A Linear-Time Algorithm to Find Four Independent Spanning Trees in Four-Connected Planar Graphs

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Abstract. Given a graph \( G \), a designated vertex \( r \) and a natural number \( k \), we wish to find \( k \) “independent” spanning trees of \( G \) rooted at \( r \), that is, \( k \) spanning trees such that, for any vertex \( v \), the \( k \) paths connecting \( r \) and \( v \) in the \( k \) trees are internally disjoint in \( G \). In this paper we give a linear-time algorithm to find four independent spanning trees in a 4-connected planar graph rooted at any vertex.

1 Introduction

Given a graph \( G = (V,E) \), a designated vertex \( r \in V \) and a natural number \( k \), we wish to find \( k \) spanning trees \( T_1, T_2, \ldots, T_k \) of \( G \) such that, for any vertex \( v \), the \( k \) paths connecting \( r \) and \( v \) in \( T_1, T_2, \ldots, T_k \) are internally disjoint in \( G \), that is, any two of them have no common intermediate vertices. Such \( k \) trees are called \( k \) independent spanning trees of \( G \) rooted at \( r \). Four independent spanning trees are drawn in Fig. 1 by thick lines. Independent spanning trees have applications to fault-tolerant protocols in networks [B96,DHSS84,IR88,OIB196].

Given a graph \( G = (V,E) \) of \( n \) vertices and \( m \) edges, and a designated vertex \( r \in V \), one can find two independent spanning trees of \( G \) rooted at any vertex in linear time if \( G \) is biconnected [BTV96,IR88], and find three independent spanning trees of \( G \) rooted at any vertex in \( O(mn) \) and \( O(n^2) \) time if \( G \) is triconnected [BTV96,CM88]. It is
Fig. 1. Four independent spanning trees $T_1, T_2, T_3$ and $T_4$ of a graph $G$ rooted at $r$. 
conjectured that, for any $k \geq 1$, every $k$-connected graph has $k$ independent spanning trees rooted at any vertex \[KS92,ZI89\]. Recently Huck has proved that every 4-connected planar graph has four independent spanning trees rooted at any vertex \[H94\]. The proof in \[H94\] yields an algorithm to actually find four independent spanning trees, but it takes time $O(n^3)$.

In this paper we give a simple linear-time algorithm to find four independent spanning trees of a 4-connected planar graph rooted at any designated vertex. Our algorithm is based on a “4-canonical decomposition” of a 4-connected planar graph \[NRN97\], which is a generalization of an $st$-numbering \[E79\], a canonical ordering \[CK93\] and a canonical 4-ordering \[KH94\].

The remainder of the paper is organized as follows. In Section 2 we introduce some definitions. In Section 3 we present our algorithm to find four independent spanning trees. Finally we put conclusion in Section 4.

2 Preliminaries

In this section we introduce some definitions.

Let $G = (V,E)$ be a connected graph with vertex set $V$ and edge set $E$. Throughout the paper we denote by $n$ the number of vertices in $G$, and we always assume that $n > 4$. An edge joining vertices $u$ and $v$ is denoted by $(u,v)$. The degree of a vertex $v$ in $G$, denoted by $d(v,G)$ or simply by $d(v)$, is the number of neighbors of $v$ in $G$. The connectivity $\kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph $K_1$. A graph $G$ is $k$-connected if $\kappa(G) \geq k$. A path in a graph is an ordered list of distinct vertices $v_1, v_2, \ldots, v_l$ such that $v_{i-1}v_i$ is an edge for all $i, 2 \leq i \leq l$. We say that two paths having common start and end vertices are internally disjoint if their intermediate vertices are disjoint. We also say that a set of paths having common start and end vertices are internally disjoint if every pair of paths in the set are internally disjoint.

A graph is planar if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A plane graph is a planar graph with a fixed embedding. The contour $C_o(G)$ of a biconnected plane graph $G$ is
the clockwise (simple) cycle on the outer face. We write $C_o(G) = (w_1, w_2, \ldots, w_h)$ if the vertices $w_1, w_2, \ldots, w_h$ on $C_o(G)$ appear in this order.

3 Algorithm

In this section we give our algorithm to find four independent spanning trees of a 4-connected planar graph rooted at any designated vertex.

Given a 4-connected planar graph $G = (V, E)$ and a designated vertex $r \in V$, we first find a planar embedding of $G$ in which $r$ is located on $C_o(G)$. Let $G' = G - \{r\}$ be the subgraph of the planar graph $G$ induced by $V - \{r\}$. In Fig. 2(a) $G$ is drawn by solid and dotted lines, and $G'$ by solid lines. Since $G$ is 4-connected, $d(r) \geq 4$.

We may assume that all the neighbors $r_1, r_2, \ldots, r_{d(r)}$ of $r$ in $G$ appear on $C_o(G')$ clockwise in this order. Let $C_o(G') = (w_1, w_2, \ldots, w_h)$, $r_1 = w_1, r_2 = w_a, r_3 = w_b$ and $r_4 = w_c$, where $1 < a < b < c \leq d(r)$.

We add to $G'$ two new vertices $r_b$ and $r_t$, join $r_b$ with $r_1$ and $r_2$, and join $r_t$ with $r_3, r_4, \ldots, r_{d(r)}$. Let $G'$ be the resulting plane graph, where vertices $r_1, r_b, r_2, r_3, r_t$ and $r_{d(r)}$ appear on $C_o(G')$ clockwise in this order. Fig. 2(b) illustrates $G'$.

Let $\Pi = (W_1, W_2, \ldots, W_m)$ be a partition of the vertex set $V - \{r\}$ of $G'$. We denote by $G'_k, 1 \leq k \leq m$, the plane subgraph of $G'$ induced by $\{r_t\} \cup W_1 \cup W_2 \cup \ldots \cup W_k$. We denote by $\overline{G}'_k, 0 \leq k \leq m - 1$, the plane subgraph of $G''$ induced by $W_{k+1} \cup W_{k+2} \cup \ldots \cup W_m \cup \{r_t\}$. We assume that if $1 \leq k \leq m$ and $W_k = \{u_1, u_2, \ldots, u_l\}$ then vertices $u_1, u_2, \ldots, u_l$ consecutively appear on $C_o(G_k)$ clockwise in this order. A partition $\Pi = (W_1, W_2, \ldots, W_m)$ of $V - \{r\}$ is called a 4-canonical decomposition of $G'$ if the following three conditions (co1)–(co3) are satisfied.

(co1) $W_1 = \{w_a, w_{a-1}, \ldots, w_1\}$ and $W_m = \{w_b, w_{b+1}, \ldots, w_c\}$;
(co2) For each $k, 1 \leq k \leq m - 1$, both $G_k$ and $\overline{G}_{k-1}$ are biconnected (See Fig. 3); and
(co3) For each $k, 1 < k < m$, one of the following three conditions holds (See Fig. 3):
   (a) $|W_k| \geq 2$, and each vertex $u \in W_k$ satisfies $d(u, G_k) = 2$ and $d(u, \overline{G}_{k-1}) \geq 3$;
(b) $|W_0| = 1$, and the vertex $u \in W_k$ satisfies $d(u, G_k) \geq 2$ and $d(u, G_{k-1}) \geq 2$; and
(c) $|W_0| \geq 2$, and each vertex $u \in W_k$ satisfies $d(u, G_k) \geq 3$ and $d(u, G_{k-1}) = 2$.

Fig. 2(b) illustrates a 4-canonical decomposition of $G' = G - \{r\}$, where $G'$ are drawn in solid lines and each set $W_i$ is indicated by an oval drawn in a dotted line. A 4-canonical decomposition is a generalization of an “st-numbering” [E79], a “canonical decomposition” [CK93] and a “canonical 4-ordering” [KH94]. Although the definition of a 4-canonical decomposition above is slightly different from one in [NR97], they are effectively equivalent each other.

Fig. 2. (a) Four-connected plane graph $G$ and (b) plane graph $G''$.

We have the following lemma.

**Lemma 1.** Let $G = (V, E)$ be a 4-connected plane graph, and let $r$ be a designated vertex on $C_0(G)$. Then $G' = G - \{r\}$ has a 4-canonical decomposition $\Pi$. Furthermore $\Pi$ can be found in linear time.

**Proof.** Similar to the proof of Lemma 3 in [NR97]. Q.E.D.
Fig. 3. Three conditions for (co3).

We need a few more definitions to describe our algorithm. For a vertex \( v \in V - \{r\} \) we write \( N(v) = \{v_1, v_2, \ldots, v_{d(v)}\} \) if \( v_1, v_2, \ldots, v_{d(v)} \) are the neighbors of vertex \( v \) in \( G'' \) and appear around \( v \) clockwise in this order. To each vertex \( v \in V - \{r\} \) we assign four edges incident to \( v \) in \( G'' \) as the left hand \( lh(v) \), the right hand \( rh(v) \), the left leg \( ll(v) \) and the right leg \( rl(v) \) as follows. We will show later that such an assignment immediately yields four independent spanning trees of \( G \). Let \( v \in W_k \) for some \( k, 1 \leq k \leq m \), then there are the following three cases to consider.

Case 1: either (i) \( 1 < k < m \) and \( W_k \) satisfies Condition (a) of (co3) or (ii) \( k = 1 \). (See Fig. 4)

Let \( W_k = \{u_1, u_2, \ldots, u_i\} \). Let \( u_0 \) be the vertex on \( C_o(G_k) \) preceding \( u_1 \), and let \( u_{i+1} \) be the vertex on \( C_o(G_k) \) succeeding \( u_i \).

For each \( u_i \in W_k \) we define \( rl(u_i) = (u_i, u_{i+1}) \), \( ll(u_i) = (u_i, u_{i-1}) \), \( lh(u_i) = (u_i, v_1) \), and \( rh(u_i) = (u_i, v_{d(u_i)} - 2) \) where we assume \( N(u_i) = \{u_{i-1}, v_1, v_2, \ldots, v_{d(u_i)} - 2, u_{i+1}\} \).

Case 2: \( W_k \) satisfies Condition (b) of (co3). (See Fig. 5)

Let \( W_k = \{u\} \), let \( u' \) be the vertex on \( C_o(G_k) \) preceding \( u \), and let \( u'' \) be the vertex on \( C_o(G_k) \) succeeding \( u \). Let \( N(u) = \{u', v_1, v_2, \ldots, v_{d(u)} - 1\} \), and let \( u'' = v_x \) for some \( x, 3 \leq x \leq
\[ d(u) - 1. \] Then \( rl(u) = (u, u'') \), \( ll(u) = (u, u') \), \( lh(u) = (u, v_1) \), and \( rh(u) = (u, v_{x-1}) \).

**Case 3:** either (i) \( 1 < k < m \) and \( W_k \) satisfies Condition (c) of (co3) or (ii) \( k = m \). (See Fig. 4)

Let \( W_k = \{u_1, u_2, \ldots, u_l\} \). Let \( u_0 \) be the vertex on \( C_o(G_{k-1}) \) succeeding \( u_1 \), and let \( u_{l+1} \) be the vertex on \( C_o(G_{k-1}) \) preceding \( u_l \). For each \( u_i \in W_k \) we define \( rl(u_i) = (u_i, v_1) \), \( ll(u_i) = (u_i, v_{d(u_i)-2}) \), \( lh(u_i) = (u_i, u_{i-1}) \), and \( rh(u_i) = (u_i, u_{i+1}) \) where we assume \( N(u_i) = \{u_{i+1}, v_1, v_2, \ldots, v_{d(u_i)-2}, u_{i-1}\} \).

![Fig. 4. Assignment for Case 1.](image)

We are now ready to give our algorithm.

**Procedure FourTrees** \((G, r)\)

```
begin
1  Find a planar embedding of \( G \) such that \( r \in C_o(G) \);
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![Fig. 5. Assignment for Case 2.](image)
2 Find a 4-canonical decomposition $\Pi = (W_1, W_2, \cdots, W_m)$ of $G - \{r\}$;
3 For each vertex $v \in V - \{r\}$ find $rl(v), ll(v), rh(v)$ and $lh(v)$;
4 Let $T_{rl}$ be a graph induced by the right legs of all vertices in $V - \{r\}$;
5 Let $T_{ll}$ be a graph induced by the left legs of all vertices in $V - \{r\}$;
6 Let $T_{lh}$ be a graph induced by the left hands of all vertices in $V - \{r\}$;
7 Let $T_{rh}$ be a graph induced by the right hands of all vertices in $V - \{r\}$;
8 Regard vertex $r_b$ in trees $T_{rl}$ and $T_{ll}$ as vertex $r$;
9 Regard vertex $r_t$ in trees $T_{lh}$ and $T_{rh}$ as vertex $r$;
10 return $T_{rl}, T_{ll}, T_{lh}$ and $T_{rh}$ as four independent spanning trees of $G$.

end

We then verify the correctness of our algorithm. Assume that $G = (V, E)$ is a 4-connected planar graph with a designated vertex $r \in V$, and that Algorithm FourTrees finds a 4-canonical decomposition $\Pi = (W_1, W_2, \cdots, W_m)$ of $G - \{r\}$ and outputs $T_{rl}, T_{ll}, T_{lh}$ and $T_{rh}$. We first have the following lemma.

Lemma 2. Let $1 \leq k \leq m$, and let $T_{rl}^k$ be a graph induced by the right legs of all vertices in $G_k - \{r_b\}$. Then $T_{rl}^k$ is a spanning tree of $G_k$.

Proof. We prove the claim by induction on $k$. 

Fig. 6. Assignment for Case 3.
Fig. 7. The four cases for Lemma 2.
Clearly the claim holds for \( k = 1 \).

We assume that \( 1 \leq k \leq m - 1 \) and \( T_{rl}^k \) is a spanning tree of \( G_k \), and we shall prove that \( T_{rl}^{k+1} \) is a spanning tree of \( G_{k+1} \). There are the following four cases to consider.

**Case 1:** \( k \leq m - 2 \) and \( W_{k+1} \) satisfies Condition (a) of (co3).

**Case 2:** \( k \leq m - 2 \) and \( W_{k+1} \) satisfies Condition (b) of (co3).

**Case 3:** \( k \leq m - 2 \) and \( W_{k+1} \) satisfies Condition (c) of (co3).

**Case 4:** \( k = m - 1 \).

For each case \( T_{rl}^{k+1} \) is a spanning tree of \( G_{k+1} \) as shown in Fig. 7; (a) for Case 1; (b) for Case 2; (c) for Case 3; and (d) for Case 4. Q.E.D.

We then have the following lemma.

**Lemma 3.** \( T_{rl}, T_{lt}, T_{lh} \) and \( T_{rh} \) are spanning trees of \( G \).

**Proof.** By Lemma 3.2 \( T_{rl}^m \) is a spanning tree of \( G_m \), and hence \( T_{rl} \) in which \( r_b \) is regarded as \( r \) is a spanning tree of \( G \).

Similarly \( T_{lt}, T_{lh} \) and \( T_{rh} \) are spanning trees of \( G \). Q.E.D.

Let \( v \) be any vertex in \( V - \{r\} \), and let \( P_{rl}, P_{lt}, P_{lh} \) and \( P_{rh} \) be the paths connecting \( r \) and \( v \) in \( T_{rl}, T_{lt}, T_{lh} \) and \( T_{rh} \), respectively. For any vertex \( u \) in \( V - \{r\} \) we write \( rank(u) = k \) if \( u \in W_k \); \( rank(r) \) is undefined. If an edge \((v, u)\) of \( G' \) is a leg of vertex \( v \), and \((v, w)\) of \( G' \) is a hand of \( v \), then \( rank(u) \leq rank(v) \leq rank(w) \) and \( rank(u) < rank(w) \).

**Lemma 4.** Each of the four pairs of paths, \( P_{rl} \) and \( P_{rh} \), \( P_{rl} \) and \( P_{lh} \), \( P_{lt} \) and \( P_{lh} \), \( P_{lt} \) and \( P_{rh} \), are internally disjoint.

**Proof.** We prove only that \( P_{rl} \) and \( P_{lh} \) are internally disjoint. Proofs for the other pairs are similar. If \( v = r_1 \) then \( P_{rl} = (v, r) \). If \( v = r_3 \) then \( P_{lh} = (v, r) \). If \( v \) is \( r_1 \) or \( r_3 \) then \( P_{rl} \) and \( P_{lh} \) are internally disjoint if \( u \) is \( r_1 \) or \( r_3 \). Thus we may assume that \( v \neq r_1, r_3 \). Let \( P_{rl} = (v, v_1, v_2, \ldots, v_l, r) \), then \( v_l = r_1 \). Let \( P_{lh} = (v, u_1, u_2, \ldots, u_l', r) \), then \( u_l' = r_3 \). The definition of a right leg implies that \( rank(v) \geq rank(v_1) \geq rank(v_2) \geq \cdots \geq rank(v_l) \), and the definition of a left hand implies that \( rank(v) \leq rank(u_1) \leq rank(u_2) \leq \cdots \leq rank(u_l') \). Thus \( rank(v_l) \leq \cdots \leq rank(v_2) \leq rank(v_1) \leq rank(v) \leq rank(u_l') \leq rank(u_2) \leq \cdots \leq rank(u_l) \). We furthermore have \( rank(v_1) < rank(u_1) \). Therefore \( P_{rl} \) and \( P_{lh} \) are internally disjoint. Q.E.D.
We next have the following lemma.

**Lemma 5.** Let \( u \in V - \{r\} \), \( ll(u) = (u, u') \), \( rl(u) = (u, u'') \), and \( N(u) = \{v_1, v_2, \ldots, v_{d(u)}\} \). One may assume that \( u' = v_1 \) and \( u'' = v_s \) for some \( s, 1 < s \leq d(u) \). Then there exists \( t, 1 \leq t \leq s \), such that \( rl(v_t) = (v_t, u) \) for each \( i, 2 \leq i \leq t - 1 \), and \( ll(v_j) = (v_j, u) \) for each \( j, t + 1 \leq j \leq s - 1 \). (Thus either (i) \( rl(v_t) = (v_t, u) \neq ll(v_t) \), (ii) \( rl(v_t) \neq (v_t, u) = ll(v_t) \), or (iii) \( rl(v_t) \neq (v_t, u) \neq ll(v_t) \). See Fig. 8.

**Fig. 8.** Illustration for Lemma 5.

**Proof.** From the definitions of a 4-canonical decomposition and a right leg, one can observe that if \( 2 \leq i \leq s - 1 \) and \( rl(v_i) = (v_i, u) \) then \( rank(v_{i-1}) < rank(v_i) \). Similarly, if \( 2 \leq i \leq s - 1 \) and \( ll(v_j) = (v_j, u) \) then \( rank(v_{j+1}) > rank(v_j) \).
Assume for a contradiction that the claim does not hold. Then $rl(v_i) = (v_i, u)$ and $ll(v_j) = (v_j, u)$ for some $i$ and $j$, $1 \leq j < i \leq s$. Let $v_i \in W_i$ and $v_j \in W_j$ for some $i'$ and $j'$, $1 \leq i', j' \leq m$. Thus $rank(v_i) = i'$, $rank(v_j) = j'$, and both $G_{i'}$ and $G_{j'}$ are biconnected.

There are the following three cases.

**Case 1:** $i' = j'$. In this case, $G_{i'}$ has edges $(u, v_j)$ and $(v_i, u)$, and all vertices in $W_{i'}$ appear on $C_o(G_{i'})$. Therefore, vertex $u$ and the vertices in $W_{i'}$ from $v_j$ to $v_i$ form a cycle in $G_{i'}$, and $G_{i'}$ has at least one vertex in the proper inside of the cycle. None of the edges of $G$ in the outside of the cycle is incident to any vertex on the cycle other than $u, v_j$ and $v_i$. Hence the removal of three vertices $u, v_j$ and $v_i$ from $G$ results in a disconnected graph, contrary to the 4-connectivity of $G$.

**Case 2:** $i' < j'$. Since $rl(v_i) = (v_i, u)$, $v_i$ precedes $u$ on $C_o(G_{i'})$. Since $ll(v_j) = (v_j, u)$, $v_j$ succeeds $u$ on $C_o(G_{j'})$. Since $G_{i'}$ is a subgraph of $G_{j'}$, $v_i$ must precede $v_j$ in $N(u)$, contrary to the assumption $j < i$.

**Case 3:** $i' > j'$. Similar to Case 2 above.

Q.E.D.

Lemma 5 immediately implies the following lemma.

**Lemma 6.** $P_{rl}$ and $P_{ll}$ may cross at a vertex $u$, but do not share a vertex $u$ without crossing at $u$.

From the definitions of a left leg and a right leg one can immediately have the following lemma.

**Lemma 7.** Let $1 \leq k \leq m$ and $u \in W_k$. Then $u$ is on $C_o(G_k)$. Let $u'$ be the succeeding vertex of $u$ on $C_o(G_k)$. Assume that the ordered set $N(u)$ starts with $u'$. Let $ll(u) = (u, v')$ and $rl(u) = (u, v'')$. Then $v''$ precedes $v'$ in $N(u)$.

We then have the following lemma.

**Lemma 8.** Each of the two pairs of paths, $P_{rl}$ and $P_{ll}$, $P_{lh}$ and $P_{rh}$, are internally disjoint.

**Proof.** We prove only that $P_{rl}$ and $P_{ll}$ are internally disjoint. Proof for the other case is similar. Suppose for a contradiction that $P_{rl}$ and $P_{ll}$ share an intermediate vertex. Let $w$ be the intermediate vertex
that is shared by $P_{rl}$ and $P_{ll}$ and appear last on the path $P_{rl}$ going from $r$ to $v$. Then $P_{rl}$ and $P_{ll}$ cross at $w$ by Lemma 6. However, the claim in Lemma 7 holds both for $k = \text{rank}(v)$ and $u = v$ and for $k = \text{rank}(w)$ and $u = w$, and hence $P_{rl}$ and $P_{ll}$ do not cross at $w$, a contradiction. \hfill Q.E.D.

By Lemmas 3, 4 and 8 we have the following lemma.

**Lemma 9.** $T_{rl}, T_{ll}, T_{lh}$ and $T_{rh}$ are four independent spanning trees of $G$ rooted at $r$.

Clearly the running time of Algorithm FourTrees is $O(n)$. Thus we have the following theorem.

**Theorem 1.** Four independent spanning trees of any 4-connected plane graph rooted at any designated vertex can be found in linear time.

### 4 Conclusion

In this paper we give a linear-time algorithm to find four independent spanning trees of a 4-connected planar graph rooted at any designated vertex. Using four independent spanning trees, one can efficiently solve the 4-path query problem for 4-connected planar graphs.

It is remained as future work to find a linear-time algorithm for a larger class of graphs, say 4-connected graphs which are not always planar.

### References


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