Abstract.

An algorithm is presented for the computation of analytical expressions for the extremal values of the $\alpha$-cuts of the fuzzy weighted average, for triangular or trapezoidal weights and attributes. Also an algorithm for the computation of the inverses of these expressions is given, providing exact membership functions of the fuzzy weighted average. Up to now, only algorithms exist for the computation of the extremal values of the $\alpha$-cuts for a fixed value of $\alpha$. To illustrate the power of our algorithms, they are applied to several examples from the literature, providing exact membership functions in each case.

1. Introduction.

In multiple criteria decision making problems, values of decision variables are weighted averages of criteria ratings. Often the rating criteria and their corresponding importance weights are vague, and therefore represented by fuzzy numbers. Then the values of the decision variables which are determined by them are fuzzy numbers as well; they are fuzzy weighted averages of the criteria ratings.

The standard approach to the calculation of fuzzy weighted averages [1–5, 7–10] is to apply the extension principle to the following weighted average function:

$$wa(x_1,..,x_n,w_1,..,w_n) = \frac{\sum_{i=1}^{n} (x_i * w_i)}{\sum_{i=1}^{n} w_i}$$  \hspace{1cm} (1)

Here $x_1,..,x_n$ are real numbers, called attributes, and $w_1,..,w_n$ are non-negative real numbers, called weights.

Let $A_1,..,A_n$ and $W_1,..,W_n$ be triangular or trapezoidal fuzzy numbers. Their $\alpha$-cuts, denoted by $A_{1\alpha},..,A_{n\alpha}$, $W_{1\alpha},..,W_{n\alpha}$, are closed intervals; the elements of $W_{1\alpha},..,W_{n\alpha}$ are non-negative. The $\alpha$-cut of the fuzzy weighted average of attributes $A_1,..,A_n$ with weights $W_1,..,W_n$ is given by the set

$$\{wa(x_1,..,x_n,w_1,..,w_n) | x_1 \in A_{1\alpha},..,x_n \in A_{n\alpha}, w_1 \in W_{1\alpha},..,w_n \in W_{n\alpha}\}$$  \hspace{1cm} (2)

This set is a closed interval and is computed by computing its extremal values. Algorithms for computing these extremal values, for fixed value of $\alpha$, are given in the literature [1–5, 7–10]. In this paper we will give an algorithm to solve this problem analytically, i.e. we will show how to
compute analytical expressions for the extremal values of eq. (2). Also an algorithm for the computation of the inverses of these expressions is given; this enables us to calculate analytically the exact membership functions of the fuzzy weighted average.

In section 2 we will present an algorithm for the computation of the extremal values of eq. (2) for a fixed value of α, which is based on previous algorithms [1–5, 7–10]. This algorithm has an intuitive geometrical interpretation, and therefore it can be easily understood.

In section 3 this algorithm is generalised to give analytical solutions, for triangular or trapezoidal weights and attributes, and it is shown how these solutions can be reversed to give exact membership functions of the fuzzy weighted average.

In section 4 the power of the algorithm is demonstrated by applying it to all examples in [1–5, 7–10], and to the case study in [6], giving exact membership functions of the fuzzy weighted average in all cases.

### 2. Computation of α-cuts of the fuzzy weighted average

Let us start with the computation of the minimal value of the set of eq. (2). We will denote the maximal and minimal value of the α–cut $A_\alpha$ by $A_\alpha^-$ and $A_\alpha^+$ respectively, so $A_\alpha^-$ is equal to the interval $[A_\alpha^-,A_\alpha^+]$. It is shown in [2,10] that the minimum of eq. (2) is among the elements where $x_i$ is equal to $A_\alpha^-$ and $w_i$ belongs to $\{W_i\alpha^-,W_i\alpha^+\}$. So the problem can be reformulated as the problem of finding the minimal element of the finite set

$$\left\{ \sum_{i=1}^{n} (A_{i\alpha^-} \cdot w_i) \middle/ \sum_{i=1}^{n} w_i \mid w_i \in \{W_{1\alpha^-},W_{1\alpha^+}\}, \ldots, w_n \in \{W_{n\alpha^-},W_{n\alpha^+}\} \right\}$$

Let $Q$ be the set of all $2^n$ mappings from the set $\{1,2,\ldots,n\}$ to the set $\{+,-\}$. These mappings can be seen as states. A state is a partition of the attributes in attributes with maximal weight and attributes with minimal weight. The set of eq. (3) can be denoted as

$$\left\{ \sum_{i=1}^{n} (A_{i\alpha^-} \cdot W_{i\alpha q(i)}) \middle/ \sum_{i=1}^{n} W_{i\alpha q(i)} \mid q \in Q \right\}$$

We define the mapping $FWA$ from states to the real numbers by

$$FWA(q) = \sum_{i=1}^{n} \frac{A_{i\alpha^-} \cdot W_{i\alpha q(i)}}{\sum_{i=1}^{n} W_{i\alpha q(i)}}$$

Now the problem has become the problem of finding the state $q$ for which $FWA(q)$ is minimal.

The first step of the algorithm consists of sorting the set $\{A_{i\alpha^-} \mid 1 \leq i \leq n\}$. From now on we assume this set is sorted. As a consequence, $A_{1\alpha^-} \leq FWA(q) \leq A_{n\alpha^-}$ for all $q \in Q$. Now let us consider what happens when $q$ changes such that, for some $i$ with $1 \leq i \leq n$, $q(i)$ changes from $-$ into $+$. This change means that the weight of $A_{i\alpha^-}$ increases, whereas the other weights remain the same. As a consequence, $FWA(q)$ will change in such a way that it moves towards $A_{i\alpha^-}$: the absolute value of $FWA(q) - A_{i\alpha^-}$ decreases, but its sign remains the same. Let $q_{\min}$ be the state for which $FWA$ takes its minimal value. It follows that $q_{\min}(i)=+$ if $A_{i\alpha^-} < FWA(q_{\min})$ and $q_{\min}(i)=-$ if $A_{i\alpha^-} > FWA(q_{\min})$. We know that $q_{\min}(1)=+$ and $q_{\min}(n)=$. Our algorithm to obtain $q_{\min}$ is now as follows. Let $q_i$ be the element of $Q$ defined by $q_i(j)=+$ for $1 \leq j \leq i$ and $q_i(j)=-$ for $i < j \leq n$. Then $q_{\min} = q_i$ for some $i$ with $2 \leq i \leq n$. To determine $q_{\min}$, start with state $q_2$. Let us consider a situation with $n=6$, which is depicted as follows:
Here the first line shows the real axis with \( A_{i\alpha} \) for \( i=1..6 \), and the fuzzy weighted average (Fwa) in the present state \( q_2 \). The second line shows the indices \( i \), and whether the corresponding weights are maximal (+) or minimal (-) in the present state. We have to change the state in such a way that Fwa becomes as small as possible. Changing the state means changing the weights from maximal to minimal or vice versa. Changing weights 1, 4, 5, or 6 would increase Fwa. Changing weights 2 or 3 decreases Fwa. Suppose we change weight 3. The value of Fwa decreases, but remains greater than \( A_{3\alpha} \). Next we change weight 2. The value of Fwa is further decreased, and can become less than \( A_{3\alpha} \). In this case Fwa would decrease even further by restoring the original weight 3. This situation would have been avoided if we had first changed weight 2. If this causes Fwa to become less than \( A_{3\alpha} \), the minimum is reached in state \( q_3 \). Otherwise, weight 3 is changed as well, and the minimum is reached in state \( q_4 \). So it is important that the weight changes are performed from left to right. In [10] the weight is changed first that causes Fwa to decrease most, which is incorrect.

Let us now return to the general case. The initial state is \( q_2 \). Compare \( \text{FWA}(q_2) \) with \( A_{2\alpha} \). If \( \text{FWA}(q_2) < A_{2\alpha} \) then \( q_{\text{min}} = q_2 \), else continue with \( q_3 \). If \( \text{FWA}(q_3) < A_{3\alpha} \) then \( q_{\text{min}} = q_3 \), else continue with \( q_4 \), et cetera. This iteration will terminate, since \( \text{FWA}(q_n) < A_{n\alpha} \). So, the algorithm to compute the minimum of \( \text{FWA}(q) \) for \( q \in Q \) is

\[
\begin{align*}
\text{sort} \{ A_{i\alpha} | 1 <= i <= n \}; \\
i = 2; \\
\text{while} \ (\text{FWA}(q_i) > A_{i\alpha}) \ i++; \\
\text{return} \ \text{FWA}(q_i);
\end{align*}
\]

The computation of the maximal value of the set of eq. (2) is quite similar. We now assume that the set \( \{ A_{i\alpha} | 1 <= i <= n \} \) is sorted. Note that this ordering may be different from the one above. In the definition of FWA (eq. (5)) we replace \( A_{i\alpha} \) by \( A_{i\alpha}^+ \). Let \( q_{\text{max}} \) be the value of \( q \) for which FWA(q) is maximal. Then \( q_{\text{max}}(i) = - \) if \( A_{i\alpha}^+ < \text{FWA}(q_{\text{max}}) \) and \( q_{\text{max}}(i) = + \) if \( A_{i\alpha}^+ > \text{FWA}(q_{\text{max}}) \). In particular, \( q_{\text{max}}(1) = - \) and \( q_{\text{min}}(n) = + \). Let \( q' \) be the element of \( Q \) defined by \( q'(j) = - \) for \( 1 <= j <= i \) and \( q'(j) = + \) for \( i < j <= n \). Then \( q_{\text{max}} = q' \) for some \( i \) with \( 1 <= i <= n-1 \). To determine \( q_{\text{min}} \), start with \( q^{(n-1)} \). Compare \( \text{FWA}(q^{(n-1)}) \) with \( A_{(n-1)\alpha}^+ \). If \( \text{FWA}(q^{(n-1)}) < A_{(n-1)\alpha}^+ \) then \( q_{\text{min}} = q^{(n-1)} \), else continue with \( q^{(n-2)} \), et cetera. So, the algorithm to compute the maximum of \( \text{FWA}(q) \) for \( q \in Q \) is

\[
\begin{align*}
\text{sort} \{ A_{i\alpha} | 1 <= i <= n \}; \\
i = n-1; \\
\text{while} \ (\text{FWA}(q_i) < A_{i\alpha}) \ i--; \\
\text{return} \ \text{FWA}(q_i);
\end{align*}
\]

Let us illustrate the algorithm with a small example, with \( n=6 \).
<table>
<thead>
<tr>
<th>i</th>
<th>$A_{i\alpha}$</th>
<th>$W_{i\alpha}$</th>
<th>FWA($q_i$)</th>
<th>i</th>
<th>$A_{i\alpha}$</th>
<th>$W_{i\alpha}$</th>
<th>FWA($q^i$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[1,4]</td>
<td>[1,3]</td>
<td></td>
<td>1</td>
<td>[1,4]</td>
<td>[1,3]</td>
<td>109/16</td>
</tr>
<tr>
<td>2</td>
<td>[2,6]</td>
<td>[1,3]</td>
<td>23/8</td>
<td>2</td>
<td>[4,5]</td>
<td>[1,3]</td>
<td>99/14</td>
</tr>
<tr>
<td>3</td>
<td>[3,7]</td>
<td>[1,3]</td>
<td>27/10</td>
<td>3</td>
<td>[2,6]</td>
<td>[1,3]</td>
<td>87/12</td>
</tr>
<tr>
<td>4</td>
<td>[4,5]</td>
<td>[1,3]</td>
<td>33/12</td>
<td>4</td>
<td>[3,7]</td>
<td>[1,3]</td>
<td>73/10</td>
</tr>
<tr>
<td>5</td>
<td>[5,8]</td>
<td>[1,3]</td>
<td>41/14</td>
<td>5</td>
<td>[5,8]</td>
<td>[1,3]</td>
<td>57/8</td>
</tr>
<tr>
<td>6</td>
<td>[6,9]</td>
<td>[1,3]</td>
<td>51/16</td>
<td>6</td>
<td>[6,9]</td>
<td>[1,3]</td>
<td></td>
</tr>
</tbody>
</table>

In the first table the $\alpha$-cuts $A_{i\alpha}$ and $W_{i\alpha}$ for some fixed $\alpha$ are shown, sorted according to $A_{i\alpha}$. The fourth column shows the values of FWA($q_i$). The minimum is seen to be 27/10, being the first value from above in the fourth column which is not greater than the corresponding value for $A_{i\alpha}$ (in the second column). In the second table the elements are sorted according to $A_{i\alpha}$. The fourth column shows the values of FWA($q^i$). The maximum is seen to be 73/10, being the first value from below in the fourth column which is not less than the corresponding value for $A_{i\alpha}$.

The extremal values obtained above could also have been found by taking the smallest resp. highest values in the fourth columns of the tables above; this is indeed the approach of [1]. So apparently we did not gain anything, except some geometrical insight, as explained above. However, as it will turn out in the next section, for obtaining an analytical solution it is crucial not to compare values of the fourth column among each other, but instead compare elements of the fourth column with elements in the second column.

The computational complexity of our algorithm is $O(n*\ln(n))$, due to the first step, the sorting of the elements. The second phase, whose computational complexity is $O(n)$, could be optimized by replacing the linear search by binary search, resulting in computational complexity $O(\ln(n))$, as in [8]; the overall computational complexity would remain $O(n*\ln(n))$, however. In [5] an algorithm with computational complexity $O(n)$ is given, in which the sorting of the elements is avoided. We have tried to keep our algorithm as simple as possible, in order to be able to generalize it to obtain an analytical solution.

### 3. Analytical Solution for the fuzzy weighted average

In this section we will show that the algorithm given in the previous section can be generalized to obtain an analytical solution for the membership function of the fuzzy weighted average. There have been two previous attempts to obtain an analytical solution. Dong and Wong [2] obtained an analytical solution for two small examples. A general method was not given, however. Their approach was to consider the partial derivatives with respect to $w_i$ of eq. (1) in order to obtain the extremal values of this equation. Kao and Liu [7] followed the same approach, and applied it to the same two examples, but also failed to provide a general solution. Our approach is different. We will generalize the algorithm of the previous section, by considering $\alpha$ to be a parameter which ranges over the interval $[0,1]$, instead of being some fixed value.

The values of the fuzzy weights and fuzzy attributes are triangular and trapezoidal fuzzy numbers. A trapezoidal fuzzy number will be denoted as a 4–tuple $(a,b,c,d)$ where $a,b,c$ and $d$ are real numbers with $a<=b<=c<=d$. The trapezoidal number $(a,b,c,d)$ has membership function $\mu$, given by
The restriction of \( \mu \) to \([a,b] \) and \([c,d] \) will be referred to as the left part resp. the right part of the trapezoidal number. A triangular fuzzy number is a trapezoidal number of the form \((a,b,b,c)\), and will be denoted by the 3–tuple \((a,b,c)\).

Our first aim is to find a function of \( \alpha \) which is the minimum of the set of eq. (4) for all \( \alpha \) in the interval \([0,1] \). The first step of the algorithm consists of sorting the set \( \{A_{i\alpha} \} \) for all \( \alpha \) in the interval \([0,1] \). However, this sorting is the same for each \( \alpha \) only if the left sides of the \( A_i \) do not cross. So, we compute all the values of \( \alpha \) for which two left sides cross. There are at most \( n(n-1)/2 \) such crosspoints. In practice, however, it turns out that there are only few crosspoints, if any at all. The crosspoints partition the interval \([0,1] \) in at most \( n(n-1)/2+1 \) subintervals. On each of these subintervals the left parts of the \( A_i \) do not cross and the set \( \{A_{i\alpha} \} \) can be sorted independent of \( \alpha \). We will consider each of these subintervals separately. So, in this step the problem has been reduced to the problem of finding the minimum of the set of eq. (4) for all \( \alpha \) in some subinterval \([min,max] \) of \([0,1] \) where the ordering of the set \( \{A_{i\alpha} \} \) is sorted independent of \( \alpha \). The next step of the algorithm is to compare \( \text{FWA}(q_2) \) with \( A_{2\alpha} \). For values of \( \alpha \) for which \( \text{FWA}(q_2) \leq A_{2\alpha} \) the minimum is \( \text{FWA}(q_2) \), for the other values of \( \alpha \) the computation will continue with \( q_3 \). Since all fuzzy numbers are trapezoidal, \( A_{2\alpha} \), \( W_{2\alpha} \), and \( W_{2\alpha}^+ \) are all linear functions of \( \alpha \). From the definition of \( \text{FWA} \) (eq. (4)) it thus follows that the equation \( \text{FWA}(q_2) \leq A_{2\alpha} \) takes the form \( (a\alpha^2+b\alpha+c)/(d\alpha+e) \leq (f\alpha+g) \), where \( a,b,c,d,e,f \) and \( g \) are real numbers. Since \( d\alpha+e \) is positive, this can be written as \( a\alpha^2+b\alpha+c \leq (d\alpha+e)(f\alpha+g) \). This can be easily solved. The roots of \( a\alpha^2+b\alpha+c = (d\alpha+e)(f\alpha+g) \) which belong to the interval \([min,max] \) partition \([min,max] \) in at most three subintervals. On each of these subintervals the result of the comparison \( \text{FWA}(q_2) \leq A_{2\alpha} \) is independent of \( \alpha \). On intervals where \( \text{FWA}(q_2) \leq A_{2\alpha} \), the analytical solution is obtained, which is equal to \( \text{FWA}(q_2) \), and has the form \( (a\alpha^2+b\alpha+c)/(d\alpha+e) \). On intervals where \( \text{FWA}(q_2) \geq A_{2\alpha} \), the computation continues in state \( q_3 \). Iteration of this process leads to the analytical solution of the minimum of the set of eq. (4) for all \( \alpha \) in the interval \([0,1] \). This solution may consist of separate solutions for a finite number of subintervals of \([0,1] \).

The algorithm can be summarized as follows:

1. Calculate the crosspoints of the left sides of the attributes;
2. Partition \([0,1] \) into subintervals according to the crosspoints;
3. For each subinterval \([min,max] \) do
   - Adapt the numbering such that \( \{A_{i\alpha} \} \) is sorted;
   - Exact solution on \([min,max] \) is \( \text{Proc} ([min,max],2) \);

where \( \text{Proc} \) is defined by
Proc (interval, i) {
    Partition the interval into subintervals according to solutions of $FWA(q_i) = A_{i\alpha-}$;
    On subintervals where $FWA(q_i) \leq A_{i\alpha-}$ the exact solution is $FWA(q)$;
    On subintervals where $FWA(q_i) \geq A_{i\alpha-}$ the exact solution is Proc (subinterval, i+1);
}

Note that the key element in this algorithm is the comparison of $FWA(q_i)$ with $A_{i\alpha-}$, which leads to a second order polynomial equation to be solved. An algorithm which compares values of $FWA(q_i)$ for different values of $i$, would have led to a third order polynomial equation, which is much more difficult to solve, if at all possible.

The solutions of $FWA(q_i) = A_{i\alpha-}$ partition the interval under consideration into at most three subintervals. It is possible that two of the subintervals need further consideration. However, this hardly occurs in practice. In all examples in the next section, all taken from $[1..10]$, it does not occur even once. It is easily shown that, if it occurs, it cannot occur a second time on the same subinterval. So, combinatorial explosion in the number of subintervals is impossible.

The algorithm to calculate a function of $\alpha$ which is the maximum of the set of eq. (4) for all $\alpha$ in the interval $[0,1]$ is quite similar. It can be summarized as follows

Calculate the crosspoints of the right sides of the attributes;
Partition $[0,1]$ into subintervals according to the crosspoints;
For each subinterval $[\text{min}, \text{max}]$ {
    Adapt the numbering such that $\{A_{i\alpha-}|1\leq i\leq n\}$ is sorted;
    Exact solution on $[\text{min}, \text{max}]$ is Proc ([\text{min}, \text{max}], n–1);
}

where Proc is defined by

Proc (interval, i) {
    Partition the interval into subintervals according to the solutions of $FWA(q^i) = A_{i\alpha+}$;
    On subintervals where $FWA(q^i) \geq A_{i\alpha+}$ the exact solution is $FWA(q)$;
    On subintervals where $FWA(q^i) \leq A_{i\alpha+}$ the exact solution is Proc (subinterval, i–1);
}

The second aim in this section is to show that the exact solutions for the minimum and maximum values of the $\alpha$-cuts of the fuzzy weighted average can be inverted to give the exact membership function of the fuzzy weighted average.
Suppose on some subinterval $[\text{min}, \text{max}]$ of $[0,1]$ the minimum of the $\alpha$-cuts has been determined with our algorithm. It is a function of $\alpha$ of the form $(a\alpha^2+b\alpha+c)/(d\alpha+e)$. The inverse of this function is the exact left part of the membership function on the interval

$$[(a\text{min}^2+b\text{min}+c)/(d\text{min}+e), (a\text{max}^2+b\text{max}+c)/(d\text{max}+e)]$$

which can be computed by solving $\alpha$ from the equation

$$x = (a\alpha^2+b\alpha+c)/(d\alpha+e)$$
Let us first consider the case where \( a \neq 0 \). Here the solution is given by

\[
\mu(x) = (dx - b \pm \sqrt{(dx - b)^2 - 4a(c - ex)}) / (2a)
\]  

(8)

where the ambiguity in the sign can be solved with the boundary conditions

\[
\begin{align*}
\mu((a*\text{min}^2 + b*\text{min} + c)/(d*\text{min} + e)) &= \text{min} \\
\mu((a*\text{max}^2 + b*\text{max} + c)/(d*\text{max} + e)) &= \text{max}
\end{align*}
\]  

(9a, 9b)

Next consider the case where \( a = 0 \) and \( dc \neq eb \). Here the equation \( x = (b\alpha + c)/(d\alpha + e) \) is solved by

\[
\mu(x) = (ex - c)/(b - dx)
\]  

(10)

on the interval

\[
[(b*\text{min} + c)/(d*\text{min} + e), (b*\text{max} + c)/(d*\text{max} + e)]
\]  

(11)

Finally consider the case where \( a = 0 \) and \( dc = eb \). In this case \( (b\alpha + c)/(d\alpha + e) \) is independent of \( \alpha \), so the inverse does not exist. Then the membership function is non–continuous in \( x = c/e \). This occurs for instance when all attributes and weights are crisp numbers (i.e. of the form \( (a,a,a) \)), leading to a crisp weighted average, whose membership function is not continuous.

In the case of the maximum of the \( \alpha \)-cuts on some subinterval \([\text{min},\text{max}]\), the computation of the right part of the membership function is almost the same, the only difference being that in the eqs. (6) and (11) the expressions for the boundary values of the intervals should be interchanged.

4. Examples

In this section we will apply our algorithms to derive exact membership functions for all fuzzy weighted averages of the examples in [1–5, 7–10] and to the case study of Kao and Liu [6]. In each case, the fuzzy attributes and fuzzy weights are listed, as well as their \( \alpha \)-cuts. This listing, and the assignment of indices from 1 to \( n \), is such that the left parts of the fuzzy attributes are properly ordered on the leftmost subinterval of \([0,1]\). In most of the cases this is the proper ordering on \([0,1]\), both for the left parts and for the right parts of the fuzzy attributes. On subintervals where the ordering is different, this is clearly indicated. Renumbering fuzzy attributes and fuzzy weights has not been given explicitely, however, so the reader should be aware of the fact that the indices in eq. (5) are with respect to the proper ordering, and not necessarily with respect to the listed ordering.

We have avoided the rounding of real numbers as much as seems reasonable. So, quotients and square roots have not been evaluated to decimal form, except where expressions would otherwise become too unwieldy. To discriminate between decimal notations which are exact and those which are approximations, the latter are followed by an asterisk (*)
Example 1: The two–term example of Dong and Wong [2].

For this example exact membership functions have been derived in [2] and [7].

\[
\begin{align*}
A_1 &= (0,1,2) \\
A_1 &= [\alpha, 2-\alpha] \\
W_1 &= (0,0.3,0.9) \\
W_1 &= [0.3\alpha, 0.9-0.6\alpha] \\
A_2 &= (2,3,4) \\
A_2 &= [2+\alpha, 4-\alpha] \\
W_2 &= (0.4,0.7,1) \\
W_2 &= [0.4+0.3\alpha, 1-0.3\alpha]
\end{align*}
\]

Since there are only two attributes, and there are no crosspoints, the calculation is trivial: the minimum of the \(\alpha\)-cuts of the fuzzy weighted average is

\[
\text{FWA}(q_2) = (-0.3\alpha^2 + 1.9\alpha + 0.8)/(-0.3\alpha + 1.3)
\]

and the maximum is \(\text{FWA}(q_1) = -1.6\alpha + 4\).

This leads to the following membership function for the fuzzy weighted average:

\[
\begin{align*}
\mu(x) &= 0 & \text{if } x \leq 8/13 \\
\mu(x) &= x/2 + 19/6 - (5/3)\sqrt{0.09x^2 - 0.42x + 4.57} & \text{if } 8/13 \leq x \leq 12/5 \\
\mu(x) &= -5x/8 + 5/2 & \text{if } 12/5 \leq x \leq 4 \\
\mu(x) &= 0 & \text{if } x \geq 4
\end{align*}
\]

Example 2: The three–term example of Dong and Wong [2].

For this example exact membership functions have been derived in [2] and [7]. The example is also treated in [3,4,10].

\[
\begin{align*}
A_1 &= (0,1,2) \\
A_1 &= [\alpha, 2-\alpha] \\
W_1 &= (0,0.3,0.9) \\
W_1 &= [0.3\alpha, 0.9-0.6\alpha] \\
A_2 &= (2,3,4) \\
A_2 &= [2+\alpha, 4-\alpha] \\
W_2 &= (0.4,0.7,1) \\
W_2 &= [0.4+0.3\alpha, 1-0.3\alpha] \\
A_3 &= (4,5,6) \\
A_3 &= [4+\alpha, 6-\alpha] \\
W_3 &= (0.6,0.8,1) \\
W_3 &= [0.6+0.2\alpha, 1-0.2\alpha]
\end{align*}
\]

Calculation of the minimum of the \(\alpha\)-cuts of the fuzzy weighted average. There are no crosspoints. \(\text{FWA}(q_2) = (-0.1\alpha^2 + 3.3\alpha + 3.2)/(-0.1\alpha + 1.9)\). \(\text{FWA}(q_2) \leq 2+\alpha\) for \(0 \leq \alpha \leq 3/8\), so \(\text{FWA}(q_2)\) is the minimum for \(0 \leq \alpha \leq 3/8\). \(\text{FWA}(q_2) \geq 2+\alpha\) for \(3/8 \leq \alpha \leq 1\). For \(3/8 \leq \alpha \leq 1\) the minimum is \(\text{FWA}(q_3) = (-0.7\alpha^2 + 2.7\alpha + 4.4)/(-0.7\alpha + 2.5)\).

Calculation of the maximum of the \(\alpha\)-cuts of the fuzzy weighted average. There are no crosspoints. \(\text{FWA}(q_2^2) = (-0.4\alpha^2 - 0.8\alpha + 7.6)/(0.4\alpha + 1.4)\). \(\text{FWA}(q_2^2) \geq 4-\alpha\) for \(0 \leq \alpha \leq 1\), so \(\text{FWA}(q_2^2)\) is the maximum for \(0 \leq \alpha \leq 1\).

This leads to the following membership function for the fuzzy weighted average:

\[
\begin{align*}
\mu(x) &= 0 & \text{if } x \leq 32/19 \\
\mu(x) &= x/2 + 16.5 - 5\sqrt{0.01x^2 - 0.1x + 12.17} & \text{if } 32/19 \leq x \leq 19/8 \\
\mu(x) &= x/2 + 27/14 - (5/7)\sqrt{0.49x^2 - 3.22x + 19.61} & \text{if } 19/8 \leq x \leq 32/9 \\
\mu(x) &= -x/2 - 1 + (5/4)\sqrt{0.16x^2 - 1.6x + 12.8} & \text{if } 32/9 \leq x \leq 38/7 \\
\mu(x) &= 0 & \text{if } x \geq 38/7
\end{align*}
\]

\[ A_1 = (0,1,2) \quad A_{1a} = [\alpha, 2-\alpha] \quad W_1 = (0.0, 0.3, 0.9) \quad W_{1a} = [0.3\alpha, 0.9-0.6\alpha] \]

\[ A_2 = (2,3,4) \quad A_{2a} = [2+\alpha, 4-\alpha] \quad W_2 = (0.4, 0.7, 1) \quad W_{2a} = [0.4+0.3\alpha, 1-0.3\alpha] \]

\[ A_3 = (4,5,6) \quad A_{3a} = [4+\alpha, 6-\alpha] \quad W_3 = (0.6, 0.8, 1) \quad W_{3a} = [0.6+0.2\alpha, 1-0.2\alpha] \]

\[ A_4 = (5,6,7) \quad A_{4a} = [5+\alpha, 7-\alpha] \quad W_4 = (0.5, 0.8, 1) \quad W_{4a} = [0.5+0.3\alpha, 1-0.2\alpha] \]

Calculation of the minimum of the \(\alpha\)-cuts of the fuzzy weighted average. There are no crosspoints. FWA(q2) = \((0.2\alpha^2 + 5.3\alpha + 5.7)/(0.2\alpha + 2.4)\). FWA(q2) <= 2+\alpha for 0<=\alpha<=1.

FWA(q3) = \((-0.4\alpha^2 + 4.7\alpha + 6.9)/(-0.4\alpha + 3)\). FWA(q3) >= 4+\alpha for 0<=\alpha<=1, so FWA(q3) is the minimum for 0<=\alpha<=1.

Calculation of the maximum of the \(\alpha\)-cuts of the fuzzy weighted average. There are no crosspoints. FWA(q3) = \((-0.6\alpha^2 – 0.4\alpha + 12.2)/(0.6\alpha + 2)\). FWA(q3) >= 6–\alpha for 0<=\alpha<=1, so FWA(q3) is the maximum for 0<=\alpha<=1.

This leads to the following membership function for the fuzzy weighted average:

\[ \mu(x) = 0 \quad \text{if } x \leq 23/10 \]

\[ \mu(x) = x/2 + 47/8 - (5/4)\sqrt{0.16x^2 - 1.04x + 33.13} \quad \text{if } 23/10 \leq x \leq 112/26 \]

\[ \mu(x) = -x/2 - 8 + (5/2)\sqrt{0.04x^2 - 0.64x + 21.92} \quad \text{if } 112/26 \leq x \leq 59/10 \]

\[ \mu(x) = -x/2 - 1/3 + (5/6)\sqrt{0.36x^2 - 4.32x + 29.44} \quad \text{if } 59/10 \leq x \leq 61/10 \]

\[ \mu(x) = 0 \quad \text{if } x \geq 61/10 \]

Example 4: 5–term example of Lee and Park[8]

\[ A_1 = (1,2,3) \quad A_{1a} = [1+\alpha, 3-\alpha] \quad W_1 = (1,2,5) \quad W_{1a} = [1+\alpha, 5-3\alpha] \]

\[ A_2 = (2,5,7) \quad A_{2a} = [2+3\alpha, 7-2\alpha] \quad W_2 = (2,2.5,3) \quad W_{2a} = [2+\alpha/2, 3–\alpha/2] \]

\[ A_3 = (6,8,9) \quad A_{3a} = [6+2\alpha, 9-\alpha] \quad W_3 = (4,7,9) \quad W_{3a} = [4+3\alpha, 9–2\alpha] \]

\[ A_4 = (7,9,10) \quad A_{4a} = [7+2\alpha, 10–\alpha] \quad W_4 = (3,4,7) \quad W_{4a} = [3+\alpha, 7-3\alpha] \]

\[ A_5 = (10,11,12) \quad A_{5a} = [10+\alpha, 12–\alpha] \quad W_5 = (3,2,4) \quad W_{5a} = [2+\alpha, 4–\alpha] \]

Calculation of the minimum of the \(\alpha\)-cuts of the fuzzy weighted average. There are no crosspoints. FWA(q2) = \((7.5\alpha^2 + 60\alpha + 74)/(2.5\alpha + 16)\). FWA(q2) >= 2+3\alpha for 0<=\alpha<=1.

FWA(q3) = \((-5\alpha^2 + 15.5\alpha + 131)/(4.5\alpha + 14)\). FWA(q3) <= 6+2\alpha for 0<=\alpha<=1, so FWA(q3) is the minimum for 0<=\alpha<=1.

Calculation of the maximum of the \(\alpha\)-cuts of the fuzzy weighted average. There are no crosspoints. FWA(q3) = \((-5\alpha^2 + 15.5\alpha + 131)/(4.5\alpha + 14)\). FWA(q3) <= 6+2\alpha for 0<=\alpha<=1, so FWA(q3) is the minimum for 0<=\alpha<=1.

FWA(q3) = \((-\alpha^2 – 28.5\alpha + 171)/(0.5\alpha + 18)\). FWA(q3) >= 9–\alpha for 0<=\alpha <= 9\sqrt{3} –15, so FWA(q3) is the maximum for 0<=\alpha <= 9\sqrt{3} –15. FWA(q3) <= 9–\alpha for 9\sqrt{3} –15<=\alpha<=1.
FWA(q^2) = (4\alpha^2 - 78.5\alpha + 216)/(-4.5\alpha + 23). FWA(q^2) \geq 7-2\alpha for 9\sqrt{3} - 15 \leq \alpha \leq 1, so FWA(q^2) is the maximum for 9\sqrt{3} - 15 \leq \alpha \leq 1.

These results are in accordance with the results by Lee and Park in [8] for \alpha=0 and \alpha=1. However, their claim that the fuzzy weighted average is a fuzzy triangular number is incorrect. Instead, the membership function of the fuzzy weighted average is calculated to be:

\[ \mu(x) = 0 \quad \text{if } x \leq 76/17 \]
\[ \mu(x) = x/6 - 61/9 + (1/9) \sqrt{2.25x^2 + 123x + 2353} \quad \text{if } 76/17 \leq x \leq 283/37 \]
\[ \mu(x) = -9x/16 + 157/16 - (1/8) \sqrt{20.25x^2 - 338.5x + 2706.25} \quad \text{if } 283/37 \leq x \leq 24-9\sqrt{3} \]
\[ \mu(x) = -x/4 - 57/4 + (1/2) \sqrt{0.25x^2 - 43.5x + 1496.25} \quad \text{if } 24-9\sqrt{3} \leq x \leq 19/2 \]
\[ \mu(x) = 0 \quad \text{if } x \geq 19/2 \]

**Example 5: 3 trapezoidal functions of Kao and Liu[7]**

\[ A_1 = (-2,1,2,3) \quad A_{1a} = [-2+3\alpha, 3-\alpha] \quad W_1 = (0,0.3,0.9) \quad W_{1a} = [0.3\alpha, 0.9-0.6\alpha] \]
\[ A_2 = (1,2,3,5) \quad A_{2a} = [1+\alpha, 5-2\alpha] \quad W_2 = (0.4,0.7,1) \quad W_{2a} = [0.4+0.3\alpha, 1-0.3\alpha] \]
\[ A_3 = (2,3,6,7) \quad A_{3a} = [2+\alpha, 7-\alpha] \quad W_3 = (0.6,0.8,1) \quad W_{3a} = [0.6+0.2\alpha, 1-0.2\alpha] \]

Calculation of the minimum of the \alpha-cuts of the fuzzy weighted average. There are no crosspoints. FWA(q_2) = (-1.3\alpha^2 + 5.6\alpha - 0.2)/(-0.1\alpha + 1.9). FWA(q_2) \leq 1+\alpha for 0 \leq \alpha \leq (19 - \sqrt{109})/12, so FWA(q_2) is the minimum for 0 \leq \alpha \leq (19 - \sqrt{109})/12.

FWA(q_2) \geq 1+\alpha for (19 - \sqrt{109})/12 \leq \alpha \leq 1, so for (19 - \sqrt{109})/12 \leq \alpha \leq 1 the minimum is FWA(q_3) = (-1.9\alpha^2 + 5.6\alpha + 0.4)/(-0.7\alpha + 2.5).

Calculation of the maximum of the \alpha-cuts of the fuzzy weighted average. There are no crosspoints. FWA(q^3) = (-0.7\alpha^2 - 0.8\alpha + 9)/(0.4\alpha + 1.4). FWA(q^3) \geq 5-2\alpha for 0 \leq \alpha \leq 1, so FWA(q^3) is the maximum for 0 \leq \alpha \leq 1.

This leads to the following membership function for the fuzzy weighted average:

\[ \mu(x) = 0 \quad \text{if } x \leq -2/19 \]
\[ \mu(x) = x/26 + 28/13 - (5/13) \sqrt{0.01x^2 - 8.76x + 30.32} \quad \text{if } -2/19 \leq x \leq (31-\sqrt{109})/12 \]
\[ \mu(x) = 7x/38 + 28/19 - (5/19) \sqrt{0.49x^2 - 11.16x + 34.4} \quad \text{if } (31-\sqrt{109})/12 \leq x \leq 41/18 \]
\[ \mu(x) = 1 \quad \text{if } 41/18 \leq x \leq 75/18 \]
\[ \mu(x) = -2x/7 - 4/7 + (5/7) \sqrt{0.16x^2 - 3.28x + 25.84} \quad \text{if } 75/18 \leq x \leq 45/7 \]
\[ \mu(x) = 0 \quad \text{if } x \geq 45/7 \]
Example 6: 8-term example of Chiao[1]

<table>
<thead>
<tr>
<th>A1</th>
<th>A2a</th>
<th>A3a</th>
<th>A4a</th>
<th>A5a</th>
<th>A6a</th>
<th>A7a</th>
<th>A8a</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.01,0.1,0.53)</td>
<td>[0.1,0.1,0.53]</td>
<td>[0.38,0.38,0.53]</td>
<td>[0.58,0.58,0.63]</td>
<td>[0.78,0.78,0.83]</td>
<td>[0.98,0.98,1]</td>
<td>[0.19,0.19,0.24]</td>
<td>[0.39,0.39,0.44]</td>
</tr>
<tr>
<td>W1</td>
<td>W2</td>
<td>W3</td>
<td>W4</td>
<td>W5</td>
<td>W6</td>
<td>W7</td>
<td>W8</td>
</tr>
<tr>
<td>(0.55,0.66,0.67)</td>
<td>(0.52,0.80,0.85)</td>
<td>(0.62,0.75,0.8)</td>
<td>(0.07,0.1)</td>
<td>(0.09,0.38,0.46)</td>
<td>(0.44,0.63,0.91)</td>
<td>(0.59,0.61,0.85)</td>
<td>(0.03,0.18,0.29)</td>
</tr>
<tr>
<td>W1n</td>
<td>W2n</td>
<td>W3n</td>
<td>W4n</td>
<td>W5n</td>
<td>W6n</td>
<td>W7n</td>
<td>W8n</td>
</tr>
<tr>
<td>(0.55+0.11a, 0.67–0.01a)</td>
<td>(0.52+0.28a, 0.85–0.05a)</td>
<td>(0.62+0.13a, 0.8–0.05 a)</td>
<td>(0.07a, 0.1–0.03a)</td>
<td>(0.09+0.29a, 0.46–0.08a)</td>
<td>(0.44+0.19a, 0.91–0.28a)</td>
<td>(0.59+0.02a, 0.85–0.24a)</td>
<td>(0.03+0.15a, 0.29–0.11a)</td>
</tr>
</tbody>
</table>

Calculation of the minimum of the α-cuts of the fuzzy weighted average. There are 4 crosspoints of left sides: A2 and A3 cross at 78/105, A2 and A4 cross at 130/131, A3 and A6 cross at 163/164 and A4 and A8 cross at 53/112. This means that we have to deal with subintervals [0.53/112, [53/112,78/105], [78/105,130/131], [130/131,163/164] and [163/164,1] separately.

Subinterval [0.53/112]: Ordering [A1,A2,A3,A4,A5,A6,A7,A8].
FWA(q3) = (1.5259α^2 + 9.7195α + 13.493)/(1.12α + 2.96). FWA(q2) >= 3.78+2.34a for 0<=α<=53/112. FWA(q3) = (0.7537α^2 + 9.2443α + 14.7404)/(0.79α + 3.29).
FWA(q3) <= 4.56+1.29a for 0<=α<=0.19463, so FWA(q3) is the minimum for 0<=α<=0.19463. FWA(q3) >= 4.56+1.29a for 0.19463<=α<=53/112.
FWA(q4) = (0.5215α^2 + 8.6557α + 15.5612)/(0.61α + 3.47). FWA(q4)<=5.08+1.03a for 0.19463<=α<=53/112, so FWA(q4) is the minimum for 0.19463<=α<=53/112.

FWA(q3) = (1.5259α^2 + 9.7195α + 13.493)/(1.12α + 2.96). FWA(q3) >= 3.78+2.34a for 53/112<=α<=78/105. FWA(q3) = (0.7537α^2 + 9.2443α + 14.7404)/(0.79α + 3.29).
FWA(q3) >= 4.56+1.29a for 53/112<=α<=78/105.
FWA(q4) = (0.5215α^2 + 8.6557α + 15.5612)/(0.61α + 3.47). FWA(q4)<=5.08+1.03a for 53/112<=α<=78/105, so FWA(q4) is the minimum for 53/112<=α<=78/105.

Subinterval [78/105,130/131]: Ordering [A1,A2,A3,A4,A5,A6,A7,A8].
FWA(q3) = (1.5259α^2 + 9.7195α + 13.493)/(1.12α + 2.96). FWA(q3) >= 4.56+1.29a for 78/105<=α<=130/131. FWA(q3) = (1.2937α^2 + 9.1309α + 14.3138)/(0.94α + 3.14).
FWA(q3) <= 3.78+2.34a for 78/105<=α<=130/131, so FWA(q3) is the minimum for 0.93436<=α<=130/131. FWA(q3) >= 3.78+2.34a for 78/105<=α<=0.93436.
FWA(q4) = (0.5215α^2 + 8.6557α + 15.5612)/(0.61α + 3.47). FWA(q4)<=5.08+1.03a for 78/105<=α<=0.93436, so FWA(q4) is the minimum for 78/105<=α<=0.93436.

Subinterval [130/131,163/164]: Ordering [A1,A2,A3,A4,A5,A6,A7,A8].
FWA(q3) = (1.5259α^2 + 9.7195α + 13.493)/(1.12α + 2.96). FWA(q3) >= 4.56+1.29a for 130/131<=α<=163/164. FWA(q3) = (1.2937α^2 + 9.1309α + 14.3138)/(0.94α + 3.14).
FWA(q3) <= 3.78+2.34a for 130/131<=α<=163/164, so FWA(q3) is the minimum for 130/131<=α<=163/164.

Subinterval [163/164,1]: Ordering [A1,A2,A3,A4,A5,A6,A7,A8].
FWA(q3) = (1.5259α^2 + 9.7195α + 13.493)/(1.12α + 2.96). FWA(q3) >= 4.56+1.29a for 163/164<=α<=1. FWA(q3) = (1.2937α^2 + 9.1309α + 14.3138)/(0.94α + 3.14).
FWA(q3) <= 3.78+2.34a for 163/164<=α<=1, so FWA(q3) is the minimum for 163/164<=α<=1.
Putting the pieces together, we have found that the minimum is
\[(0.7537\alpha^2 + 9.2443a + 14.7404)/(0.79\alpha + 3.29).\] if \(0 \leq \alpha \leq 0.19463^*.
\[(0.5215\alpha^2 + 8.6557a + 15.5612)/(0.61\alpha + 3.47).\] if \(0.19463^* \leq \alpha \leq 0.93436^*.
\[(1.2937\alpha^2 + 9.1309a + 14.3138)/(0.94\alpha + 3.14).\] if \(0.93436^* \leq \alpha \leq 1.

Calculation of the maximum of the \(\alpha\)-cuts of the fuzzy weighted average. There are 2 crosspoints of right sides: \(A_2\) and \(A_6\) cross at \(136/245\) and \(A_4\) and \(A_6\) cross at \(61/171.\) This means that we have to deal with subintervals \([0,61/171],\) \([61/171,136/245]\) and \([136/245,1]\) separately.

Subinterval \([0,61/171]\): Ordering \([A_1,A_3,A_4,A_5,A_6,A_8,A_7].\)
FWA(q) = \((-1.8193\alpha^2 + 4.4713a + 22.0864)/(0.98\alpha + 3.1).\) FWA(q) <= 10–0.74\(\alpha\) for \(0 \leq \alpha \leq 61/171.\) FWA(q) = \((-1.6269\alpha^2 + 1.6789a + 24.6864)/(0.72\alpha + 3.36).\) FWA(q) <= 2.37\(\alpha\) for \(0 \leq \alpha \leq 61/171.\)

Subinterval \([61/171,136/245]\): Ordering \([A_1,A_3,A_4,A_5,A_6,A_8,A_7].\)
FWA(q) = \((-0.75\alpha^2 - 2.7463a + 28.2347)(0.35\alpha + 3.73).\) FWA(q) <= 8.63–2.51\(\alpha\) for \(0 \leq \alpha \leq 136/245.\)

Subinterval \([136/245,1]\): Ordering \([A_1,A_3,A_4,A_5,A_6,A_8,A_7].\)
FWA(q) = \((-0.7218\alpha^2 - 6.1914a + 31.6516)/(–0.12\alpha + 4.2).\) FWA(q) <= 8.63–2.51\(\alpha\) for \(136/245 \leq \alpha \leq 1.\)

Putting the pieces together, we have found that the maximum is:
\[(0.2553\alpha^2 - 7.3875a + 31.8706)(–0.08\alpha + 4.16)\] if \(0 \leq \alpha \leq 0.25062^*,\)
\[(0.2835\alpha^2 - 10.8326a + 35.2875)/(–0.55\alpha + 4.63)\] if \(0.25062^* \leq \alpha \leq 1.\)
This leads to the following membership function for the fuzzy weighted average:

\[
\mu(x) = \begin{cases} 
0 & \text{if } x \leq 4.480^* \\
0.524^* x - 6.133^* + 0.663^* \sqrt{0.6241x^2 - 4.687^* x + 41.018} & \text{if } 4.480^* < x \leq 4.811^* \\
0.585^* x - 8.299^* + 0.959^* \sqrt{0.3721x^2 - 3.322^* x + 42.460} & \text{if } 4.811^* < x \leq 5.966^* \\
0.363^* x - 3.529^* + 0.386^* \sqrt{0.8836x^2 - 0.917^* x + 9.302} & \text{if } 5.966^* < x \leq 6.063^* \\
-0.970^* x + 19.105^* - 1.764^* \sqrt{0.0064x^2 + 3.066^* x + 22.029} & \text{if } 6.063^* < x \leq 7.255^* \\
-0.157^* x + 14.468^* - 1.958^* \sqrt{2.51^2 + 0.378^* x + 1.78^*} & \text{if } 7.255^* < x \leq 7.661^* \\
0 & \text{if } x \geq 7.661^* 
\end{cases}
\]

Example 7: Case study of Kao and Liu[6]

In [6], Kao and Liu apply fuzzy weighted average to determine the competitiveness of manufacturing firms. For each of 15 firms, its competitiveness index is the average of its fuzzy technology index and its fuzzy management index. Its fuzzy management index is a fuzzy weighted average of the fuzzy values of 18 criteria. Its fuzzy technology index is a fuzzy weighted average of the fuzzy values of 4 criteria. From the 15 firms, we consider only firm 15 here. We first establish its fuzzy management index. Among the 18 fuzzy values for the management criteria, there are only 7 different ones. Equal values need to be considered only once, provided their weights are added. This leads to the following table of fuzzy attributes and fuzzy weights:

<table>
<thead>
<tr>
<th>A_1 = (0.2,0.4)</th>
<th>A_1_α = [0.2α, 0.4–0.2α]</th>
<th>W_1 = (0.65,0.84,0.95)</th>
<th>W_1_α = [0.65+0.19α, 0.95–0.11α]</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_2 = (0.3,0.5)</td>
<td>A_2_α = [0.3+0.2α, 0.7–0.2α]</td>
<td>W_2 = (0.66,0.85,0.96)</td>
<td>W_2_α = [0.66+0.19α, 0.96–0.11α]</td>
</tr>
<tr>
<td>A_3 = (0.4,0.6)</td>
<td>A_3_α = [0.4+0.2α, 0.8–0.2α]</td>
<td>W_3 = (2.51,3.25,3.71)</td>
<td>W_3_α = [2.51+0.74α, 3.71–0.46α]</td>
</tr>
<tr>
<td>A_4 = (0.5,0.7)</td>
<td>A_4_α = [0.5+0.2α, 0.9–0.2α]</td>
<td>W_4 = (1.16,1.53,1.78)</td>
<td>W_4_α = [1.16+0.37α, 1.78–0.25α]</td>
</tr>
<tr>
<td>A_5 = (0.57,0.7)</td>
<td>A_5_α = [0.57+0.2α, 0.97–0.2α]</td>
<td>W_5 = (0.61,0.8,0.91)</td>
<td>W_5_α = [0.61+0.19α, 0.91–0.11α]</td>
</tr>
<tr>
<td>A_6 = (0.6,0.8)</td>
<td>A_6_α = [0.6+0.2α, 1–0.2α]</td>
<td>W_6 = (4.46,5.77,6.46)</td>
<td>W_6_α = [4.46+1.31α, 6.46–0.69α]</td>
</tr>
<tr>
<td>A_7 = (0.8,1)</td>
<td>A_7_α = [0.8+0.2α,1]</td>
<td>W_7 = (1.4,1.78,1.95)</td>
<td>W_7_α = [1.4+0.38α,1.95–0.17α]</td>
</tr>
</tbody>
</table>

Calculation of the minimum of the α-cuts of the fuzzy weighted average. There are no crosspoints. FWA(q_2) = (0.614α^2 + 4.0863α + 5.9257)/(3.07α + 11.75). FWA(q_3) = (0.554α^2 + 4.0563α + 6.0157)/(2.77α + 12.05). FWA(q_4) = (0.314α^2 + 3.8163α + 6.4957)/(1.57α + 13.25). FWA(q_5) = (0.19α^2 + 3.6303α + 6.8057)/(0.95α + 13.87). FWA(q_6) = (–0.598α^2 + 0.4483α + 10.7757)/(2.82α + 12). FWA(q_7) = (–0.198α^2 – 1.9517α + 12.7757)/(0.82α + 14). FWA(q_8) = (–0.138α^2 – 2.3027α + 13.0667)/(0.52α + 14.3). FWA(q_9) = (–0.970α^2 + 19.105α – 1.764α + 12.105)/(2.82α + 12). FWA(q_10) = (–0.138α^2 – 2.3027α + 13.0667)/(0.52α + 14.3). FWA(q_11) = (–0.970α^2 + 19.105α – 1.764α + 12.105)/(2.82α + 12). FWA(q_12) = (–0.138α^2 – 2.3027α + 13.0667)/(0.52α + 14.3). FWA(q_13) = (–0.970α^2 + 19.105α – 1.764α + 12.105)/(2.82α + 12). FWA(q_14) = (–0.138α^2 – 2.3027α + 13.0667)/(0.52α + 14.3). FWA(q_15) = (–0.970α^2 + 19.105α – 1.764α + 12.105)/(2.82α + 12).

Calculation of the maximum of the α-cuts of the fuzzy weighted average. There are no crosspoints. FWA(q_1) = (0.614α^2 + 4.0863α + 5.9257)/(3.07α + 11.75). FWA(q_2) = (0.554α^2 + 4.0563α + 6.0157)/(2.77α + 12.05). FWA(q_3) = (0.314α^2 + 3.8163α + 6.4957)/(1.57α + 13.25). FWA(q_4) = (0.19α^2 + 3.6303α + 6.8057)/(0.95α + 13.87). FWA(q_5) = (–0.598α^2 + 0.4483α + 10.7757)/(2.82α + 12). FWA(q_6) = (–0.198α^2 – 1.9517α + 12.7757)/(0.82α + 14). FWA(q_7) = (–0.138α^2 – 2.3027α + 13.0667)/(0.52α + 14.3). FWA(q_8) = (–0.970α^2 + 19.105α – 1.764α + 12.105)/(2.82α + 12). FWA(q_9) = (–0.138α^2 – 2.3027α + 13.0667)/(0.52α + 14.3). FWA(q_10) = (–0.970α^2 + 19.105α – 1.764α + 12.105)/(2.82α + 12). FWA(q_11) = (–0.138α^2 – 2.3027α + 13.0667)/(0.52α + 14.3). FWA(q_12) = (–0.970α^2 + 19.105α – 1.764α + 12.105)/(2.82α + 12). FWA(q_13) = (–0.138α^2 – 2.3027α + 13.0667)/(0.52α + 14.3). FWA(q_14) = (–0.970α^2 + 19.105α – 1.764α + 12.105)/(2.82α + 12). FWA(q_15) = (–0.970α^2 + 19.105α – 1.764α + 12.105)/(2.82α + 12).

This leads to the following membership function for the fuzzy weighted average:
\[ \mu(x) = 0 \quad \text{if } x \leq 0.490^* \]
\[ \mu(x) = 2.5x - 6.077^* + 1.592^* \sqrt{2.4649x^2 + 4.659^* x + 6.406^*} \quad \text{if } 0.490^* \leq x \leq 0.568^* \]
\[ \mu(x) = 2.5x - 9.553^* + 2.632^* \sqrt{0.9025x^2 + 3.644^* x + 8.007^*} \quad \text{if } 0.568^* \leq x \leq 0.717^* \]
\[ \mu(x) = -1.884^* x - 8.343^* + 3.623^* \sqrt{0.2704x^2 - 5.499^* x + 12.515^*} \quad \text{if } 0.717^* \leq x \leq 0.914^* \]
\[ \mu(x) = 0 \quad \text{if } x \geq 0.914^* \]

Next we establish the fuzzy technology index of the 15th firm. The table with fuzzy attributes and fuzzy weights for the fuzzy technological index is:

\[
\begin{align*}
A_1 &= (0.5, 0.7, 0.87) \\
A_2 &= (0.5, 0.75, 1) \\
A_3 &= 0.83 \\
A_4 &= 0.86
\end{align*}
\]

Calculation of the minimum of the \( \alpha \)-cuts of the fuzzy weighted average. There are no crosspoints. \( \text{FWA}(q_2) = (0.012^* \alpha^2 + 0.5165^* \alpha + 1.062)/(0.41^* \alpha + 1.59) \). \( \text{FWA}(q_2) \geq 0.5 + 0.25^* \alpha \) for \( 0 \leq \alpha \leq 1 \). \( \text{FWA}(q_3) = (-0.0855^* \alpha^2 + 0.4190^* \alpha + 1.257)/(0.02^* \alpha + 1.98) \). \( \text{FWA}(q_3) \leq 0.83 \) for \( 0 \leq \alpha \leq 1 \), so \( \text{FWA}(q_3) \) is the minimum for \( 0 \leq \alpha \leq 1 \).

Calculation of the maximum of the \( \alpha \)-cuts of the fuzzy weighted average. There are 4 crosspoints of right sides: \( A_1 \) and \( A_3 \) intersect at \( \alpha = 4/17 \); \( A_1 \) and \( A_4 \) intersect at \( \alpha = 1/17 \); \( A_2 \) and \( A_3 \) intersect at \( \alpha = 17/25 \) and \( A_2 \) and \( A_4 \) intersect at \( \alpha = 14/25 \). This means that we have to deal with subintervals \([0, 1/17], [1/17, 4/17], [4/17, 14/25], [14/25, 17/25]\) and \([17/25, 1]\) separately.

Subinterval \([0, 1/17]\): Ordering \([A_3, A_4, A_1, A_2]\).
\[ \text{FWA}(q_3) = (0.0152^* \alpha^2 + 0.1243^* \alpha + 1.451)/(0.4^* \alpha + 1.6) \]. \( \text{FWA}(q_3) \geq 0.87–0.17^* \alpha \) for \( 0 \leq \alpha \leq 1/17 \), so \( \text{FWA}(q_3) \) is the maximum for \( 0 \leq \alpha \leq 1/17 \).

Subinterval \([1/17, 4/17]\): Ordering \([A_3, A_4, A_1, A_2]\).
\[ \text{FWA}(q_3) = (0.0152^* \alpha^2 + 0.1243^* \alpha + 1.451)/(0.4^* \alpha + 1.6) \]. \( \text{FWA}(q_3) \geq 0.86 \) for \( 1/17 \leq \alpha \leq 4/17 \), so \( \text{FWA}(q_3) \) is the maximum for \( 1/17 \leq \alpha \leq 4/17 \).

Subinterval \([4/17, 14/25]\): Ordering \([A_1, A_3, A_4, A_2]\).
\[ \text{FWA}(q_3) = (0.0152^* \alpha^2 + 0.1243^* \alpha + 1.451)/(0.4^* \alpha + 1.6) \]. \( \text{FWA}(q_3) \geq 0.86 \) for \( 4/17 \leq \alpha \leq 0.34984^* \), so \( \text{FWA}(q_3) \) is the maximum for \( 4/17 \leq \alpha \leq 0.34984^* \). \( \text{FWA}(q_3) \leq 0.86 \) for \( 0.34984^* \leq \alpha \leq 14/25 \). \( \text{FWA}(q_3) = (0.0152^* \alpha^2 - 0.2111^* \alpha + 1.7864)/(0.01^* \alpha + 1.99) \). \( \text{FWA}(q_3) \geq 0.83 \) for \( 0.34984^* \leq \alpha \leq 14/25 \), so \( \text{FWA}(q_3) \) is the maximum for \( 0.34984^* \leq \alpha \leq 14/25 \).

Subinterval \([14/25, 17/25]\): Ordering \([A_1, A_3, A_2, A_4]\).
\[ \text{FWA}(q_3) = (-0.0823^* \alpha + 0.2764^* \alpha + 1.3964)/(0.4^* \alpha + 1.6) \]. \( \text{FWA}(q_3) \leq 1–0.25^* \alpha \) for \( 14/25 \leq \alpha \leq 17/25 \). \( \text{FWA}(q_3) = (0.0152^* \alpha^2 - 0.2111^* \alpha + 1.7864)/(0.01^* \alpha + 1.99) \). \( \text{FWA}(q_3) \geq 0.83 \) for \( 14/25 \leq \alpha \leq 0.64255 \), so \( \text{FWA}(q_3) \) is the maximum for \( 14/25 \leq \alpha \leq 0.64255^* \). \( \text{FWA}(q_3) \leq 0.83 \) for \( 0.64255^* \leq \alpha \leq 17/25 \). \( \text{FWA}(q_3) = (0.0152^* \alpha^2 - 0.5099^* \alpha + 2.0852)/(–0.35^* \alpha + 2.35) \). \( \text{FWA}(q_3) \) is the maximum for \( 0.64255^* \leq \alpha \leq 17/25 \).

Subinterval \([17/25, 1]\): Ordering \([A_1, A_2, A_3, A_4]\).
FWA(q^3) = (-0.0823\alpha^2 + 0.2764\alpha + 1.3964)/(0.4\alpha + 1.6). FWA(q^3) <= 0.83 for 17/25<=\alpha<=1.

FWA(q^2) = (-0.0823\alpha^2 - 0.0224\alpha + 1.6952)/(0.04\alpha + 1.96). FWA(q^2) >= 1–0.25\alpha for 0.70278<=\alpha<=1, so FWA(q^2) is the maximum for 0.70278<=\alpha<=1.

FWA(q^1) = (0.0152\alpha^2 - 0.5099\alpha + 2.0852)/(-0.35\alpha + 2.35). FWA(q^1) is the maximum for 17/25<=\alpha<=0.70278.

Putting the pieces together, we have found that the maximum is:
(0.0152\alpha^2 + 0.1243\alpha + 1.451)/(0.4\alpha + 1.6) if 0<=\alpha<=0.34984
(0.0152\alpha^2 - 0.2111\alpha + 1.7864)/(0.04\alpha + 1.99) if 0.34984<=\alpha<=0.64255
(-0.0823\alpha^2 - 0.0224\alpha + 1.6952)/(0.04\alpha + 1.96) if 0.64255<=\alpha<=0.70278

This leads to the following membership function for the fuzzy weighted average:

$$\mu(x) = \begin{cases} 0 & \text{if } x <= 0.635^* \\ -0.117^*x + 2.450^* - 5.848^*\sqrt{\frac{0.0004x^2 - 0.69392x + 0.605455}{\sqrt{0.1225x^2 - 0.21405x + 0.133^*}}} & \text{if } 0.635^* <= x <= 0.79525 \\ -0.243^*x - 0.136^* + 6.075^*\sqrt{0.0016x^2 - 0.64344x + 0.559^*} & \text{if } 0.79525 <= x <= 0.824^* \\ -11.513^*x + 16.773^* - 32.895^*\sqrt{0.1225x^2 - 0.21405x + 0.133^*} & \text{if } 0.824^* <= x <= 0.83 \\ 0.329^*x + 6.944^* - 32.895^*\sqrt{0.0001x^2 + 0.125214x - 0.064^*} & \text{if } 0.83 <= x <= 0.906875 \\ 13.159^*x - 4.089^* - 32.895^*\sqrt{0.16x^2 - 0.00216x - 0.0728^*} & \text{if } 0.906875 <= x <= 0.906875 \end{cases}$$

Having obtained the analytical expressions for the \(\alpha\)-cuts of the fuzzy management index and the fuzzy technology index, their average gives the \(\alpha\)-cuts of the competitiveness index. Where the \(\alpha\)-cuts for the competitiveness index is obtained easily, it is not easy to obtain its membership function. This could be obtained by calculation of the inverse of the functions for the \(\alpha\)-cuts; however this would imply the solution of polynomial equations of degree 3. It could also be obtained by applying Zadeh's extension of the average operation to the fuzzy management index and the fuzzy technology index. However, this also leads to the problem of polynomial equations of degree 3. So, we must be content to have found an analytical expression for the \(\alpha\)-cuts of the competitiveness index, observing that the final ranking of the firms is done using \(\alpha\)-cuts.

References


