Interfacial energies on Penrose lattices

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1 Introduction

Penrose lattices are discrete sets of the plane (which are also subsets of a regular Bravais lattice), whose underlying tassellations of the plane by rhomboidal tiles with angles multiple of $\pi/5$ (with vertices the points of the Penrose lattice itself) are a prototype of quasicrystalline a-periodic structures (see Figure 1). In this paper we consider "discrete" energies directly

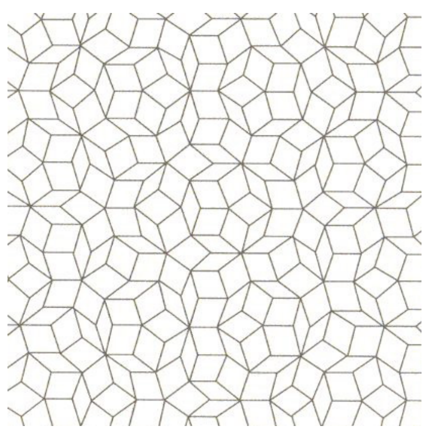

Figure 1: A Penrose tassellation

defined on a Penrose lattice $\mathcal{P}$, and examine their overall behaviour via a Γ-convergence approach. Such a treatment combines homogenization issues and a passage from discrete systems to continuous variational problems.

The energies we study are the analog of the energies of *Ising type systems* on a regular lattice $\mathcal{L}$ in $\mathbb{R}^2$. Those can be written as set functionals depending on subsets A of a scaled copy $\varepsilon\mathcal{L}$ of $\mathcal{L}$; i.e.,

$$E_\varepsilon(A) = \varepsilon\,\#\{(i,j) : i \in A, j \notin A, |i-j| = \varepsilon\}.$$

Note that, up to multiplicative constants, the energies E_ε can be rewritten as *ferromagnetic energies* depending on a *spin* variable $u : \varepsilon\mathcal{L} \to \{-1,+1\}$

$$E_\varepsilon(u) = -\varepsilon \sum \Big\{ (u(i)u(j) - 1) : i,j \in \varepsilon\mathcal{L}, |i-j| = \varepsilon \Big\}$$

(with the identification, e.g., of A with $\{i : u(i) = 1\}$). Their asymptotic behaviour as $\varepsilon \to 0$ describes the behaviour of large sets in $\mathcal{L}$, through the computation of a limit surface energy.

If $\mathcal{L}$ is a periodic lattice, then periodic homogenization techniques for surface energies have been applied to obtain a limit energy as ε to 0 which is defined on sets of finite perimeter (upon identification of a set $A \subset \varepsilon\mathcal{L}$ with a suitable subset of $\mathbb{R}^2$ in a way that, with an abuse of notation, precisely $E_\varepsilon(A) = \mathcal{H}^1(\partial A)$). A general theory for this type of homogenization can be found in [9]. In the case of the square lattice a direct computation can be found in [1], giving the limit energy

$$F(A) = \int_{\partial^* A} \|\nu\|_1 d\mathcal{H}^1,$$

where $\partial^* A$ denotes the essential boundary of A and ν its inner normal. The anisotropic energy density $\|\nu\|_1 = \|(\nu_1, \nu_2)\|_1 = |\nu_1| + |\nu_2|$ describes the macroscopic effect of the geometry of the underlying lattice.

We will give a description of the behaviour of the energies E_ε when $\mathcal{L} = \mathcal{P}$ is a Penrose lattice. Even though those lattices are not periodic, they can be generated by a projection procedure from a (periodic) five-dimensional cubic lattice on a particular two-dimensional plane in $\mathbb{R}^5$. That construction both highlights the pentagonal symmetries of the Penrose tassellations and suggests that they enjoy some "quasi-periodic" properties. Those properties can be formalized in an analytic way as follows: if we label with an index $i \in \{1, \dots, 10\}$ the ten possible types of tiles of the tassellation (the ten equivalence classes of the rhombi up to translations) and define the measurable function $i(x)$ which maps a point x of the plane to the index of the rhombus to which x belongs, then the map i is *Besicovitch almost periodic* (actually, W^1-*almost periodic*, see [10]). This is sufficient to characterize for example the limit as $\varepsilon \to 0$ of bulk integrals of the form

$$F_\varepsilon(u) = \int_\Omega f\Big(i\Big(\frac{x}{\varepsilon}\Big), Du\Big)\,dx,$$

which expresses the overall bulk properties of a composite medium whose geometry follows a Penrose tassellation [10], since it can be seen as a particular case of an energy of the form

$$\int_\Omega g\Big(\frac{x}{\varepsilon}, Du\Big)\,dx,$$

with g a Besicovitch almost-periodic function in the space variable [7].

Unfortunately, contrary to the bulk case, these almost-periodic properties of the Penrose tassellations are not sufficient to directly adapt the general results on homogenization of surface integrals [2] to this discrete case, since those results require the uniform almost periodicity of the coefficients (which clearly cannot hold for the function i defined above since it is not even continuous). This requirement cannot be weakened to W^1-almost periodicity, since for surface integrals changes of the energy on sets of small measure may change the value of the limit. We then have to prove additional properties that derive from the “quasi periodicity” of the lattice $\mathcal{P}$, which loosely speaking state that there exists a “large” set of translations such that the lattice is invariant under those translations, up to *well-separated* “small” regions. Since these small regions are far apart, by controlling their size (and their perimeter) we can then neglect their effect on the overall energy and reason as if the above-mentioned translations were periods of the function i. In this way a homogenization theorem can be proved characterizing an energy density φ such that the Γ-limit of the energies E_ε is of the form

$$F(A) = \int_{\partial^* A} \varphi(\nu) d\mathcal{H}^1.$$

Note that all the reasoning can be repeated for more complex energies of the form

$$E_\varepsilon(A) = \varepsilon \sum \Big\{ f\Big(\frac{i-j}{\varepsilon}\Big) : i \in A, j \notin A, |i-j| = \varepsilon \Big\},$$

which may take into account also the microscopical orientation of the interactions and not only their number, and by localizing all interactions to a fixed subset Ω of $\mathbb{R}^2$.

2 Statement of the result

2.1 Generation of Penrose lattices by projection

Let Π be the two-dimensional plane in $\mathbb{R}^5$ spanned by the vectors

$$v_1 = \sum_{k=1}^{5} \sin\left(\frac{2(k-1)\pi}{5}\right) e_k \quad \text{and} \quad v_2 = \sum_{k=1}^{5} \cos\left(\frac{2(k-1)\pi}{5}\right) e_k, \tag{1}$$

where e_k is the unit vector on the k-th axis. We note that, considering the matrix M whose action is the permutation of all the coordinate axes in order, then Π is the plane of the vectors v such that the action of M on v is a rotation of $2\pi/5$. Then, we consider the set $\mathcal{Z}$ of the points $z \in \mathbb{Z}^5$ such that $z + (0,1)^5 \cap \Pi \neq \emptyset$, and the function $\phi\colon \mathbb{Z}^5 \to \mathbb{R}^2$ defined as $\phi(z) = \sum_{k=1}^5 z_k e^{\frac{ik\pi}{5}}$. We set $\phi(\mathcal{Z}) = \mathcal{P}$.

Remark 1 (characterization of Penrose tilings)**.** The tiling obtained by joining p and p' in $\mathcal{P}$ by an edge if and only if $|p - p'| = 1$ is a Penrose tiling (as defined, e.g., in [14]). Hence, the construction above will be our definition of a *Penrose lattice.*

We note that in the original construction of de Bruijn [12] the tiling is obtained, in an equivalent way, by projecting onto Π the points $z \in \mathbb{Z}^5$ such that $z + (0,1)^5 \cap \Pi \neq \emptyset$. Moreover, the construction gives a Penrose tiling for any parallel plane $\gamma + \Pi$ with γ such that $\sum_{k=1}^5 \gamma_k = 0 \pmod 1$.

We denote by $\mathcal{T}$ the set of the Penrose “cells” of the tiling in $\mathbb{R}^2$; we get two possible shapes of rhombi for the cells $T \in \mathcal{T}$, each one with five possible orientations. Then, we can define a function $a\colon \mathbb{R}^2 \to \{1, \ldots, 10\}$ in $L^\infty(\mathbb{R}^2)$ associating to each x in the inner part of a Penrose cell an index giving the shape and the orientation of the cell. Moreover, in order to fix for each cell one of the vertices, we define $v\colon \mathbb{R}^2 \to \mathcal{P}$ as the function which associates to each $x \in T$ (where T is an open cell) one of the two vertices $p_1 = (x_1, y_1), p_2 = (x_2, y_2)$ corresponding to the angle of $\pi/5$ (or $2\pi/5$) so that $v(x) = p_i$ if $\|y_i\| < \|y_j\|$ or, when $\|y_i\| = \|y_j\|$, if $\|x_i\| < \|x_j\|$.

2.2 Surface energies on the Penrose lattice

Given a discrete set A such that $A \subset \varepsilon\mathcal{P} \subset \mathbb{R}^2$, where $\varepsilon > 0$, we define

$$E_\varepsilon(A) = \varepsilon \# \{(i,j) : i \in A, j \notin A, |i-j| = \varepsilon\}\,. \tag{2}$$

Moreover, for any open set $\Omega \subset \mathbb{R}^2$ we define

$$E_\varepsilon(A;\Omega) = \varepsilon \# \{(i,j) : i \in A, j \notin A, |i-j| = \varepsilon, i,j \in \Omega\}\,. \tag{3}$$

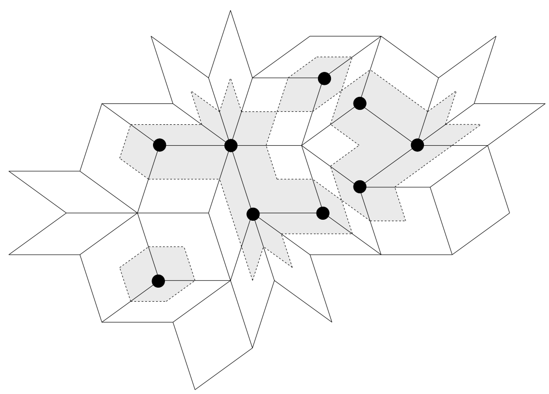

Figure 2: The discrete set A (dotted points) and the corresponding set $i(A)$ (shaded region).

We identify a set $A \subset \varepsilon\mathcal{P}$ with a subset $i(A)$ of $\mathbb{R}^2$ in the following way. Given $p \in \varepsilon\mathcal{P}$, we consider the set of the Penrose cells $T^1_\varepsilon, \dots, T^{N(p)}_\varepsilon$ such that p is a vertex of T^j_ε. We set

$$C_\varepsilon(p) = \bigcup_{j=1}^{N(p)} p + \frac{T^j_\varepsilon - p}{2}$$

so that we can define $i(A) = \bigcup_{p\in A} C_\varepsilon(p)$ (see Figure 2). Such a set will be called a ε-*Penrose set*. Note that with this identification we have

$$E_\varepsilon(A) = \mathcal{H}^1\left(\partial(i(A))\right).$$

Moreover, for a set $A \subset \varepsilon\mathcal{P}$ we denote by $\tilde{\partial}A$ the set of points $i \in A$ such that there exists $j \in \mathcal{P}$ with $|i-j| = \varepsilon$ and $j \notin A$.

Proposition 2 (compactness). *Given a sequence $\{A_\varepsilon\}$ with $A_\varepsilon \subset \varepsilon\mathcal{P}$ such that*

$$\sup_\varepsilon E_\varepsilon(A_\varepsilon) < +\infty,$$

then (up to subsequences) there exists $A \subset \mathbb{R}^2$ of finite perimeter such that

$$\chi_{i(A_\varepsilon)} \to \chi_A \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^2).$$

Proof. The proof follows immediately from the compactness of families with equibounded perimeter, once we remark that the condition $\sup_\varepsilon E_\varepsilon(A_\varepsilon) < +\infty$ gives $\sup_\varepsilon \mathcal{H}^1\left(\partial(i(A_\varepsilon))\right) < +\infty$. □

The previous proposition shows in particular that it is sufficient to identify the Γ-limit for sets with finite perimeter.

Definition 3. *In the following we say that a sequence $\{A_\varepsilon\}$ with $A_\varepsilon \subset \varepsilon\mathcal{P}$ converges to a set A (as $\varepsilon \to 0$) if the characteristic functions of the corresponding sets $i(A_\varepsilon)$ converge to χ_A in $L^1_{\text{loc}}(\mathbb{R}^2)$.*

2.3 The Homogenization Theorem

We will prove that the energies E_ε converge to a limit interfacial energy. In order to define such limit functional, we will need first to define its energy density (the *surface tension*).

We denote by $Q^\nu_T(x)$ the square with centre x, side length T and two faces orthogonal to ν, and $H^\nu(x_0) = \{x : \langle x - x_0, \nu\rangle < 0\}$ the half plane with boundary the line through x_0 and orthogonal to ν.

Given a set $B \subset \mathcal{P}$, we say that $B \sim H^\nu(x_0)$ on $\partial Q^\nu_T(x_0)$ if $B = \mathcal{P} \cap H^\nu(x_0)$ in $Q^\nu_T(x_0) \setminus Q^\nu_{T-\delta}(x_0)$ for some $\delta > 1$.

Proposition 4 (definition of the surface tension)**.** *Given a sequence $\{x_T\}$, there exists the limit*

$$\varphi(\nu) = \lim_{T\to+\infty} \frac{1}{T} \inf\left\{E(B; Q^\nu_T(x_T)) : B \subset \mathcal{P}, B \sim H^\nu_T \text{ on } \partial Q^\nu_T(x_T)\right\} \tag{4}$$

where $H^\nu_T = H^\nu(x_T)$ and

$$E(B; Q^\nu_T(x_T)) = E_1(B; Q^\nu_T(x_T)) = \#\{(i,j) : i \in B, j \notin B, |i-j| = 1, i,j \in Q^\nu_T(x_T)\}.$$

The proof of this proposition will be given in Section 3.

We can now state the homogenization result as follows (for a quick introduction to the notation for sets of finite perimeter we refer to [4]).

Theorem 5 (Homogenization of surface energies on Penrose lattices)**.** *The sequence of functionals E_ε defined in* (2) *Γ-converges as $\varepsilon \to 0$, with respect to the set convergence in Definition* 3*, to the functional*

$$E(A) = \int_{\partial^* A} \varphi(\nu)\, d\mathcal{H}^1, \tag{5}$$

defined on sets with finite perimeter in $\mathbb{R}^2$, where ν stands for the inner normal to the reduced boundary $\partial^* A$ of the set of finite perimeter A, and φ is defined in (4)*.*

3 Existence of the surface tension

We now prove Proposition 4. It will be obtained by using some properties deduced from the "quasi-periodicity" of $\mathcal{P}$. We start by showing the following proposition.

Proposition 6. *Let $\eta > 0$. Then there exists R_η with $R_\eta \to +\infty$ as $\eta \to 0$ such that for any $x, y \in \mathbb{Z}^5$ satisfying*

$$dist(x, \Pi) < \eta \quad and \quad 0 < |x - y| < R_\eta,$$

we have $dist(y, \Pi) > \eta$.

Proof. By contradiction, we assume that for all $\eta > 0$ there exist x_η and y_η in $\mathbb{Z}^5$ such that $x_\eta - y_\eta = z$ with $z \in \mathbb{Z}^5 \setminus \{0\}$ (this can be assumed without loss of generality by the finiteness of the set $\{z \in \mathbb{Z}^5 \setminus \{0\} : |z| \leq R\}$, up to subsequences), and $\mathrm{dist}(x_\eta, \Pi) < \eta$, $\mathrm{dist}(y_\eta, \Pi) < \eta$. Then, $\mathrm{dist}(z, \Pi) < 2\eta$ for any η, so that $z \in \Pi \cap \mathbb{Z}^5 \setminus \{0\}$, giving the contradiction. □

Proof of Proposition 4. The function $x \mapsto \mathrm{dist}(\pi(x), \mathbb{Z}^5)$ is uniformly almost periodic in $\mathbb{R}^2$ (see [10]); then, by the characterization of uniformly almost-periodic functions in [3], the set $\widetilde{T}_\eta = \{x \in \mathbb{R}^2 \ : \ \mathrm{dist}(\pi(x), \mathbb{Z}^5) < \eta\}$ is relatively dense in $\mathbb{R}^2$, and the set $T_\eta = \widetilde{T}_\eta \cap \mathcal{P}$ is relatively dense too (since the points in this set are the projections of the points in $\mathbb{Z}^5$ with distance less than η from Π); i.e., for any η there exists an inclusion lenght $L_\eta > 0$ such that $T_\eta + [0, L_\eta)^2 = \mathbb{R}^2$.

We start by proving the thesis for $x_T = 0$ for all T; eventually, we will show that the limit is independent of the sequence of the centers of the squares by a comparison argument.

Without loss of generality it is sufficient to give the proof for $\nu = (0, 1)$, the difference from the general case resulting only in a more complex notation. We denote by H the half plane $\{(x, y) : y < 0\}$.

Let $Q_T = \left(-\frac{T}{2}, \frac{T}{2}\right)^2$; B_T will be a subset of $\mathcal{P}$ such that $B_T = \mathcal{P} \cap H$ in a neighbourhood U_T of ∂Q_T. It is not restrictive to assume that B_T is such that $i(B_T)$ is simply connected with $\partial(i(B_T)) \cap (Q_T \setminus U_T)$ a connected polygonal curve. Indeed, if it is not, then we can consider, in the place of B_T, the set $\mathcal{P} \cap (Q_T \setminus C_T)$, where C_T is the connected component of the complement of $i(B_T)$ containing $(\mathcal{P} \setminus H) \cap U_T$.

Now, we consider $S >> T$ and the square Q_S, and we define a set B_S as follows. Fixed $\eta > 0$, let L_η be the inclusion lenght given by the relative density of T_η as above, and denote by Q^k with $k = 1, \ldots, \left[\frac{S}{T+2L_\eta}\right]$ disjoint coordinate squares with side lenght $T + 2L_\eta$ included in $\left(-\frac{S}{2}, \frac{S}{2}\right) \times \left(-\frac{T}{2} - L_\eta, \frac{T}{2} + L_\eta\right)$. The relative density ensures that for any k there exists a translation vector $\tau_\eta^k \in T_\eta$ such that $Q_T(\tau_\eta^k) \subset Q^k$, where $Q_T(\tau_\eta^k)$ is the square centered in τ_η^k with side lenght T. For each k we set $B^k = \left(i(B_T) + \tau_\eta^k\right) \cap \mathcal{P}$. We define the set B_S as

$$B_S = \begin{cases} B^k & \text{in } \ Q_T(\tau_\eta^k) \\ \mathcal{P} \cap H & \text{in } Q_S \setminus \bigcup_k Q_T(\tau_\eta^k). \end{cases} \tag{6}$$

Now we want to estimate the length of the boundary of $i(B^k)$, which can be decomposed into a part which is the translation of the boundary of $i(B_T)$, and a second part, which is modified after the translation. This second part can be estimated by counting the number of

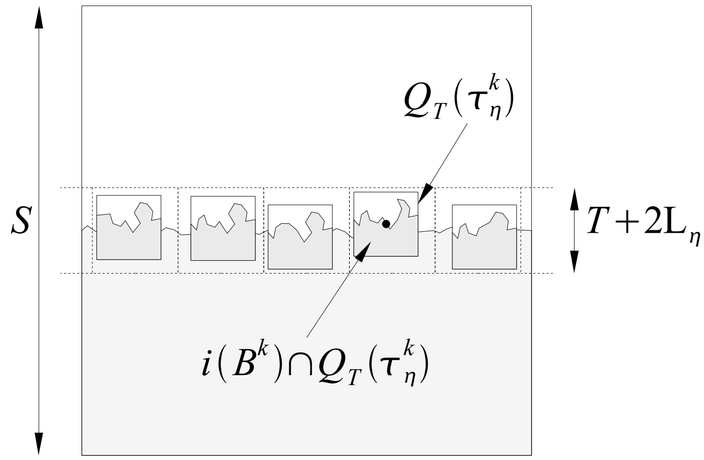

Figure 3: Construction of the set B_S.

points in the "boundary" of B_T whose translation does not belong to $\mathcal{P}$. Note that for each one of these points we have at most 10 (the number of nearest neighbours in $\mathbb{Z}^5$) connections changing the value of the energy; it follows that

$$E(B^k;Q_T(\tau^k_\eta)) \leq E(B_T;Q_T) + 10\#\left(\left(\tilde{\partial}B_T+\tau^k_\eta\right)\setminus\mathcal{P}\right).$$

Recalling the construction of the Penrose tilings (Section 2.1), we have that if $y\in\mathbb{R}^2$ belongs to a Penrose cell such that all the vertices correspond to points $z\in\mathbb{Z}^5$ such that $\mathrm{dist}(z,\Pi)\geq\eta$, then, for a given $\tau\in T_\eta$, we get that $y+\tau$ belongs to a translated cell with the same shape and orientation. It follows that if a point $p\in\mathcal{P}$ is such that for some $\tau\in T_\eta$ the translate $p+\tau$ does not belong to $\mathcal{P}$, then there exists a point in $\{p\}\cup\{p'\in\mathcal{P}:|p-p'|=1\}$ corresponding to a $z\in\mathbb{Z}^5$ such that $\mathrm{dist}(z,\Pi)<\eta$. We deduce that given two points p and q in $\left(\tilde{\partial}B_T+\tau^k_\eta\right)\setminus\mathcal{P}$, then there exist $p'\in\mathcal{P}$ with $|p-p'|\leq1$ and $q'\in\mathcal{P}$ with $|q-q'|\leq1$ such that $\mathrm{dist}(\phi^{-1}(p'),\Pi)<\eta$ and $\mathrm{dist}(\phi^{-1}(p'),\Pi)<\eta$. Now, Proposition 6 ensures that the distance in $\mathbb{Z}^5$ between $\phi^{-1}(p')$ and $\phi^{-1}(q')$ is greater than R_η. Hence, since $i(B_T)$ is simply connected, there exists $c>0$ independent on T,k and η such that

$$10\ \#\left(\left(\tilde{\partial}B_T+\tau^k_\eta\right)\setminus\mathcal{P}\right)\leq c\ \frac{E(B_T;Q_T)}{R_\eta}.$$

The contributions to $E(B_S,Q_S)$ outside the union of the squares B^k can be easily estimated since there we have $B_S=\mathcal{P}\cap\mathcal{H}$. Hence, the set B_S defined in (6) is a subset of $\mathcal{P}$

such that $B_S \sim H$ on ∂Q_S and

$$\begin{aligned} E(B_S;Q_S) \quad &\le \quad \left[\frac{S}{T+2L_\eta}\right]\left(E(B_T;Q_T)+c\frac{E(B_T;Q_T)}{R_\eta}+cL_\eta\right) \\ &\quad +c(T+2L_\eta)\left(\frac{S}{T+2L_\eta}-\left[\frac{S}{T+2L_\eta}\right]\right). \end{aligned}$$

Thus, taking the upper limit as $S\to+\infty$,

$$\begin{aligned} \limsup_{S\to+\infty}\frac{1}{S}E(B_S;Q_S) \quad &\le \quad \frac{1}{T+2L_\eta}\left(E(B_T;Q_T)+c\frac{E(B_T;Q_T)}{R_\eta}+cL_\eta\right) \\ &\le \quad \frac{1}{T}\left(E(B_T;Q_T)+c\frac{E(B_T;Q_T)}{R_\eta}+cL_\eta\right) \end{aligned}$$

and, taking the lower limit as $T\to+\infty$,

$$\limsup_{S\to+\infty}\frac{1}{S}E(B_S;Q_S)\le\liminf_{T\to+\infty}\frac{1}{T}E(B_T;Q_T)\left(1+\frac{c}{R_\eta}\right).$$

Now, for a given $\sigma>0$, let $B_T^\sigma\subset\mathcal{P}$ and such that $B_T^\sigma\sim H$ on ∂Q_T and

$$E(B_T^\sigma;Q_T)\le\inf\{E(B,Q_T): B\subset\mathcal{P}, B\sim H \text{ on } \partial Q_T\}+\sigma.$$

Then

$$\begin{aligned} &\limsup_{S\to+\infty}\frac{1}{S}\inf\{E(B,Q_S): B\subset\mathcal{P}, B\sim H \text{ on } \partial Q_S\} \\ &\qquad\le\left(1+\frac{c}{R_\eta}\right)\liminf_{T\to+\infty}\frac{1}{T}\inf\{E(B,Q_T): B\subset\mathcal{P}, B\sim H \text{ on } \partial Q_T\}. \end{aligned}$$

Letting $\eta\to0$, since $R_\eta\to+\infty$ by Proposition 6, we have

$$\begin{aligned} &\limsup_{S\to+\infty}\frac{1}{S}\inf\{E(B,Q_S): B\subset\mathcal{P}, B\sim H \text{ on } \partial Q_S\} \\ &\qquad\le\liminf_{T\to+\infty}\frac{1}{T}\inf\{E(B,Q_T): B\subset\mathcal{P}, B\sim H \text{ on } \partial Q_T\}, \end{aligned}$$

and we conclude that there exists the limit

$$\varphi(\nu,0)=\lim_{T\to+\infty}\frac{1}{T}\inf\left\{E(B;Q_T^\nu): B\subset\mathcal{P}, B\sim H^\nu \text{ on } \partial Q_T^\nu\right\}.$$

Now we prove that the limit defining $\varphi(\nu)$ exists for any sequence $\{x_T\}\subset\mathbb{R}^2$.

Given $x_T=(x_T^1,x_T^2)$, the relative density of the set of the admissible translations ensures that for any $\eta>0$ there exists $\tau_\eta^T\in T_\eta$ such that $Q_T(\tau_\eta^T)\subset Q_{T+2L_\eta}(x_T)$. Let $B_T\subset\mathcal{P}$ with $B_T\sim H_T$ on $\partial Q_T(x_T)$. Reasoning as above, we can assume that $i(B_T)$ is simply connected, and we set

$$B'_{\eta,T}=\begin{cases}\left(i(B_T)+\tau_\eta^T\right)\cap\mathcal{P} & \text{in } Q_T(\tau_\eta^T)\\ H_T\cap\mathcal{P} & \text{in } Q_{T+2L_\eta}(x_T)\setminus Q_T(\tau_\eta^T),\end{cases}$$

where $H_T = \{(x,y) : y - x_T^2 < 0\}$. It follows that

$$E(B'_{\eta,T}; Q_T(\tau_\eta^T)) \leq E(B_T; Q_T)\left(1 + \frac{c}{R_\eta}\right)$$

and then we deduce

$$\begin{aligned}\frac{1}{T+2L_\eta} E(B'_{\eta,T}; Q_{T+L_\eta}(x_T)) &\leq \frac{1}{T+2L_\eta} E(B'_{\eta,T}; Q_T(\tau_\eta^T)) + c' \frac{L_\eta}{T+2L_\eta} \\ &\leq \frac{1}{T} E(B_T; Q_T)\left(1 + \frac{c}{R_\eta}\right) + c' \frac{L_\eta}{T}.\end{aligned}$$

Taking the infimum over the admissible sets we get

$$\begin{aligned}&\inf \frac{1}{T+2L_\eta} \left\{E(B; Q_{T+2L_\eta}(x_T)) \ : \ B \subset \mathcal{P}, B \sim H_T \text{ on } \partial Q_{T+2L_\eta}(x_T)\right\} \\ &\quad \leq \inf \frac{1}{T} \{E(B; Q_T) \ : \ B \subset \mathcal{P}, B \sim H \text{ on } \partial Q_T\} \left(1 + \frac{c}{R_\eta}\right) + c' \frac{L_\eta}{T},\end{aligned}$$

and this implies

$$\begin{aligned}&\limsup_{T \to +\infty} \inf \frac{1}{T} \{E(B; Q_T(x_T)) \ : \ B \subset \mathcal{P}, B \sim H_T \text{ on } \partial Q_T(x_T)\} \\ &\quad \leq \lim_{T \to +\infty} \left(\inf \frac{1}{T} \{E(B; Q_T) \ : \ B \subset \mathcal{P}, B \sim H \text{ on } \partial Q_T\} \left(1 + \frac{c}{R_\eta}\right) + c' \frac{L_\eta}{T}\right) \\ &\quad = \varphi(\nu, 0) + o(1)_{\eta \to 0}. \qquad (7)\end{aligned}$$

In an analogous way we can construct, for any $\eta > 0$, a set $B''_{\eta,T} \subset \mathcal{P}$ such that $B''_{\eta,T} \sim H_T$ on $\partial Q_T(x_T)$ and

$$E(B_T; Q_T) \leq E(B''_{\eta,T}; Q_T(\tau))\left(1 + \frac{c}{R_\eta}\right),$$

obtaining

$$\begin{aligned}&\liminf_{T \to +\infty} \inf \frac{1}{T} \{E(B; Q_T(x_T)) \ : \ B \subset \mathcal{P}, B \sim H_T \text{ on } \partial Q_T(x_T)\} \left(1 + \frac{c}{R_\eta}\right) \\ &\quad \geq \lim_{T \to +\infty} \inf \frac{1}{T} \{E(B; Q_T) \ : \ B \subset \mathcal{P}, B \sim H \text{ on } \partial Q_T\}.\end{aligned}$$

Then

$$\liminf_{T \to +\infty} \inf \frac{1}{T} \{E(B; Q_T(x_T)) \ : \ B \subset \mathcal{P}, B \sim H_T \text{ on } \partial Q_T(x_T)\} \geq \varphi(\nu, 0) + o(1)_{\eta \to 0},$$

concluding the proof. □

4 Proof of the homogenization result

We now prove Theorem 5. As usual, this is divided into proving a lower and an upper bound separately.

4.1 Lower bound

The proof of the lower bound will be achieved through the use of the blow-up technique of Fonseca and Müller [13] (see also [8] for an analysis of this method for homogenization problems).

Let A be of finite perimeter; let $\{A_\varepsilon\}$ be a sequence of admissible sets such that $A_\varepsilon \to A$. It is not restrictive to assume that there exists the limit $\lim\limits_{\varepsilon\to 0^+} E_\varepsilon(A_\varepsilon)$ and that it is finite.

We set

$$D_\varepsilon = \left\{ k = \frac{i+j}{2} : i \in A_\varepsilon,\ j \in \varepsilon\mathcal{P} \setminus A_\varepsilon,\ |i-j| = \varepsilon \right\}$$

and define the measures $\mu_\varepsilon = \sum_{k\in D_\varepsilon} \varepsilon\delta_k$. In this way,

$$\mu_\varepsilon(\mathbb{R}^2) = E_\varepsilon(A_\varepsilon),$$

so that the sequence $\{\mu_\varepsilon\}$ is bounded, and, up to subsequences, we can assume

$$\mu_\varepsilon \overset{*}{\rightharpoonup} \mu.$$

We want to prove that the inequality

$$\frac{d\mu}{d\mathcal{H}^1 \llcorner \partial^* A}(x_0) \geq \varphi(\nu_A(x_0)) \tag{8}$$

holds for $\mathcal{H}^1$-a.a. $x_0 \in \partial^* A$.

With fixed $x_0 \in \partial^* A$ we set $\nu = \nu_A(x_0)$, and define $Q^\nu_\varrho(x_0)$ as any cube with centre x_0, side length ϱ and two faces orthogonal to ν. Note that, for almost every ϱ, $\mu(Q^\nu_\varrho(x_0)) = \liminf_{\varepsilon\to 0} \mu_\varepsilon(Q^\nu_\varrho(x_0))$.

By the Besicovitch differentiation theorem it follows that $\mathcal{H}^1$-almost every $x_0 \in \partial^* A$ is a Lebesgue point for μ with respect to $\mathcal{H}^1 \llcorner \partial^* A$. Hence, for $\mathcal{H}^1$-a.a. $x_0 \in \partial^* A$ there exists the limit

$$\frac{d\mu}{d\mathcal{H}^1 \llcorner \partial^* A}(x_0) = \lim_{\varrho\to 0} \frac{\mu(Q^\nu_\varrho(x_0))}{\mathcal{H}^1(Q^\nu_\varrho(x_0) \cap \partial^* A)}.$$

We can assume that x_0 is such that $\dfrac{A - x_0}{\varrho} \to H^\nu$ as $\varrho \to 0$.

We may therefore assume that $x_0 \in \partial^* A$ satisfies the properties above. Since μ is finite it is also possible to choose an infinitesimal sequence ϱ_ε such that

$$\frac{d\mu}{d\mathcal{H}^1 \llcorner \partial^* A}(x_0) = \liminf_{\varepsilon\to 0} \frac{\mu_\varepsilon(Q^\nu_{\varrho_\varepsilon}(x_0))}{\mathcal{H}^1(Q^\nu_{\varrho_\varepsilon}(x_0) \cap \partial^* A)},$$

satisfying the asymptotic conditions $T_\varepsilon = \dfrac{\varrho_\varepsilon}{\varepsilon} \to +\infty$ and

$$\frac{\left|\frac{1}{\varepsilon}(i(A_\varepsilon) - x_0)\Delta H^\nu\right|}{T_\varepsilon^2} = o(1)_{\varepsilon\to 0}.$$

We set $x_{T_\varepsilon} = \dfrac{x_0}{\varepsilon}$, so that

$$\frac{\left|i\left(\frac{A_\varepsilon}{\varepsilon}\right)\Delta H^\nu_{T_\varepsilon}\right|}{T_\varepsilon^2} = o(1)_{\varepsilon\to 0}, \tag{9}$$

where we recall that $H^\nu_{T_\varepsilon} = \{x : \langle x - x_{T_\varepsilon}, \nu\rangle < 0\}$. Then we get

$$\begin{aligned}\liminf_{\varepsilon\to 0}\frac{\mu_\varepsilon(Q^\nu_{\varrho_\varepsilon}(x_0))}{\mathcal{H}^1(Q^\nu_{\varrho_\varepsilon}(x_0)\cap\partial^* A)} &= \liminf_{\varepsilon\to 0}\frac{E_\varepsilon(A_\varepsilon; Q^\nu_{\varrho_\varepsilon}(x_0))}{\varrho_\varepsilon}\\ &= \liminf_{\varepsilon\to 0}\frac{E\left(\frac{A_\varepsilon}{\varepsilon}; Q^\nu_{T_\varepsilon}(x_{T_\varepsilon})\right)}{T_\varepsilon}.\end{aligned}$$

Now, by modifying the "boundary values" of the sets $\dfrac{A_\varepsilon}{\varepsilon}$ we construct a sequence $B_\varepsilon \subset \mathcal{P}$, such that $B_\varepsilon \sim H^\nu_{T_\varepsilon}$ on $\partial Q^\nu_{T_\varepsilon}(x_{T_\varepsilon})$, and

$$\liminf_{\varepsilon\to 0}\frac{E(B_\varepsilon; Q^\nu_{T_\varepsilon}(x_{T_\varepsilon}))}{T_\varepsilon} \le \liminf_{\varepsilon\to 0}\frac{E\left(\frac{A_\varepsilon}{\varepsilon}; Q^\nu_{T_\varepsilon}(x_{T_\varepsilon})\right)}{T_\varepsilon}. \tag{10}$$

For surface energies this modification is usually obtained by a use of the coarea formula (see [2]). Here we will mimick that reasoning in a discrete setting.

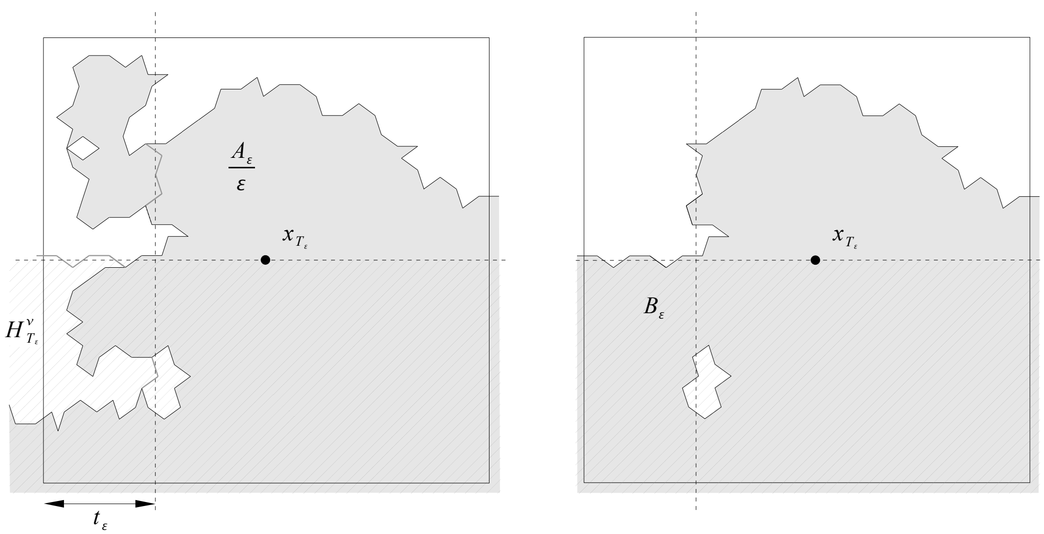

Figure 4: Construction of B_ε by changing the "boundary values".

Since

$$|(i(\mathcal{P}\cap H^\nu_{T_\varepsilon})\Delta\ H^\nu_{T_\varepsilon})\cap Q^\nu_{T_\varepsilon}(x_{T_\varepsilon})| = O(T_\varepsilon)_{\varepsilon\to 0}$$

and also

$$|(i(\mathcal{P}\cap Q^\nu_{T_\varepsilon}(x_{T_\varepsilon}))\Delta\ Q^\nu_{T_\varepsilon}(x_{T_\varepsilon}))| = O(T_\varepsilon)_{\varepsilon\to 0},$$

setting

$$M_\varepsilon = \left| \left(i \left(\frac{A_\varepsilon}{\varepsilon} \right) \Delta\ i(\mathcal{P} \cap H^\nu_{T_\varepsilon}) \right) \cap i(\mathcal{P} \cap Q^\nu_{T_\varepsilon}(x_{T_\varepsilon})) \right|$$

and recalling (9), it follows that $M_\varepsilon/T_\varepsilon^2 = o(1)_{\varepsilon\to 0}$. The set $\left(i\left(\frac{A_\varepsilon}{\varepsilon}\right) \Delta\ i(\mathcal{P}\cap H^\nu_{T_\varepsilon})\right) \cap i(\mathcal{P} \cap Q^\nu_{T_\varepsilon}(x_{T_\varepsilon}))$ can be written as a finite disjoint union of rescaled Penrose cells R with side lenght 1/2; that is,

$$\left(i \left(\frac{A_\varepsilon}{\varepsilon} \right) \Delta\ i(\mathcal{P} \cap H^\nu_{T_\varepsilon}) \right) \cap i(\mathcal{P} \cap Q^\nu_{T_\varepsilon}(x_{T_\varepsilon}) = \bigcup_{R \in \mathcal{R}_\varepsilon} R.$$

We fix $\delta > 0$ and get

$$\begin{aligned} M_\varepsilon &= \left| \bigcup_{R\in\mathcal{R}_\varepsilon} R \right| \\ &\geq \frac{1}{c_1} \int_{\frac{\delta}{2}T_\varepsilon}^{\delta T_\varepsilon} \#\{R \in \mathcal{R}_\varepsilon : R \cap \partial Q^\nu_{T_e - 2t}(x_{T_\varepsilon}) \neq \emptyset\}\, dt. \end{aligned}$$

Then there exists $t_\varepsilon \in \left[\frac{\delta}{2}T_\varepsilon, \delta T_\varepsilon\right]$ such that

$$\#\{R \in \mathcal{R}_\varepsilon : R \cap \partial Q^\nu_{T_e - 2t_\varepsilon}(x_{T_\varepsilon}) \neq \emptyset\} \leq \frac{2c_1}{\delta T_\varepsilon} M_\varepsilon = \frac{2c_1}{\delta} T_\varepsilon o(1)_{\varepsilon\to 0}. \tag{11}$$

Defining

$$B_\varepsilon = \begin{cases} \dfrac{A_\varepsilon}{\varepsilon} & \text{in } Q^\nu_{T_\varepsilon - 2t_\varepsilon}(x_{T_\varepsilon}) \\ \mathcal{P} \cap H^\nu_{T_\varepsilon} & \text{otherwise in } Q^\nu_{T_\varepsilon}(x_{T_\varepsilon}) \end{cases}$$

it follows that

$$\begin{aligned} \frac{E(B_\varepsilon; Q^\nu_{T_\varepsilon}(x_{T_\varepsilon}))}{T_\varepsilon} &\leq \frac{E\left(\frac{A_\varepsilon}{\varepsilon}; Q^\nu_{T_\varepsilon}(x_{T_\varepsilon})\right)}{T_\varepsilon} + \frac{c}{\delta} o(1)_{\varepsilon\to 0} + c\frac{t_\varepsilon}{T_\varepsilon} \\ &\leq \frac{E\left(\frac{A_\varepsilon}{\varepsilon}; Q^\nu_{T_\varepsilon}(x_{T_\varepsilon})\right)}{T_\varepsilon} + \frac{c}{\delta} o(1)_{\varepsilon\to 0} + c\delta, \end{aligned}$$

so that we get for any $\delta > 0$ the inequality

$$\liminf_{\varepsilon\to 0} \frac{E(B_\varepsilon; Q^\nu_{T_\varepsilon}(x_{T_\varepsilon}))}{T_\varepsilon} \leq \liminf_{\varepsilon\to 0} \frac{E\left(\frac{A_\varepsilon}{\varepsilon}; Q^\nu_{T_\varepsilon}(x_{T_\varepsilon})\right)}{T_\varepsilon} + c\delta.$$

Then, taking the limit for $\delta \to 0$, the required inequality (10) follows.

Recalling the definition of φ, we get

$$\liminf_{\varepsilon\to 0} \frac{\mu_\varepsilon(Q^\nu_{\varrho_\varepsilon}(x_0))}{\mathcal{H}^1(Q^\nu_{\varrho_\varepsilon}(x_0) \cap \partial^* A)} \geq \liminf_{\varepsilon\to 0} \frac{E(B_\varepsilon; Q^\nu_{T_\varepsilon}(x_{T_\varepsilon}))}{T_\varepsilon} \geq \varphi(\nu_A(x_0)); \tag{12}$$

i.e., (8). Using the lower semicontinuity of the total measure and the positiveness of μ, we have

$$\liminf_{\varepsilon\to 0^+} E_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon\to 0^+} \mu_\varepsilon(\mathbb{R}^2) \geq \mu(\mathbb{R}^2) \geq \int_{\partial^* A} \frac{d\mu}{d\mathcal{H}^1 \llcorner \partial^* A} d\mathcal{H}^1.$$

We then conclude the proof of the Γ-$\liminf$ inequality by integrating (8) on $\partial^* A$.

4.2 Upper bound

The proof of the upper bound will be given by density. We start by proving the existence of a recovery sequence when A is a polyhedral set, the general case following then by a density argument.

Proposition 7. *Let $A \subset \mathbb{R}^2$ be a polyhedral set. Then there exists a sequence $\{A_\varepsilon\}_\varepsilon$ such that $A_\varepsilon \subset \varepsilon\mathcal{P}$, $A_\varepsilon \to A$ as $\varepsilon \to 0$ (in the sense of the Definition 3), and*

$$\lim_{\varepsilon\to 0} E_\varepsilon(A_\varepsilon) = E(A).$$

Proof. We fix $\sigma > 0$. For any $T > 0$, let $\widetilde{C}^\sigma_T(\nu)$ be a 1-Penrose set such that $i(\widetilde{C}^\sigma_T(\nu))$ is simply connected, $\widetilde{C}^\sigma_T(\nu) \sim H^\nu$ in ∂Q^ν_T and

$$E(\widetilde{C}^\sigma_T(\nu); Q^\nu_T) < \inf\{E(B; Q^\nu_T) : B \subset \mathcal{P}, B \sim H^\nu \text{ in } \partial Q^\nu_T\} + \sigma.$$

Note that

$$E_\varepsilon(\varepsilon\widetilde{C}^\sigma_T(\nu); Q^\nu_{\varepsilon T}) = \varepsilon\, E(\widetilde{C}^\sigma_T(\nu); Q^\nu_T).$$

Now we choose $T = \frac{1}{\sqrt{\varepsilon}}$, and define

$$C^\sigma_\varepsilon(\nu) = \varepsilon\, \widetilde{C}^\sigma_{1/\sqrt{\varepsilon}}(\nu).$$

In the following computation, we omit the dependence on σ and denote $C^\sigma_\varepsilon(\nu)$ by $C_\varepsilon(\nu)$ and $\widetilde{C}^\sigma_T(\nu)$ by $\widetilde{C}_T(\nu)$.

We will use a construction localized close to each edge of A. To that end, we start by considering the set $\widetilde{F} = \{x + t\nu : x \in F, 0 < t < \delta\}$ where F is a segment with lenght L orthogonal to ν, and $\delta > 0$. Let x_0 be the center of F, and define $S = \{x + t\nu : x \in F, -\delta < t < \delta\}$.

Reasoning as in the proof of Proposition 4, for any fixed $\eta > 0$ the relative density of the set T_η of the "admissible translations" ensures that there exists an inclusion lenght L_η such that for each point $\dfrac{x_0 + i\sqrt{\varepsilon}\nu^\perp}{\varepsilon}$, with $i = -M, \dots, M$ and $M = \left[\frac{L/2}{\sqrt{\varepsilon}+2\varepsilon L_\eta}\right]$ we can find a translation $\widehat{\tau}^{i,\eta}_\varepsilon \in T_\eta$ such that

$$\left|\frac{x_0 + i\sqrt{\varepsilon}\nu^\perp}{\varepsilon} - \widehat{\tau}^{i,\eta}_\varepsilon\right| < L_\eta.$$

Then, setting $\tau^i_\varepsilon = \varepsilon\widehat{\tau}^{i,\eta}_\varepsilon$ (where we omit the dependence on η), it follows that

$$Q^i_{\sqrt{\varepsilon}} := Q^\nu_{\sqrt{\varepsilon}}(\tau^i_\varepsilon) \subset Q^\nu_{\sqrt{\varepsilon}+2\varepsilon L_\eta}(x_0 + i\sqrt{\varepsilon}\nu^\perp) \qquad i = -M, \dots, M.$$

Now, for each $i = -M, \dots, M$ we set $B^i_\varepsilon = i(C_\varepsilon(\nu) + \tau^i_\varepsilon) \cap \varepsilon\mathcal{P}$, and

$$F_\varepsilon = \begin{cases} B^i_\varepsilon & \text{in } Q^i_{\sqrt{\varepsilon}} \\ \widetilde{F} \cap \varepsilon\mathcal{P} & \text{in } S \setminus \bigcup_i Q^i_{\sqrt{\varepsilon}}. \end{cases}$$

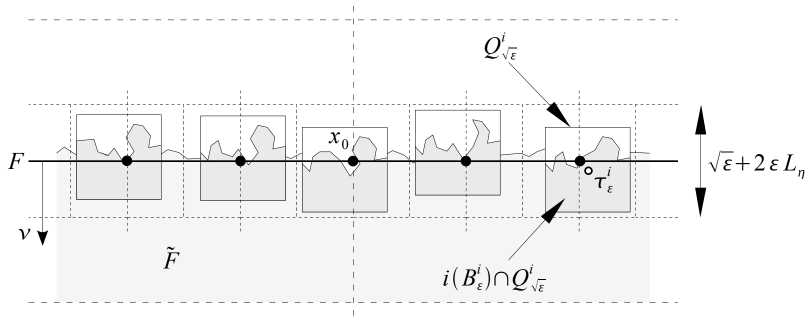

Figure 5: Recovery sequence for $\widetilde{F}$.

Note that $F_\varepsilon \Delta \widetilde{F} \subset S_\varepsilon = \{x + t\nu : x \in F, -\sqrt{\varepsilon} - 2\varepsilon L_\eta < t < \sqrt{\varepsilon} + 2\varepsilon L_\eta\}$. We then get

$$E_\varepsilon(F_\varepsilon; S) \le \sum_i E_\varepsilon(B^i_\varepsilon; Q^i_{\sqrt{\varepsilon}}) + c(\sqrt{\varepsilon} + 2\varepsilon L_\eta)\left(\frac{L}{\sqrt{\varepsilon} + 2\varepsilon L_\eta} - \left[\frac{L}{\sqrt{\varepsilon} + 2\varepsilon L_\eta}\right]\right).$$

Following the proof of the Proposition 4, we obtain for each i the estimate

$$E_\varepsilon(B^i_\varepsilon; Q^i_{\sqrt{\varepsilon}}) \le E_\varepsilon(C_\varepsilon(\nu); Q_{\sqrt{\varepsilon}})\left(1 + \frac{c}{R_\eta}\right)$$

with c independent of ε, so that

$$\begin{aligned} E_\varepsilon(F_\varepsilon; S) \quad &\le \quad \left[\frac{L}{\sqrt{\varepsilon}}\right] E_\varepsilon(C_\varepsilon(\nu); Q_{\sqrt{\varepsilon}})\left(1 + \frac{c}{R_\eta}\right) \\ &\quad + c(\sqrt{\varepsilon} + 2\varepsilon L_\eta)\left(\frac{L}{\sqrt{\varepsilon} + 2\varepsilon L_\eta} - \left[\frac{L}{\sqrt{\varepsilon} + 2\varepsilon L_\eta}\right]\right). \end{aligned}$$

Since

$$\begin{aligned} E_\varepsilon(C_\varepsilon(\nu); Q_{\sqrt{\varepsilon}}) \quad &= \quad \varepsilon E(\widetilde{C}_{1/\sqrt{\varepsilon}}(\nu); Q^\nu_{1/\sqrt{\varepsilon}}) \\ &< \quad \varepsilon \inf\left\{E(B; Q^\nu_{1/\sqrt{\varepsilon}}) : B \subset \mathcal{P}, B \sim H^\nu \text{ in } \partial Q^\nu_{1/\sqrt{\varepsilon}}\right\} + \varepsilon\sigma, \end{aligned}$$

it follows that

$$\begin{aligned} E_\varepsilon(F_\varepsilon; S) \quad &\le \quad \frac{L}{1/\sqrt{\varepsilon}}\left(\inf\left\{E(B; Q^\nu_{1/\sqrt{\varepsilon}}) : B \subset \mathcal{P}, B \sim H^\nu \text{ in } \partial Q^\nu_{1/\sqrt{\varepsilon}}\right\} + \sigma\right)\left(1 + \frac{c}{R_\eta}\right) \\ &\quad + c(\sqrt{\varepsilon} + 2\varepsilon L_\eta)\left(\frac{L}{\sqrt{\varepsilon} + 2\varepsilon L_\eta} - \left[\frac{L}{\sqrt{\varepsilon} + 2\varepsilon L_\eta}\right]\right) \end{aligned}$$

Taking the limit for $\varepsilon \to 0$ we get the inequality

$$\limsup_{\varepsilon\to 0} E_\varepsilon(F_\varepsilon; S)$$
$$\leq L \lim_{\varepsilon\to 0} \frac{1}{1/\sqrt{\varepsilon}} \inf \Big\{ E(B; Q^\nu_{1/\sqrt{\varepsilon}}) : B \subset \mathcal{P}, B \sim H^\nu \text{ in } \partial Q^\nu_{1/\sqrt{\varepsilon}} \Big\} \left(1 + \frac{c}{R_\eta}\right)$$

so that, recalling the definition of φ, it follows

$$\limsup_{\varepsilon\to 0} E_\varepsilon(F_\varepsilon; S) \leq L\varphi(\nu)\left(1 + \frac{c}{R_\eta}\right).$$

For $\eta \to 0$, since $R_\eta \to +\infty$, we find

$$\limsup_{\varepsilon\to 0} E_\varepsilon(F_\varepsilon; S) \leq |F|\varphi(\nu). \tag{13}$$

Let A be a polyhedral subset of $\mathbb{R}^2$, with edges $F^1, \ldots, F^N$; we denote by ν^j the inner normal to A in F^j, and by H^j the half plane $\{\langle x - x^j, \nu_j\rangle < 0\}$, where $x^j \in F^j$. Moreover, let S^j be the set $\{x + t\nu_j : x \in F^j, -\delta < t < \delta\}$ and $S^j_\varepsilon = \{x + t\nu_j : x \in F^j, -\sqrt{\varepsilon} - 2\varepsilon L_\eta < t < \sqrt{\varepsilon} + 2\varepsilon L_\eta\}$.

Let F^j_ε be the set constructed as in the previous step, with $F^j = F$ and $A \cap S^j = \widetilde{F}$. We define $W_\varepsilon = \bigcup_{j\neq k} S^j_\varepsilon \cap S^k_\varepsilon$, and set

$$A_\varepsilon = \begin{cases} F^j_\varepsilon & \text{in } S^j_\varepsilon \setminus W_\varepsilon \\ A \cap \varepsilon\mathcal{P} & \text{otherwise in } \mathbb{R}^2. \end{cases}$$

Recalling (13), it follows that

$$\limsup_{\varepsilon\to 0} E_\varepsilon(A_\varepsilon) \leq \sum\nolimits_j |L^j|\varphi(\nu^j) = \int_{\partial A} \varphi(\nu)\, d\mathcal{H}^{n-1}.$$

Since $|W_\varepsilon|$ goes to 0 as $\varepsilon \to 0$, then $A_\varepsilon \to A$ and A_ε is a recovery sequence for A. $\Box$

We can conclude the proof of the upper bound for a general A. Given A of finite perimeter, there exists a sequence of polyhedral sets A_k such that $|A_k \Delta A| \to 0$ and $|D\chi_{A_k}|(\mathbb{R}^2) \to |D\chi_A|(\mathbb{R}^2)$ as $k \to +\infty$ (see [4]). Then, if we prove that the function φ is continuous, we can apply the Reshetnyak Theorem (see, e.g., [4] Theorem 1.3.2) to obtain

$$\lim_{k\to+\infty} E(A_k) = E(A),$$

concluding the proof of the upper bound thanks to the lower semicontinuity of the upper Γ-limit.

Therefore, we conclude by showing the continuity of φ. Given a unit vector $\nu = (\cos\theta, \sin\theta)$, $T > 0$ and $x_0 \in \mathbb{R}^2$, let B^T_ν be a 1-Penrose set such that $B \sim H^\nu$ on $\partial Q^\nu_T(x_0)$ and

$$E(B^T_\nu; Q^\nu_T(x_0)) < \inf\{E(B; Q^\nu_T(x_0)) : B \subset \mathcal{P}, B \sim H^\nu \text{ on } \partial Q^\nu_T(x_0)\} + \sigma.$$

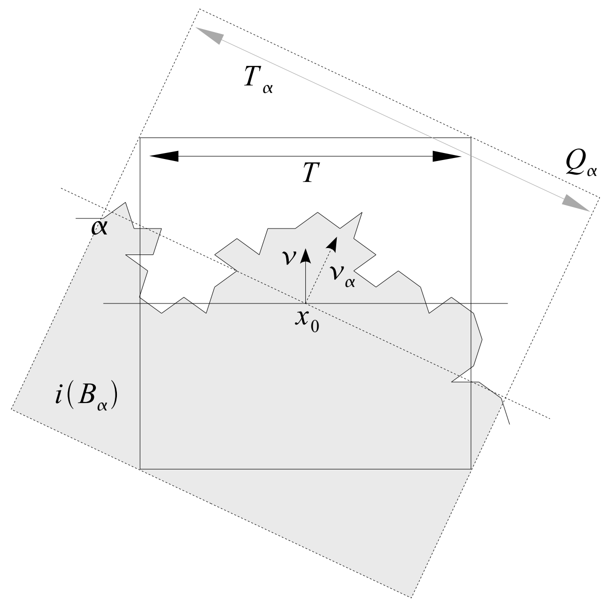

Figure 6: Construction of B_α.

It is not restrictive for our purpose to fix $\nu = (0,1)$. Now, for $\alpha \in (0, \pi/4)$ let $\nu_\alpha = (\sin\alpha, \cos\alpha)$. We consider the square $Q_\alpha = Q^{\nu_\alpha}_{T_\alpha}(x_0)$ where $T_\alpha = T(\sin\alpha + \cos\alpha)$ (see Figure 6), and set:

$$B_\alpha = \begin{cases} B^T_\nu & \text{in } Q^\nu_T(x_0) \\ Q_\alpha \cap \mathcal{P} & \text{otherwise in } Q_\alpha. \end{cases}$$

Then, B_α is a 1-Penrose set satisfying the boundary condition $B_\alpha \sim H^{\nu_\alpha}$ on ∂Q_α. By construction, it follows that

$$E(B_\alpha; Q_\alpha) = E(B^T_\nu; Q^\nu_T(x_0)) + cT\left(\sin\alpha + \tan\alpha(1 - \sin\alpha)\right)$$

where c is independent on T, α, σ, so that

$$\frac{1}{T_\alpha} E(B_\alpha; Q_\alpha) = \frac{T}{T_\alpha}\left(\frac{1}{T} E(B^T_\nu; Q^\nu_T(x_0)) + o(1)_{\alpha\to 0}\right).$$

It follows that

$$\lim_{\alpha\to 0} \varphi(\nu_\alpha) = \varphi(\nu) + \sigma;$$

the arbitrariness of $\sigma > 0$ allows to deduce the continuity of φ.

5 Generalizations

The computation performed above is based on the “quasi-periodic” properties of the Penrose lattice. As it does not depend on the particular type of interaction between points and all the arguments used are local, the result can be generalized in some directions as follows.

5.1 More general energies

We can consider bond energies depending on the orientation of the bonds, of the form

$$E_\varepsilon(A) = \varepsilon \sum \left\{ c_{(j-i)/\varepsilon} : i \in A, j \notin A, |i-j| = \varepsilon \right\}, \tag{14}$$

where c_z (defined for $z = e^{ik\pi/5}$, $k = 0, \dots, 9$) are strictly positive weights (the previous case is obtained by taking all $c_z = 1$). Theorem 5 holds exactly as is, upon considering the localized energy

$$E(A; \Omega) = \sum \{ c_{j-i} : i \in A, j \notin A, |i-j| = 1, i, j \in \Omega \} \tag{15}$$

in the definition of φ.

The proof follows exactly the one of Theorem 5, with only a heavier notation.

5.2 Localized functionals

The Homogenization Theorem can be proved directly for the localized functionals in (3), upon requiring the Lipschitz regularity of $\partial\Omega$ (otherwise standard counterexamples can be adapted). In this case the limit energy is also localized and reads as

$$E(A; \Omega) = \int_{\Omega \cap \partial^* A} \varphi(\nu)\, d\mathcal{H}^1, \tag{16}$$

defined on sets with finite perimeter in Ω, with the same definition of φ.

The proof of the lower bound is the same as for the case in the whole $\mathbb{R}^2$, while for the upper bound only the statement of Proposition 7 must be modified by adding the requirement that A is a polyhedral set in $\mathbb{R}^2$ with $\mathcal{H}^1(\partial A \cap \partial\Omega) = 0$. The proof of the proposition remains unchanged, as the rest of the proof of the upper bound.

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