat embedding property. This characterization works even for the infinite dimensional case, if we take weak structures. But in all cases it only adresses the countable case [14], [15].

Both classes are non elementary for all dimensions > 2. For quasipolyadic algebras of infinite dimensions, however, it is not known whether atomic algebras are completely representable or not. This is anther result for which there is an unbased feeling in the air that it is true. Both classes are psuedo-elementary.

But now we show that there is a very important diference. An atom structure which is completely representable, have all atomic algebras based on it completely representable, but this is not the case for neat reducts. The former class is not elementary, and it seems that the class of atom structures for which algebras based are neat reducts is also not- elementary. (We are a little bit careless about the number of extra dimensions in the neat reduct, but its variation leads to the richness of the problem. We could require that both algebras have the same nuber of extra dimensions, but we can also not asume that. We have not pursued this matter any further). Next we give results results concerning neat reducts, for cylindric and polyadic algebras. An atom structure of dimension α is (strongly) neat if (every) some algebra based on this atom structure is in $\mathfrak{Nr}_\alpha\mathbf{CA}_{\alpha+\omega}$.

Theorem 4.1. *For every ordinal $\alpha > 1$, there exists a neat atom structure of* *dimension α, that is not strongly neat.*

Proof. Let $\alpha > 1$ and $\mathfrak{F}$ is field of characteristic 0. Let

$$V = \{s \in {}^{\alpha}\mathfrak{F} : |\{i \in \alpha : s_i \neq 0\}| < \omega\},$$

Note that $V$is a vector space over the field $\mathfrak{F}$. We will show that V is a weakly neat atom structure that is not strongly neat. Indeed V is a concrete atom structure $\{s\} \equiv_i \{t\}$ if $s(j) = t(j)$ for all $j \neq i$, and $\{s\} \equiv_{ij} \{t\}$ if $s \circ [i, j] = t$.

Let $\mathfrak{C}$ be the full complex algebra of this atom structure, that is

$$\mathfrak{C} = (\wp(V), \cup, \cap, \sim, \emptyset, V, \mathsf{c}_i, \mathsf{d}_{ij}, \mathsf{s}_{ij})_{i,j\in\alpha}.$$

Then clearly $\wp(V) \in \mathfrak{Nr}_\alpha \mathbf{CA}_{\alpha+\omega}$. Indeed Let $W = {}^{\alpha+\omega}\mathfrak{F}^{(0)}$. Then $\psi : \wp(V) \to \mathfrak{Nr}_\alpha \wp(W)$ defined via

$$X \mapsto \{s \in W : s \upharpoonright \alpha \in X\}$$

is an isomomorphism from $\wp(V)$ to $\mathfrak{Nr}_\alpha \wp(W)$. We shall construct an algebra $\mathfrak{A}$ such that $\mathsf{At}\mathfrak{A} \cong V$ but $\mathfrak{A} \notin \mathfrak{Nr}_\alpha \mathbf{CA}_{\alpha+1}$.

Let y denote the following α-ary relation:

$$y = \{s \in V : s_0 + 1 = \sum_{i>0} s_i\}.$$

Note that the sum on the right hand side is a finite one, since only finitely many of the s_i's involved are non-zero. For each $s \in y$, we let y_s be the singleton containing s, i.e. $y_s = \{s\}$. Define $\mathfrak{A} \in WQEAs_\alpha$ as follows:

$$\mathfrak{A} = \mathfrak{Sg}^{\mathfrak{C}}\{y, y_s : s \in y\}.$$

We shall prove that

$$\mathfrak{Rd}_{SC}\mathfrak{A} \notin \mathfrak{Nr}_\alpha SC_{\alpha+1}.$$

That is for no $\mathfrak{P} \in SC_{\alpha+1}$, it is the case that $\mathfrak{Sg}^{\mathfrak{C}}X$ exhausts the set of all α dimensional elements of $\mathfrak{P}$. So assume, seeking a contradiction, that $\mathfrak{Rd}_{SC}\mathfrak{A} \in \mathfrak{Nr}_\alpha SC_{\alpha+1}$. Let $X = \{y_s : s \in y\}$. Of course every element of X, being a singleton, is an atom. Next we show that $\mathfrak{A}$ is atomic, i.e evey non-zero element contains a minimal non-zero element. Towards this end, let $s \in {}^{\alpha}\mathfrak{F}^{(\mathbf{0})}$ be an arbitrary sequence. Then

$$\langle s_0, s_0 + 1 - \sum_{i>1} s_i, s_i \rangle_{i>1}$$

and

$$\langle \sum_{0<i<\alpha} s_i - 1, s_i \rangle_{i\geq 1}$$

are elements in y. Since

$$\{s\} = \mathsf{c}_1\{\langle s_0, s_0 + 1 - \sum_{i>1} s_i, s_i\rangle_{i>1}\} \cap \mathsf{c}_0\{\langle \sum_{0\neq i<\alpha} s_i - 1, s_i\rangle_{i\geq 1}\},$$

It follows that $\mathfrak{A}$ has the same atom structure.

■

5 Stronger Logics

We now show that logics like $L_{\kappa,\omega}$ and $L_{\infty,\omega}$ cannot characterize the class of neat reducts.The second case is of course much stronger. The first cast can be destilled from the case in [13], by simple modifications. First we let our language have κ^+ predicate symbols (instead just countably many). In this case $\mathfrak{A}_u$, as defined in [13] will have cardinality κ^+. Then we alter the *uth* component, and its permuted versions, by inserting in a Boolean algebra that $L_{\kappa,\omega}$ equivalent to $\mathfrak{A}$, whose cardinality is κ. The rest of the proof works.

But now we prove the stronger result and this needs a more drastic change. We will make our components *atomic* Boolean algebras, and for this we require that the basic relations defined in [13] not only distinct, but *disjoint.*. This is necessary if we want atomic algebras. We use a different more basic method to contruct our desired model, which has apperaed in previous publications of ours , in related contexts; and has proved to be quite a nut cracker in these kinds of problems. We include proof for the readers conveniance.

$(R, +)$ denotes an arbitray uncountable group, and $n = \{0, \cdots, n-1\}$ denotes a fixed finite ordinal > 1.

Definition 5.1. Let $k < \omega$. Then $S(n, k)$ denotes the set of sequences $\langle i_0, \cdots, i_{n-1}\rangle$ such that $i_0 \leq i_1 \cdots \leq i_{n-1} = k$. $Cof^+(R)$ denotes the set of all nonempty finite or cofinite subsets of R, i.e.

$$Cof^+(R) = \{X \subseteq R : X \text{ is non empty, and } X \text{ or } R - X \text{ is finite}\}.$$

Let C_r be an n-ary relation symbol for every $r \in R$. For any finite $X \subseteq R$, we define the formulas:

$$\eta(X) = \vee\{C_r(x_0, \cdots, x_{n-1}) : r \in X\}, \quad \text{and}$$

$$\eta(R - X) = \neg\eta(X) = \wedge\{\neg C_r(x_0, \cdots, x_{n-1}) : r \in X\}.$$

Let U be a set and E an equivalence relation on U. Then we write xEy if $(x, y) \in E$. We write $xE'y$ if $(x, y) \notin E$. Suppose that E has distinct n equivalence classes, or blocks . Then we write $D_E(x_0, x_1 \cdots x_{n-1})$ for the formula $\bigwedge_{0\leq i<j<n} x_iE'x_j$ asserting that x_i, x_j are pairwise unrelated according to E, for all $i < j < n$. That is for all $s \in {}^nU$, $D_E(s_0, \cdots, s_{n-1})$ iff the s_i's belong to distinct blocks.

Theorem 5.2. *There are a set W, an equivalence relation E with n blocks on W, and n-ary relations $C_r \subseteq {}^nW$ for all $r \in R$, such that conditions (i)-(v) below hold:*

(i) $C_r(w_0, w_1 \cdots, w_{n-1})$ implies $D_E(w_0, w_1 \cdots, w_{n-1})$ for all $r \in R$, and for all $w_0, \cdots, w_{n-1} \in W$.

(ii) $C_r(w_0, \cdots, w_{n-1})$ implies $C_r(w_{\pi(0)}, \cdots, w_{\pi(n-1)})$ for all $r \in R$, $w_0, \cdots, w_{n-1} \in W$ and permutation π of n.

(iii) For all $r \in R$ and for all $w_0, w_1, \cdots w_{n-2}$ in W such that $w_i E' w_j$ whenever $i < j < n-1$, there exists $w_{n-1} \in W$ such that $C_r(w_0, w_1, \cdots w_{n-1})$.

(iv) For all $k \in \omega$, for all distinct $w, w_0 \cdots, w_{k-1} \in W$, and for any function

$f : S(n,k) \to Cof^+(R)$, there is a $w_k \in W \setminus \{w_0, \cdots, w_{k-1}\}$ such that $w_k E w$, i.e. w_k is in the same block as w, and

$$\bigwedge \{D(w_{i_0}, w_{i_1}, \cdots, w) \implies \eta(f(i))[w_{i_0}, w_{i_1}, \cdots, w_{i_{n-1}}] : i \in S(n,k)\}.$$

(v) The C_r's are pairwise disjoint.

(vi) $C_{r_1}; C_{r_2} = C_{r_1+r_2}$

Proof. We shall construct the structure $\langle W, C_r \rangle_{r\in R}$ by a routine step by step fashion. We note that condition (iii) follows from (iv). Often, however, we will only need (and refer to) the weaker condition (iii), hence the redundancy in the formulation of Lemma 1. Let $I({}^k(|R|))$ be the set of all injections, i.e. one to one functions from k to $|R|$. Let

$$Q = \cup\{I({}^k(|R|)) \times^{S(n,k)} Cof^+(R) : k < \omega\}.$$

Then $|Q| = |R| = \mu$, say. Roughly Q stands for the set of all tasks that we have to exhaust. We will construct a set W with cardinality μ. Q is intended to represent all the instances of condition (iv) as follows: An element of Q is of the form $\langle \alpha_0, \cdots, \alpha_{k-1}, f\rangle$, where $\alpha_0, \cdots, \alpha_{k-1} < \mu$ and $f : S(n,k) \to Cof^+(R)$. This represents the instance of (iv) where we take $k, w_{\alpha_0}, \cdots, w_{\alpha_{k-1}}, f$ as the *concrete* values of the quantified items in (iv). Let ρ be an enumeration of Q such that: for all $l < \mu$, for all $q \in Q$, there exists j, with $l < j < \mu$, such that $\rho(j) = q$. Such a ρ clearly exists. Fix a well ordering $\prec$ of R. Let $l < \mu$, and suppose that for all $i < l$ we have already defined the element w_i, and the n-ary relation $C_r^i \subseteq {}^nW_i$, where $W_i = \{w_k : k < i\}$, and C_r^i and W_i satisfy all the conditions with the possible exception of (iv). In the l'th step we will make the $\rho(l)$'th instance of (iv) true. Assume that $\rho(l) = \langle \alpha_0, \cdots, \alpha_{k-1}, f\rangle$. Then

$\alpha_0, \cdots, \alpha_{k-1} < \mu$ and $f : S(n,k) \to Cof^+R$, for some k. ($k = 0$ is allowed, too). Let w_l be an element not in $\cup\{W_i : i < l\}$. If there exists $i < k$ such that $l \leq \alpha_i$, then for all $r \in R$, we define $C_r^l = \cup\{C_r^i : i < l\}$. Else, $l > \alpha_i$ for all $i < k$. In this case, let $v_0 = w_{\alpha_0}, \cdots,\ v_{k-1} = w_{\alpha_{k-1}}$ and $v_k = w_l$. For all $r \in R$ we define:

$$\begin{aligned} X_r^l = \{\langle v_{i_0}, \cdots, v_{i_{n-1}}\rangle \ &:\ i_0 \leq \cdots \leq i_{n-1} = k \\ &\quad \text{and } r \text{ is the } \prec\text{-least element of } f(\langle i_0, \cdots, i_{n-1}\rangle)\} \end{aligned}$$

and

$$C_r^l = \cup\{C_r^i \cup \{\langle s_{\pi(0)}, \cdots, s_{\pi(n-1)}\rangle : s \in X_r^l, \pi \text{ is a permutation of } n\} \ :\ i < l\}.$$

Finally we set

$$W = \cup\{W_l : l < \mu\} \quad \text{and } C_r = \cup\{C_r^l : l < \mu\}.$$

Now we are going to check that the structure $\langle W, C_r\rangle_{r\in R}$, so defined, satisfies conditions (i)-(v). To this end, for any $l < \mu$ and $r \in R$ let

$$Y_r^l = \{\langle s_{\pi(0)}, \cdots, s_{\pi(n-1)}\rangle : s \in X_r^l \text{ and } \pi \text{ is a permutation on } n\}$$

and

$$D_r^l = \cup\{C_r^j : j < l\}.$$

Then for all $r \in R$ we have $C_r^l = Y_r^l \cup D_r^l$. Also, the following are not difficult to check:

(1) $w_l \in Rgs$ (the range of s) if $s \in Y_r^l$, and $w_l \notin Rgs$ if $s \in D_r^l$.

(2) $C_r^j \subseteq C_r^m$ if $j \leq m < \mu$.

It is easy to show by induction on $l < \mu$ that for all $r \in R$, C_r^l is symmetric, and if $s \in C_r^l$ then s satisfies D_E, i.e. $s(i)$ and $s(j)$ are in distinct blocks for $0 \leq i < j < n$. Thus C_r satrisfies (i) and is symmetric and so C_r satisfies (ii), too. Now let $r, p \in R$ be distinct. We want to show by induction on $l < \mu$ that C_r^l and C_p^l are disjoint. Now D_r^l and D_p^l are disjoint by the induction hypothesis and by (2). By (1), it is therefore, enough to show that Y_r^l and Y_p^l are disjoint. Assume $s \in Y_r^l$, and let $\rho(l) = \langle \alpha_0, \cdots, \alpha_{k-1}, f\rangle$ and $v = \langle v_0, \cdots, v_k\rangle = \langle w_{\alpha_0}, \cdots, w_{\alpha_{k-1}}, w_l\rangle$. Then v is one to one, since the w_{α_j}'s are pairwise distinct and $w_l \neq w_{\alpha_j}$ for all $j < k$, by its very choice. It follows thus that there are a unique $i \in S(n,k)$ and permutation π of n such that $s = \langle z_{\pi(0)}, \cdots, z_{\pi(n-1)}\rangle$, where $z = \langle v_{i_0}, \cdots, v_{i_{n-1}}\rangle \in X_r^l$. Thus r is the $\prec$-least element of $f(i)$, by $z \in X_r^l$. Since $p \neq r$, we get that $z \notin X_p^l$, and so $z \notin Y_p^l$. We have shown that C_r^l and C_p^l are disjoint. By (2) the C_r's are pairwise disjoint, i.e. condition (v) holds. Finally we check condition(s) (iv) (and (iii)): Let $k < \omega$, $w_{\alpha_0}, \cdots, w_{\alpha_{k-1}} \in W$ be distinct and let $w \in W$. Let

$f : S(n,k) \to Cof^+R$. Let $l < \mu$ be such that $\rho(l) = \langle \alpha_0, \cdots, \alpha_{k-1}, f \rangle$ and $l > \alpha_0, \cdots, l > \alpha_{k-1}$. Such an l exists by the properties of ρ. Then it is not difficult to check that we constructed the w_l so that it satisfies ϕ

$$= \bigwedge \{ D(w_{\alpha_{i_0}}, w_{\alpha_{i_1}} \cdots, w_l) \implies \eta(f(i))(w_{\alpha_{i_0}}, w_{\alpha_{i_1}}, \cdots, w_{\alpha_{i_{n-2}}}, x_{i_{n-1}}) : i \in S(n,k) \}$$

in $\langle W_l, C^l_r \rangle_{r \in R}$. By $C^l_r = {}^n W_l \cap C_r$ we get that ϕ is satisfied in $\langle W, C_r \rangle_{r \in R}$, as well. By this the proof of Lemma 1 is complete. ■

Notice that by the construction of W, $|W| = |R|$. In particular, W is also an uncountable set. We have excluded the empty set from Cof^+R in order that (iv) can be satisfied, because $\eta(\emptyset)$ is false for any relations C_r. Notice that condition (iv) in Lemma 1, is a "saturation condition" on W. It will be used in the proof of fact 3.1 below, to show that the structure $\langle W, C_r \rangle_{r \in R}$ admits elimination of quantifiers in a rather strong sense. The saturation condition (iv) in words. If we have k distinct elements of W , then for any block, say W_i, of E, and for any *prescription*, there is an element of this block W_i satisfying this prescription. A prescription is the following: Given any $n-1$ elements of the pre-selected k elements, if these are in distinct blocks from each other and from W_i then one of $C_r : r \in X$ holds for them, or none of $C_r : r \in X$ hold for them, where X is a finite subset of R.

Let $U = R \times n$. Let $p(u,r) = \{((s_i, u_i) : i < n) \in {}^n U : (s_0 \ldots s_{n-1}) \in C_r\}$ and let $1_u = p(u, T)$.

Let

$$\mathfrak{A}(n) = \mathfrak{S}g^{\mathfrak{C}} \{ p(u,r) : r \in R \}.$$

Let $1_u = E(u,T)$. For $u \in V$, let $\mathfrak{A}_u$ denote the relativisation of $\mathfrak{A}$ to 1_u i.e

$$\mathfrak{A}_u = \{ x \in A : x \leq 1_u \}.$$

$\mathfrak{A}_u$ is a boolean algebra. Also $\mathfrak{A}_u$ is uncountable for every $u \in V$ Define a map $f : \mathfrak{B}\mathfrak{l}\mathfrak{A} \to \mathfrak{P} = \prod_{u \in V} \mathfrak{A}_u$, by

$$f(a) = \langle a \cdot 1_u \rangle_{u \in V}.$$

Now each $\mathfrak{A}_u \cong Cof(R)$ and hence is atomic. Also clearly the $\prod A_u$ is also atomic, its atoms are $(s_i : i < V)$ such that $s_i \neq 0$ for all except some j where s_j is an atom of A_j.

Let $u_0, u_1 \in S_3$ be distinct and $u_2 = u_1 \circ u_0$. Let $J = \{u_0, u_1, s_{[i,j]} u_3, i, j < n\}$. Take $\mathfrak{B} = \prod_{u = u_0, u_1} A_u \times B_{s_{[i,j]} u_2} \times_{u \notin J} A_u$ where $\mathfrak{B}_v$ is the algebra $Cof(\mathcal{N})$, for N is an elementary subgroup of R. It is easy to show we expand the language of boolean algebras with constants $1_u : u \in V$ and $d_{i,j}$, The algebra $\mathfrak{A}$ becomes first order interpretable with a one dimensional quantifier free interpretation in $\mathfrak{P}$, and under this interpretaion $\mathfrak{B}$ becomes a polyadic equality algebra elementary equivalent to $\mathfrak{A}(n)$ but is not a neat reduct; we denote it by $\mathfrak{B}(n)$.

Now we play a game: we devise a game between $\forall$ (male) and $\exists$(female). We imagine that $\forall$ wants to prove that $\mathfrak{A}(n)$ is different from $\mathfrak{B}(n)$ while $\exists$ tries to show that $\mathfrak{A}(n)$ is the same as $\mathfrak{B}(n)$. So their conversation has the form of a game. Player $\forall$ wins if he manages to find a difference between $\mathfrak{A}(n)$ and $\mathfrak{B}(n)$ before the play is over; otherwise $\exists$ wins. The game is played in $\mu \leq \omega$ steps. At the ith step of a play, player $\forall$ takes one of the structures $\mathfrak{A}(n)$, $\mathfrak{B}(n)$ and chooses an atom of this structure; then $\exists$ chooses an atom of the other structure. So between them they choose an atom a_i of $\mathfrak{A}(n)$ and an atom b_i of $\mathfrak{B}(n)$. Apart from the fact that player $\exists$ must choose from the other structure from player $\forall$ at each step, both players have complete freedom to choose as they please; in particular, either player can choose an element which was chosen at an earlier step. Player $\exists$ is allowed to see and remember all previous moves in the play. (As the game theorists would say, this is a game of perfect information.) At the end of the play sequences $\bar{a} = (a_i : i < \mu)$ and $\bar{b} = (b_i : i < \mu)$ have been chosen. The pair $(\bar{a}, \bar{b})$ is known as the play. We count the play $(\bar{a}, \bar{b})$ as a win for player $\exists$, and we say that $\exists$ wins the play, if there is an isomorphism $f : \mathfrak{S}g^{\mathfrak{A}(n)}ran(\bar{a}) \to \mathfrak{S}g^{\mathfrak{B}(n)}ran(\bar{b})$ such that $f\bar{a} = \bar{b}$. Let us denote this game by $EF_{\mu}(\mathfrak{A}(n), \mathfrak{B}(n))$. (It is an instance of an Ehrenfeuch-Fraisse game.) The more $\mathfrak{A}(n)$ is like $\mathfrak{B}(n)$, the better chance player $\exists$ has of wining these games. For example if player $\exists$ knows about an isomorphism $i : \mathfrak{A}(n) \to \mathfrak{B}(n)$ then she can be sure of winning every time. All she has to do to follow the rule is: Choose $i(a)$ whenever player $\forall$ has just chosen an element a of $\mathfrak{A}(n)$ and $i^{-1}(b)$ whenever player $\forall$ has just chosen b from $\mathfrak{B}(n)$. A strategy for a player in a game is a set of rules which tell the player exactly how to move, depending on what has happened earlier in the play. We say that the player uses the strategy σ in a play if each of his or her moves obeys the rules of σ. We say that σ is a winning strategy if the player wins every play in which he or she uses σ. The game generalizes verbatim to atomic boolean algebras with operators.

Definition 5.3. Two atomic structures $\mathfrak{A}$ and $\mathfrak{B}$ are back and forth equivalent if $\exists$ has a winning strategy for the game $EFA_{\omega}(\mathfrak{A}, \mathfrak{B})$.

Let $At\mathfrak{D}$ denotes the set of atoms of $\mathfrak{D}$. There is a useful criterion for two structures to be back and forth equivalent.

Definition 5.4. A back and forth system from $\mathfrak{A}$ to $\mathfrak{B}$ is a set I of pairs $(\bar{a}, \bar{b})$ of tuples $\bar{a}$ from $At\mathfrak{A}$ and $\bar{b}$ from $At\mathfrak{B}$, such that

(i) If $(\bar{a}, \bar{b})$ is in I, then $\bar{a}$ and $\bar{b}$ have the same length and $(\mathfrak{A}, \bar{a})$ and $(\mathfrak{B}, \bar{b})$ satisfies the same quantifier free formulas.

(ii) I is not empty.

(iii) For every pair $(\bar{a}, \bar{b})$ in I and every atom c of $\mathfrak{A}$ there is an atom d of $\mathfrak{B}$ such that $(\bar{a}c, \bar{bd})$ is in I and

(iv) For every pair $(\bar{a}, \bar{b})$ in I and every atom d of $\mathfrak{B}$ there is an atom c of A such that $(\bar{a}c, \bar{bd})$ is in I.

Note that by (i) if $\bar{a}$ and $\bar{b}$ is in I then there is an isomorphism $f : \mathfrak{S}g(ran\bar{a}) \to \mathfrak{S}g(ran\bar{b})$ such that $f(\bar{a}) = \bar{b}$.

We write I^* for the set of all such functions corresponding to pairs of tuples of atoms in I. The above conditions imply the following for $J = I^*$.

(i) each $f \in J$ is an isomorphism from a finitely generated substructure of $\mathfrak{A}$ to a finitely generated substructure of $\mathfrak{B}$.

(ii) J is non empty

(iii) for every $f \in J$ and $c \in At\mathfrak{A}$ there is $g \supseteq f$ such that $g \in J$ and $c \in dom(g)$

(iv) for every $f \in J$ and $d \in At\mathfrak{B}$ there is $g \supseteq f$ such that $g \in J$ and $d \in ran(g)$

And conversely, it is not hard to see, that if J is any set satisfying then there is a back and forth system I such that $J = I^*$. The following Theorem is intuitive.

Theorem 5.5. *$\mathfrak{A}$ and $\mathfrak{B}$ are back-and forth equivalent if and only if there is a back and forth system from $\mathfrak{A}$ to $\mathfrak{B}$.*

Proof. Suppose that $\mathfrak{A}$ is back and forth equivalent to $\mathfrak{B}$, so that player $\exists$ has a winning strategy σ for the game $EF_\omega(\mathfrak{A}, \mathfrak{B})$. Then define I to consist of all pairs of tuples of atoms which are of the form $(\bar{c} \upharpoonright n, \bar{d} \upharpoonright n)$ for some $n < \omega$ and some paly $(\bar{c}, \bar{d})$ in which $\exists$ uses σ. The set I is a back and forth system from $\mathfrak{A}$ to $\mathfrak{B}$. First putting $n = 0$ in the definition of I , we see that I contains the pair of 0 tuples $(\langle\rangle, \langle\rangle)$. This establishes (ii). Next (iii) and (iv) express that σ tells player $\exists$ what to do at each step of this game. And finally (i) holds because the strategy of σ is winning. In the other direction, suppose that there exists a back and forth system I from $\mathfrak{A}$ to $\mathfrak{B}$. Define the set I^* of maps as above, and choose an arbitrary well ordering of I^*. Consider the following strategy σ for player $\exists$ in the game $EF_\omega(\mathfrak{A}, \mathfrak{B})$. At each step if the play is so far $(\bar{a}, \bar{b})$ and $\forall$ has just chosen an element c from $\mathfrak{A}$, find the first map f in I^* such that $\bar{a}$ and c are in the domain of f and $f(\bar{a}) = f(\bar{b})$ and then choose d to be fc, likewise in the other direction. ∎

This strategy makes $\exists$ win. Coming back to our algebras we have:

Theorem 5.6. *(i) $\exists$ has a winning strategy in $EEF_\omega(\mathfrak{A}(n), \mathfrak{B}(n))$.*

(ii) $\mathfrak{A}(n) \equiv_{\infty,\omega} \mathfrak{B}(n)$.

Proof. Both $\mathfrak{A}(n)$ and $\mathfrak{B}(n)$ are atomic . So $\mathfrak{A}(n)$ and $\mathfrak{B}(n)$ are identical in all components except for the components "coloured " by 1_u, $u \in T_n = V \sim J$ beneath which $\mathfrak{A}(n)$ has uncountably many atoms and $\mathfrak{B}(n)$ has countably many atoms. Now for the game. At each step, if the play so far $(\bar{a}, \bar{b})$ and $\forall$ chooses an atom a in one of the substructures, we have one of two case. Either $a.1_u = a$ for some $u \notin T_n$ in which case $\exists$ chooses the same atom in the other structure. Else $a \leq 1_u$ for some $u \in T_n$. Then $\exists$ chooses a new atom below 1_u (distinct from a and all atoms played so far.) This is possible since there finitely many atoms in play and there are infinitely many atoms below 1_u. This strategy makes $\exists$ win. Let J be a back and forth system which exists by Theorem 6 and (i). Order J by reverse inclusion, that is $f \leq g$ if f extends g. $\leq$ is a partial order on J. For $g \in J$, let $[g] = \{f \in J : f \leq g\}$. Then $\{[g] : g \in J\}$ is the base of a topology on J. Let $\mathfrak{C}$ be the complete Boolean algebra of regular open subsets of J with respect to the topology defined on J. Form the boolean extension $\mathfrak{M}^{\mathfrak{C}}$. We want to define an isomorphism in $\mathfrak{M}^{\mathfrak{C}}$ of $\breve{\mathfrak{A}}$ to $\breve{\mathfrak{B}}$. We shall use the following for $s \in \mathfrak{M}^{\mathfrak{C}}$, (1):

$$||(\exists x \in \breve{s})\phi(x)|| = \sum_{a \in s} ||\phi(\breve{a})||.$$

Define G by (2):

$$||G(\breve{a}, \breve{b})|| = \{f \in J : f(a) = b\}$$

for $c \in \mathfrak{A}$ and $d \in \mathfrak{B}$. If the right-hand side, is not empty, that is it contains a function f, then let f_0 be the restriction of f to the substructure of $\mathfrak{A}$ generated by $\{a\}$. Then $f_0 \in J$. Also

$$\{f \in J : f(c) = d\} = [f_0] \in \mathfrak{C}.$$

G is therefore a $\mathfrak{C}$-valued relation. Now let $u, v \in \mathfrak{M}$. Then

$$||\breve{u} = \breve{v}|| = 1 \text{ iff } u = v,$$

and

$$||\breve{u} = \breve{v}|| = 0 \text{ iff } u \neq v$$

Therefore

$$||G(\breve{a}, \breve{b}) \wedge G(\breve{a}, \breve{c})|| \subseteq ||\breve{b} = \breve{c}||.$$

for $a \in \mathfrak{A}$ and $b, c \in \mathfrak{B}$. Therefore "G is a function." is valid. It is one to one because its converse is also a function. (This can be proved the same way). Finally we show that that $\mathfrak{A}(n) \equiv_{\infty\omega} \mathfrak{B}(n)$ using "soft model theory" as follows: Form a boolean extension $\mathfrak{M}^*$ of $\mathfrak{M}$ in which the cardinailities of $\mathfrak{A}(n)$ and $\mathfrak{B}(n)$ collapse to ω. Then $\mathfrak{A}(n)$ and $\mathfrak{B}(n)$ are still back and forth equivalent in $\mathfrak{M}^*$. Then $\mathfrak{A}(n) \equiv_{\infty\omega} \mathfrak{B}(n)$ in $\mathfrak{M}^*$, and hence also in $\mathfrak{M}$ by absoluteness of $\models$. $\square$

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