Robust Control of Decomposable LPV Systems
Under Time-Invariant and Time-Varying Interconnection Topologies (Part 2)

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Abstract—In this paper and its companion paper [1], an approach to distributed control of a class of interconnected systems is proposed. The systems under consideration are known as decomposable systems, which include multi-agent systems as a special case. It is assumed that the interconnection topology is uncertain but belongs to a known set of topologies. In the first part of this series of two papers, sufficient conditions for stability analysis are derived for different cases of directed time-invariant and time-varying topologies. In this paper, the second part, these conditions are extended to decomposable LPV systems. Recent work on state feedback controller synthesis is extended here to robust output feedback controller synthesis. The distributed controllers proposed here are robust in a twofold sense: first against uncertain topologies and second against the LPV dynamics of the subsystems.

I. INTRODUCTION

There is an increasing interest in the field of interconnected systems due to their wide range of applications. One class of interconnected systems are distributed systems: groups of identical subsystems, that are connected, either physically or virtually, e.g. by information exchange. Examples for the former class can be found in the framework of spatially-distributed systems [2], whereas multi-agent systems, which are of considerable interest in cooperative control [3], are an example of the latter.

In [4] the framework of decomposable systems has been introduced as a tool to model interconnected systems, which encompasses physically coupled systems as well as multi-agent systems. Based on this framework [5] proposes a convex controller synthesis for distributed state feedback control, which involve the use of the full block S-procedure, a tool widely used in LPV control [6], [7]. The present work is based on this framework.

Linear parameter varying (LPV) systems are a well known framework to model nonlinear or uncertain systems [8]. Thus in combination with decomposable systems, they make it possible to model a group of nonlinear and/or time-varying subsystems, where the nonlinear or time-varying behavior is represented by uncertain parameters as in [9], [10], [11]. In [9] this approach is used for synchronization of multi-agent systems via interpolation techniques. In [10] an approach for state feedback controller synthesis for affinely dependent subsystems is proposed, where a distinction is made between homogeneous and heterogeneous parameter dependence. While in the heterogeneous case they may differ. In the context of multi-agent systems, homogeneous dependence on scheduling parameters could be e.g. on altitude or speed for a formation of UAVs, whereas heterogeneous dependence may represent local states of agents which are modeled as quasi-LPV systems. In [11] the framework of decomposable systems is extended to LPV systems and an output feedback controller synthesis method is proposed. It is shown, that the interaction may be allowed to switch between arbitrary symmetric topologies. In both cases gain-scheduling controller synthesis is performed, which requires the LPV scheduling parameters to be measurable online.

A. Contribution of this Paper

The work presented here is divided into two parts. The first part, presented in the companion paper [1], is focused on the interconnection topology with LTI subsystems. While in [5] diagonalizable time-invariant and in [11] symmetric time-varying topologies are considered, this work presents analysis conditions for time-invariant and time-varying non-symmetric interconnection topologies for decomposable LTI-systems. This paper represents part two of this work, where the approach is extended to LPV-systems with rational homogeneous parameter dependency, and a method for robust output-feedback controller synthesis is derived. Robustness here refers on one hand to the uncertain and possibly time-varying topology, and on the other hand to the parameter uncertainty of the subsystems. While for gain-scheduled output feedback controller synthesis convex methods exists, for the robust control problem considered here an iterative process similar to [12] is proposed.

B. Outline of the Paper

Section II reviews the notion of decomposable systems and summarizes the results from part 1 in [1]. Section III extends the results obtained so far from decomposable LTI systems to LPV systems with rational parameter dependency. In Section IV the output feedback controller synthesis is derived. Numerical examples presented in Section V illustrate the main results of that work. Finally, the paper is concluded in Section VI.

II. DECOMPOSABLE SYSTEMS

Consider a linear dynamic system

\[ \dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_p \\ \dot{z}_p \\ \dot{w}_p \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B}_p & \hat{B}_u \\ \hat{C}_p & \hat{D}_{pp} & \hat{D}_{pu} \\ \hat{C}_y & \hat{D}_{yp} & 0 \end{bmatrix} \begin{bmatrix} x_p \\ z_p \\ w_p \end{bmatrix} + \begin{bmatrix} \dot{u} \end{bmatrix} \]

(1)
with $\bar{x} \in \mathbb{R}^{n_x}$, $\bar{u} \in \mathbb{R}^{n_u}$, $\bar{y} \in \mathbb{R}^{n_y}$, the disturbance $\bar{w}_p \in \mathbb{R}^{n_{wp}}$ and the performance output $\bar{z}_p \in \mathbb{R}^{n_{zp}}$. The time dependency of the signals has been omitted here for the sake of brevity.

A system (5) is called decomposable [4], if there exists a pattern matrix $P$, such that each of the model matrices in (1) is in the set $\mathcal{D}(P)$, which is defined as

$$
\mathcal{D}(P) := \left\{ \hat{M} \in \mathbb{R}^{n_p \times n_q} \mid \exists M^d, M^i \in \mathbb{R}^{n \times n} : \hat{M} = I_N \otimes M^d + P \otimes M^i \right\}.
$$

The notation $\circledast$ indicates that a matrix belongs to the set $\mathcal{D}(P)$. The superscript $d$ labels the decentralized and the superscript $i$ the interconnected part.

A decomposable system consists of $N$ identical subsystems, which are interconnected by the pattern matrix $P$. This decomposable system is to be controlled by a distributed controller, which inherits the interconnection topology of the plant and is thus decomposable itself. The decomposable system (1) is to be controlled by a distributed output-feedback controller

$$
\hat{K}_{OF} : \begin{bmatrix} \hat{x}_K \\ \hat{y} \end{bmatrix} = \begin{bmatrix} \hat{A} \\ \hat{C}_K \end{bmatrix} \begin{bmatrix} \hat{B} \\ \hat{D} \end{bmatrix} \begin{bmatrix} \hat{x}_K \\ \hat{y} \end{bmatrix}
$$

(3)

with $\hat{x}_K \in \mathbb{R}^{n_{xK}}$. Substituting (3) into (1) and assuming that $B_u = 0$, $D_{pu} = 0$, $C_y = 0$ and $D_{yp} = 0$ leads to the closed-loop system

$$
\hat{T} : \begin{bmatrix} \hat{x}_K \\ \hat{z}_p \end{bmatrix} = \begin{bmatrix} \hat{A} + B_u \hat{D}_y \hat{C}_y & B_u \hat{D}_x \hat{C}_x & B_p + B_u \hat{D}_y \hat{D}_{yp} \\ \hat{C}_{xp} + D_{pu} \hat{D}_y \hat{C}_y & \hat{D}_{xp} \hat{C}_x + D_{py} \hat{D}_y & \hat{D}_{xp} + D_{pu} \hat{D}_y \hat{D}_{yp} \end{bmatrix} \begin{bmatrix} \hat{x}_K \\ \hat{z}_p \end{bmatrix}.
$$

(4)

The closed loop system (4) can be brought into decomposable form by application of the following Lemma:

**Lemma 1:** Every Matrix in the set

$$
\mathcal{D}_{pq}(P) := \left\{ \hat{M} = \begin{bmatrix} \hat{M}_{11} \cdots \hat{M}_{1q} \\ \vdots \\ \hat{M}_{p1} \cdots \hat{M}_{pq} \end{bmatrix} \mid \hat{M}_{ij} \in \mathcal{D}(P), M_{ij}^d, M_{ij}^i \in \mathbb{R}^{n \times n} \right\}
$$

can be permuted such that it is decomposable

$$
\hat{M} = I_N \otimes 
\begin{bmatrix} M_{11}^d & \cdots & M_{1q}^d \\ \vdots & \ddots & \vdots \\ M_{p1}^d & \cdots & M_{pq}^d \end{bmatrix} + P \otimes 
\begin{bmatrix} M_{11}^i & \cdots & M_{1q}^i \\ \vdots & \ddots & \vdots \\ M_{p1}^i & \cdots & M_{pq}^i \end{bmatrix} = T_p \hat{M} T_q
$$

with $T_p$ and $T_q$ being permutation matrices. The $rs$-block of $T_p$ is defined as

$$
[T_p]_{rs} = \begin{cases} I_p, & \text{if } r = pk - (p - i), \quad s = N(i - 1) + k, \quad k = 1, \ldots, N \\ 0, & \text{otherwise} \end{cases}
$$

and $T_q$ is defined similarly by interchanging $p$ with $q$.

**Proof:** This can easily be shown by performing the permutation and applying the rules of the Kronecker product.

With Lemma 1 a permutation of (4) leads to the decomposable system

$$
\bar{T} : \begin{bmatrix} \bar{x} \\ \bar{z}_p \end{bmatrix} = \begin{bmatrix} \bar{A} \\ \bar{C}_p \end{bmatrix} \begin{bmatrix} \bar{B}_p \\ \bar{D}_{pp} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{z}_p \end{bmatrix}
$$

(5)

with $\bar{x} \in \mathbb{R}^{n_x}$ and $n_x = n_x + n_x^K$, as the permuted closed-loop state vector.

The assumption $B_u^d = 0$, $D_{pu}^d = 0$, $C_y^d = 0$ and $D_{yp}^d = 0$ is necessary, such that (4) can be permuted to be decomposable, but it is not restrictive. It is a matter of interpretation, whether the interconnection is placed in controller output or system input; both lead to identical closed-loop matrices.

In the case of time-invariant topologies a signal transformation can be used to transform $T$ into a block-diagonal form $\hat{T}$, for details see [1]. This transformation maintains stability but not necessarily performance.

The results of Theorem 1, 2 and 3 in [1] on robust stability and performance in the presence of uncertain and possibly time-varying interconnection topologies can be summarized as follows:

**Lemma 2:** Given a known set of topologies $\mathcal{P}$ and a decomposable system $\hat{T}$, with either an uncertain time-invariant topology $P \in \mathcal{P}$ or an uncertain time-varying topologies $P(t)$ that switches between elements of $\mathcal{P}$, s.t. $P(t) \in \mathcal{P}$ for all $t$. The system $\hat{T}$ is stable and, in the

(i) time-invariant case, $\hat{T}$ has an $H_\infty$ norm less than $\gamma$, 
(ii) time-varying case, $\hat{T}$ has an induced $L_2$ gain less than $\gamma$, 

if there exist $X^d > 0$ and $Q^d = Q^d^T$, $R^d = R^d^T$ and $S^d$ such that

$$
\begin{bmatrix} I \otimes [0] & 0 \\ 0 \otimes I \otimes [0] & A^d \\ 0 \otimes -I \otimes [0] & B^d \end{bmatrix} < 0
$$

(6)

and in the

(i) time-invariant case

- with all $P \in \mathcal{P}$ being diagonalizable

$$
\begin{bmatrix} X^d & 0 \\ S^d \otimes R^d \end{bmatrix} \begin{bmatrix} \lambda_{\mathcal{P}} \\ I_{n_P} \end{bmatrix} > 0 \quad \forall \lambda \in \mathcal{L}_g,
$$

(7)

- with any $P \in \mathcal{P}$ being non-diagonalizable

$$
\begin{bmatrix} I \otimes [0] & 0 \\ 0 \otimes I \otimes [0] & A^d \\ 0 \otimes -I \otimes [0] & B^d \end{bmatrix} < 0
$$

(8)

(ii) time-varying case

$$
\begin{bmatrix} I \otimes [0] & 0 \\ 0 \otimes I \otimes [0] & A^d \otimes Q^d \end{bmatrix} \begin{bmatrix} \lambda_{\mathcal{P}} \\ I_{n_P} \end{bmatrix} > 0 \quad \forall P(t) \in \mathcal{P},
$$

(9)

with $n_P = n_x + n_x^K$. Here $\mathcal{L}_g$ is the union of the sets of eigenvalues of all $P \in \mathcal{P}$ defined as $\mathcal{L}_g = \{ \text{eig } P \mid P \in \mathcal{P} \}$. The set $\mathcal{J}_g$ is analogously defined as the set of all Jordan blocks of all $P \in \mathcal{P}$ defined as

$$
\mathcal{J}_g = \{ J_{ki}(P) \mid k = 1, \ldots, m, \forall i = 1, \ldots, \gamma(\lambda_k) \forall P \in \mathcal{P} \}
$$

where $m$ is the number of different eigenvalues of $P$ and for each eigenvalue $\lambda_k$ there exist $\gamma(\lambda_k)$ (geometric multiplicity) Jordan blocks of dimension $m_{ki} \times m_{ki}$.
In the following, equations (7), (8) and (9) are all represented by
\[
[\star]^T \left[ I_{\psi} \otimes Q^d \ I_{\psi} \otimes S^d \ I_{\psi} \otimes \rho \right] \Psi [I_{\psi + \rho}] > 0 \ \forall \psi \in \Psi, \quad (10)
\]
where \( \psi, \Psi \) and \( \Psi \) have to be replaced appropriately, depending on which case is being considered.

### III. Decomposable LPV-Systems

The decomposable systems considered so far have been LTI systems. But we regarded the interconnection structure as uncertainty, and solved for a gain-scheduled controller, which inherits the scheduling policy of the plant, i.e. inherits the interconnection topology. Thus the full block S-procedure can be applied to assesses stability and performance, as summarized in Lemma 2. In the following, decomposable LPV systems are considered. By decomposable LPV systems we mean decomposable systems with LPV rather than LTI dynamics (1). Consequently the decomposed systems will comprise LPV subsystems. It is assumed that each subsystem displays the same functional dependency on the parameters. Furthermore, we distinguish between systems where the scheduled parameters take the same values for all subsystems, referred to as homogeneous dependency, and systems where the parameters take different values, referred to as heterogeneous dependency. Here only homogeneous dependency is considered, in contrast to [11], where heterogeneous dependencies are discussed.

Assume now that the closed-loop matrices in (5) are LFT-dependent on a parameter vector \( \theta(t) \) with \( \theta(t) \in \Theta \) for all \( t \), such that they are in \( \mathcal{D}(P) \) for all \( \theta(t) \in \Theta \). Then the closed-loop system is decomposable and can be rewritten in LFT form as

\[
\mathcal{A}_{\tilde{T}}: \begin{cases}
\tilde{x}_\hat{\lambda} = \begin{bmatrix}
I_N \otimes A^d & 0 & 0 & 0 & 0 \\
I_N \otimes A^i & 0 & 0 & 0 & 0 \\
I_N \otimes C^l_n & 0 & 0 & 0 & 0 \\
I_N \otimes C^l_p & 0 & 0 & 0 & 0 \\
I_N \otimes C^l_{\psi} & 0 & 0 & 0 & 0 \\
\tilde{u}_{\hat{\lambda}} = \text{diag} \left( \begin{bmatrix} P \otimes I_{\hat{\xi}} \otimes I_{\hat{\eta}}, P \otimes I_{\hat{\eta}}, P \otimes I_{\hat{\eta}} \otimes I_{\hat{\theta}} \end{bmatrix} \right) \right.
\end{cases}
\end{array}
\]

where in addition to the uncertainty channel resulting from the uncertain topology an uncertainty channel representing the scheduling parameters \( \Theta \) appears. Here \( \Theta(\theta(t)) \) is a matrix-valued mapping that maps \( \theta(t) \in \mathbb{R}^{n_{\theta}} \) into \( \mathbb{R}^{n_{\eta} \times n_{\eta}} \), and \( \Theta \) is a set of vertices of a polytope that contains all admissible values of \( \Theta(t) \). Since the uncertainty can appear in the decentralized as well as in the interconnected part, the size of the uncertainty channel increases by \( Nn_{\theta} \) as well.

To enforce that the closed loop system is decomposable it is necessary that \( D^d_{\psi,\theta} = 0 \), by \( D^d_{\psi,\theta} = 0 \) and e.g. \( D^d_{\psi,\theta} = 0 \) and \( D^i_{\psi,\theta} = 0 \). Furthermore it has to be assumed e.g. that \( D^d_{\theta,\theta} = 0 \), \( D^d_{\psi,\theta} = 0 \), \( D^i_{\psi,\theta} = 0 \), \( D^d_{\theta,\theta} = 0 \), \( D^i_{\theta,\theta} = 0 \).

**Theorem 1:** Given is a known set of topologies \( \mathcal{P} \) and a decomposable LPV system \( \check{T} \), with either an uncertain time-invariant topology \( \check{P} \in \mathcal{P} \) or an uncertain time-varying topology \( \check{P}(t) \in \mathcal{P} \). System \( \check{T} \) is stable and, in the

(i) time-invariant case, \( \check{T} \) has an induced \( L_2 \) gain less than \( \gamma \),

(ii) time-varying case, \( \check{T} \) has an induced \( L_2 \) gain less than \( \gamma \),

if there exist \( X^d > 0 \) and \( Q = Q^T, R = R^T \) and \( S \) such that

\[
[\star]^T \begin{bmatrix} Q \otimes S \ \ I_{\psi} \otimes \rho \end{bmatrix} \begin{bmatrix}
\Psi & I_{\psi} \otimes \theta \\
0 & I_{\psi} \otimes \theta
\end{bmatrix} > 0 \ \forall \psi \in \Psi, \quad (12)
\]

with

\[
\begin{bmatrix} Q^d \ \ Q^i \end{bmatrix} = \begin{bmatrix} Q^d_{\psi} \ \ Q^d_{\theta} \end{bmatrix}
\]

and \( n_{\lambda} = n_{\xi} + n_{\eta} + n_{\theta} \). The matrices \( S_{\psi}, R_{\theta} \) and \( S \) and \( R \) are partitioned respectively.

**Proof:** The proof is similar to that of Theorems 1, 2 and 3 in [1]. In case of time-invariant topologies the signal transformation is applied. Because of the larger uncertainty channel of the interconnection, the multiplier matrices have to be partitioned into three parts with dimensions \( n_{\xi}, n_{\theta} \) and \( n_{\eta} \), respectively. In addition, a multiplier has to be introduced for the LPV-uncertainty channel. This also has to be partitioned according to the multiplier for the interconnection, such that the multiplier condition can be permuted and (12) and (13) are achieved.

### IV. Controller Synthesis

In the following the synthesis of a controller that satisfies Theorem 1, is presented. We will focus on finding a distributed output feedback controller \( \check{K}_{\text{OF}} \), but we show in the second part of this section how the synthesis simplifies if state feedback can be used as in [5].

#### A. Output Feedback Controller

According to Theorem 1, a stabilizing controller for \( \tilde{T} \) that guarantees an induced \( L_2 \) gain of \( \tilde{T} \) less than \( \gamma \) for

all \( \Theta \in \Theta \) exists, if \( X^d > 0 \), \( Q = Q^T, R = R^T \), \( S \) and \( A^{\lambda}, A^{\iota}, B^{\lambda}, B^{\iota}, C^{\lambda}, C^{\iota}, D^{\lambda}, D^{\iota} \)

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The closed-loop matrices in (13) are given by

\[
\begin{bmatrix}
A^i & B^i & D_{i\delta}^i \\
C_{i\alpha} & D_{i\alpha} & D_{i\alpha\delta}
\end{bmatrix}
\]

The matrix inequality (13) is not linear in the variables. Therefore a congruence transformation of (13) by \(\text{diag}(T_Y, I, I)\) is performed, leading to

\[
[x]^T \begin{bmatrix}
0 & I \\
I & Z
\end{bmatrix}
\begin{bmatrix}
0 & I \\
I & Z
\end{bmatrix}^{-1}
\begin{bmatrix}
\begin{bmatrix}
    0 & I \\
    I & 0
\end{bmatrix} & \begin{bmatrix}
    A^i & B^i & D_{i\delta}^i \\
    C_{i\alpha} & D_{i\alpha} & D_{i\alpha\delta}
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
    0 & I \\
    I & 0
\end{bmatrix} & \begin{bmatrix}
    A^i & B^i & D_{i\delta}^i \\
    C_{i\alpha} & D_{i\alpha} & D_{i\alpha\delta}
\end{bmatrix}
\end{bmatrix}
< 0
\]

with the transformation matrices

\[
T_Y = \begin{bmatrix}
    Y \\
    V^T
\end{bmatrix},
\quad
T_Z = \begin{bmatrix}
    I \\
    0
\end{bmatrix}
\]

and \(U\) and \(V\) non-singular and \(I - Z Y = U V^T\). The relationship between the original Lyapunov matrix \(X^d\) and the transformed matrices is given by \(X^dT_Y = T_Z X^d\).

Considering only the outer factors in (16), the congruence transformation of (13) transforms the closed-loop variables into

\[
T_Y^2 \begin{bmatrix}
    A & B & D \\
    C & D & D_{\alpha\delta}
\end{bmatrix}^T = \begin{bmatrix}
    A & B & D \\
    C & D & D_{\alpha\delta}
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
    0 & I \\
    I & 0
\end{bmatrix} < 0
\]

are satisfied. Here (20) ensures that the transformed Lyapunov matrix is positive definite, while (21) is the transformation of (13).

Condition (21) is still not linear in the variables, since there are product terms between the multiplier variables and \(C_{i\alpha}, D_{i\alpha\delta}\) and \(D_{i\alpha\delta}\). In [12] a procedure is proposed, which is adapted here to the special problem of finding a distributed controller. The special aspect here is that in fact two controllers have to be found simultaneously, the decentralized one and the interconnected one.

The first step is to find a feasible solution of (12), (20) and (21); this is followed by an iterative procedure to optimize the performance.

1) Feasible Initial Solution: The initial step is the nominal design for \(\Delta = 0\) for all \(\Delta \in \Delta\). This leads to solutions for \(Z, Y, U\) and \(K^d, L^d, M^d\) and \(N^d\). This step can be interpreted as cutting all existing interconnections and considering only the nominal plant without any parameter dependency. The task is then to find a decentralized nominal controller that stabilizes the decentralized nominal part of the plant. If \((A^d, B^d)\) is stabilizable and \((A^d, C^d)\) is detectable, a feasible solution is known to exist, because those properties are invariant under congruence transformation.

The next step, which is called (II) in the following, is to fix \(Z\) and \(U\) and and solve for \(Q, S, R\) and \(Y, K^d, L^d, M^d, N^d, K^i, L^i, M^i\) and \(N^i\) for the scaled uncertainties \(r\Delta\) for all \(\Delta \in \Delta\). The scaling factor \(r\) is to be maximized by bisection. Since in (21) the interconnected transformed
controller variables appear in \( C_\Delta \) and \( D_\Delta p \) and thus are multiplied by the multiplier, (21) is not linear in \( Q, S \) and \( R \) and \( K, L, M \) and \( N \). Using the Dualization Lemma [7] the scaled versions of (12) and (21), which are linear in the dual unknowns, are

\[
[s]^T \begin{bmatrix}
Q_{\infty} & S_{\infty} & R_{\infty} \\
S_{\infty}^T & I & 0 \\
R_{\infty}^T & 0 & I
\end{bmatrix} \begin{bmatrix}
I & -r_\Delta T \\
-r_\Delta T & 0
\end{bmatrix} < 0 \quad \forall \Delta \in \Delta (22)
\]

\[
\begin{bmatrix}
0 & I & 0 \\
I & 0 & 0 \\
Q & S & R
\end{bmatrix}^T \begin{bmatrix}
-A^T & -C^T & -C_p^T \\
C^T & I & 0 \\
-B^T & -D^T & -D_p^T
\end{bmatrix} \begin{bmatrix}
0 & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix} \begin{bmatrix}
0 & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix} > 0 (23)
\]

with \( \frac{Q}{S} R = \begin{bmatrix} Q & S & R \end{bmatrix}^{-1} \).

Having found \( Q, S, R \) and \( Y, K, L, M, N \) that maximize \( r > r_{\text{max}} \), the multiplier variables are fixed in the second step (I2), and again by bisection values of the remaining parameters are found that maximize \( r \). Here (20) has to be fulfilled together with a scaled version of (23), which is

\[
\begin{bmatrix}
0 & I & 0 \\
I & 0 & 0 \\
\hat{Q} & \hat{S} & \hat{R}
\end{bmatrix}^T \begin{bmatrix}
-A^T & -C^T & -C_p^T \\
C^T & I & 0 \\
-B^T & -D^T & -D_p^T
\end{bmatrix} \begin{bmatrix}
0 & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix} \begin{bmatrix}
0 & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix} > 0 (24)
\]

with \( \hat{r} = \frac{r}{r_{\text{max}}} \). By this scaled condition, it is ensured that the dual of (12), which is

\[
[s]^T \begin{bmatrix}
Q_{\infty} & S_{\infty} & R_{\infty} \\
S_{\infty}^T & I & 0 \\
R_{\infty}^T & 0 & I
\end{bmatrix} \begin{bmatrix}
I & -\Delta T \\
-\Delta T & 0
\end{bmatrix} < 0 \quad \forall \Delta \in \Delta (25)
\]

is satisfied, such that (25) does not need to be considered in this step. It can be seen that in (24) we have quadratic terms in \( B_\Delta \) and \( B_{\Delta p} \). Here the inverse Schur complement can be used for the inner block \( Q \) and \( -\frac{1}{\gamma} I \), since both are negative definite.

After maximizing \( r \), we repeat iterating between (I1) and (I2) until a value \( r \geq 1 \) is reached, that gives an initial feasible solution. Now the performance can be optimized.

2) Performance Optimization: The performance is optimized in an iterative process as well, by minimizing \( \gamma \) while satisfying the conditions (20), (23) and (25). In the first step (O1) of the iteration \( Z, Y, U, K, L, M, N \) are fixed and \( \gamma \) is minimized subject to (23) and (25). In the second step (O2) \( Q, S, \hat{R} \) and \( K, L, M, N \) are fixed and \( \gamma \) is minimized subject to (20) and (23). As before, the inverse Schur complement is used to eliminate quadratic terms, thus \( Q \) has to be inverted, which may lead to numerical problems. To avoid these it can be helpful to impose an upper bound on \( Q \) by the extra condition \( Q < \alpha I \) with a suitable \( \alpha > 0 \).

B. Dynamic State Feedback Controller

If access to the system states is available and a dynamic state feedback controller can be used, the synthesis simplifies considerably, as also discussed in [5]. In this case we substitute in (15) \( C_d \equiv I \) and \( D_{y_2} = 0 \). As in the output feedback case the inequality (6) is nonlinear, but here no iteration is necessary since the problem can be convexified: first the congruence transformation \( \text{diag}(Y, I, I) \) with \( Y = X^{-1} \) is applied to (13). Applying the dualization Lemma [7] to the transformed versions of (13) and to (7), followed by a change of variables, leads to linear matrix inequalities in the parameters \( Y > 0, M_1, M_2, M_3 \) and \( M_4 \). The controller matrices are then reconstructed according to

\[
\begin{bmatrix}
D_{Kd} & C_{Kd} \\
D_{Ki} & C_{Ki}
\end{bmatrix} = M_1 Y^{-1} \begin{bmatrix}
B_{Kd} & A_{Kd} \\
B_{Ki} & A_{Ki}
\end{bmatrix} = M_2 Y^{-1} (26)
\]

V. Numerical Results

The controller synthesis approach proposed in the previous section is tested on a numerical example. For this purpose, the distributed system given in equation (27) from [5] is considered, and a linear fractional uncertainty is added to allow a robust controller synthesis. The uncertainty is chosen to be rather small, in order to be able to find a feasible solution.

The procedure of output feedback synthesis is performed for the system (27) with a symmetric interconnection pattern \( P_{\text{sym}} \), where \( \lambda = (-2.2361, 2.2361) \), as shown in Figure 1. For comparison, the controller synthesis is performed for the LPV system (27) as well as for its LTI part by setting \( \Theta = 0 \). These cases are indicated in the following by indices LPV and LTI, respectively. For numerical reasons we set \( \alpha = 10^3 \). To find an initial solution the iterative process is performed for \( r = [0, 1.1] \) and is stopped if \( r \geq 1 \). In both cases a feasible solution is found after one iteration of step I1. Figure 2 shows the improvement of \( \gamma \) in the following optimization process. The optimization is stopped if the improvement between two iteration steps is less than 0.01. In the LTI case 27 iteration steps are performed, leading to \( \gamma_{\text{LTI}} = 0.2796 \) with \( \|T\|_\infty = 0.2710 \). For the LPV case 32 iteration steps lead to \( \gamma_{\text{LPV}} = 0.2802 \). This value is slightly larger than that of the LTI case, but not significantly due to the relatively small size of the uncertainty.

\[1\] The matrix \( P \) in [5] contains a typo; but the eigenvalues given there are consistent with the graph and \( P \), used here.

\[
P_{\text{sym}} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Fig. 1. Numerical example: Graph and corresponding pattern matrix
Fig. 2. Numerical example: Iteration to optimize the output feedback design

\[ \dot{\bar{z}} = \begin{bmatrix} \dot{x} \\ \\ \Theta \dot{z} \end{bmatrix} = \begin{bmatrix} I_N \otimes 0.1 & 0.2 & 0.7 \\ 0 & -0.9 & 0.6 & -0.4 \\ -0.9 & 0.6 & -0.5 & -0.1 & -0.1 & 0.1 & 0.1 & 0.1 & 0.7 \\ 0 & -0.3 & -0.3 & -0.3 & -0.3 \\ 0 & -0.2 & -0.2 & -0.2 & -0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} P \otimes 0.1 & 0.1 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1 \end{bmatrix} \otimes \begin{bmatrix} 0 \end{bmatrix} \otimes 0 \]

\[ \bar{w}_{\Theta} = \Theta \dot{z}, \Theta \in [-1, 1]. \]

(27)

VI. CONCLUSIONS

In this work decomposable systems are considered, whose subsystems can include LPV dynamics with homogeneous rational dependency on uncertain parameters. Conditions for stability of these decomposable LPV systems are derived.

An existing approach to distributed state feedback controller synthesis is extended to distributed output feedback controller synthesis for decomposable LPV systems, whose interconnection pattern is allowed to vary within a known set of topologies. Here the results of the first part of this work [1] are used to deal with time-invariant and time-varying topologies.

REFERENCES


