On maximum planar induced subgraphs

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Abstract

The nonplanar vertex deletion or vertex deletion \(vd(G)\) of a graph \(G\) is the smallest nonnegative integer \(k\), such that the removal of \(k\) vertices from \(G\) produces a planar graph \(G'\). In this case \(G'\) is said to be a maximum planar induced subgraph of \(G\). We solve a problem proposed by Yannakakis: find the threshold for the maximum degree of a graph \(G\) such that, given a graph \(G\) and a nonnegative integer \(k\), to decide whether \(vd(G) \leq k\) is NP-complete. We prove that it is NP-complete to decide whether a maximum degree 3 graph \(G\) and a nonnegative integer \(k\) satisfy \(vd(G) \leq k\). We prove that unless \(P = NP\) there is no polynomial-time approximation algorithm with fixed ratio to compute the size of a maximum planar induced subgraph for graphs in general. We prove that it is Max SNP-hard to compute \(vd(G)\) when restricted to a cubic input \(G\). Finally, we exhibit a polynomial-time \(3/4\)-approximation algorithm for finding a maximum planar induced subgraph of a maximum degree 3 graph.

Keywords: Topological graph theory; Nonplanar vertex deletion; Nonplanar edge deletion; Nonapproximability; Maximum planar induced subgraph; Cubic graphs; Planarity invariants; NP-complete and Max SNP-hard problems

1. Introduction

Measures for nonplanarity have an important place in the study of planar graphs due to many industrial and combinatorial applications which involve planarity concepts. There are several important measures for the nonplanarity of a graph, for instance, the minimum number of crossings in an embedding in the plane, the genus, the minimum number of edges whose removal defines a planar graph, the minimum number of edge-disjoint planar subgraphs whose sets of edges partition the set of edges of the graph. The corresponding decision problems for most of these invariants are known to be NP-complete [7,9,13–15]. Even methods like polynomial time approximation schemes in some cases are not likely to exist [4,6,8].

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The nonplanar vertex deletion or vertex deletion $vd(G)$ of a graph $G$ is the smallest nonnegative integer such that the removal of $vd(G)$ vertices from $G$ produces a planar graph $G'$. The graph $G'$ is a maximum planar induced subgraph of $G$. The vertex deletion decision problem (VD) consists in deciding, given a graph $G$ and a nonnegative integer $k$, whether $vd(G) \leq k$. The minimization problem of finding $vd(G)$ of a given graph $G = (V, E)$ is denoted by MINVD. The maximization problem of finding the number of vertices of a maximum planar induced subgraph of a given graph $G = (V, E)$ is denoted by MAXPIS.

With respect to special classes of graphs, vertex deletion is known for complete graphs and complete bipartite graphs: $vd(K_n) = n - 4$ if $n > 4$ and 0 otherwise; and $vd(K_{n,m}) = \min(n, m) - 2$ if $\min(n, m) > 2$ and 0 otherwise. We have established recently the vertex deletion values for all $C_n \times C_n$ graphs [11].

Yannakakis [15,14] proved that VD is NP-complete. The special instance constructed by Yannakakis in the NP-completeness reduction is not a graph with maximum degree 3. A natural question in the study of the complexity of a graph-theoretical decision problem is to determine the best possible bounds on the vertex degrees for which the problem remains NP-complete [15,14]. In his fundamental paper, Yannakakis [14] proposed the following problem: find the threshold for the maximum degree of a graph $G$ such that, given a graph $G$ and a nonnegative integer $k$, to decide whether $vd(G) \leq k$ is NP-complete.

The nonplanar edge deletion decision problem (ED) consists in deciding, given a graph $G = (V, E)$ and a nonnegative integer $k$, whether there is a subset $E' \subset E$, such that the graph $G' = (V, E \setminus E')$ is planar and $|E'| \leq k$. When $k$ is the smallest nonnegative integer, such that there exists $E' \subset E$, where $G' = (V, E \setminus E')$ is planar with $|E'| = k$, then $G'$ is said to be a maximum planar subgraph of $G$. The minimization problem of finding the minimum number of edges whose removal from $G = (V, E)$ defines a planar graph is denoted by MINED. The maximization problem of finding the number of edges of a maximum planar subgraph of a given graph $G = (V, E)$ is denoted by MAXPS.

Liu and Geldmacher [10] and independently Yannakakis [15,14] proved that ED is NP-complete. Faria et al. [7] proved that ED is NP-complete for cubic graphs. Calinescu et al. [4] proved that MINED and MAXPS are Max SNP-hard and exhibited a polynomial-time $\frac{3}{4}$-approximation algorithm for MAXPS for general graphs. Subsequently, Faria et al. [6,8] proved that MINED is Max SNP-hard for cubic graphs.

In this paper, we consider the computation of vertex deletion and the complexity of the corresponding decision and optimization problems. We prove that even restricted to cubic graphs, VD is NP-complete. We prove that unless $P = NP$, there is no polynomial-time approximation algorithm with a fixed ratio for MAXPIS for graphs in general. With respect to special classes of graphs, we use the concept of the L-reductions of Papadimitriou and Yannakakis [12] to prove that MINVD is Max SNP-hard for cubic graphs, meaning that there is a constant $\varepsilon > 0$, such that the existence of a polynomial-time approximation algorithm for MINVD restricted to cubic graphs with performance ratio at most $1 + \varepsilon$ implies that $P = NP$ [2,12]. Our NP-completeness and Max SNP-hardness results are optimum with respect to the allowed maximum vertex degree, because a graph with maximum degree 2 is a collection of paths and circuits that define a planar graph. We also present a polynomial-time $\frac{3}{4}$-approximation algorithm for MAXPIS for graphs with maximum degree 3.

For obtaining that VD is NP-complete for cubic graphs we use that ED is NP-complete for cubic graphs [7]. We use the negative result, due to Arora and Safra [3] that unless $P = NP$, there is no polynomial-time approximation algorithm with a fixed ratio for MAXIMUM INDEPENDENT SET in order to prove that unless $P = NP$, there is no polynomial-time approximation algorithm with a fixed ratio for MAXPIS for graphs in general. Specifically, we show that the existence of an $\varepsilon$-approximation for MAXPIS implies the existence of an $\frac{\varepsilon}{2}$-approximation for MAXIMUM INDEPENDENT SET. To prove that MINVD is Max SNP-hard for cubic graphs we L-reduce [12] the Max SNP-hard problem MINED for cubic graphs [6,8] to MINVD for cubic graphs.

This article is organized as follows. Sections 2 and 3 establish notation, definitions, and main properties. Sections 4–7 present our complexity results. Section 8 proposes our polynomial-time $\frac{3}{4}$-approximation algorithm. Section 9 concludes with final remarks and related open problems.

2. Notation and definitions

Let $G = (V, E)$ be a graph, $v \in V$ and $S \subset V$. The subgraph of $G$ induced by $S$ is the maximal subgraph of $G$ with vertex set $S$. The graph $G - v$ is the subgraph of $G$ induced by $V \setminus \{v\}$. The graph $G - S$ is the subgraph of $G$ induced by $V \setminus S$. 
Lemma 6. Let $G$ be a graph. A set $\Sigma = \{v_1, v_2, v_3, \ldots, v_{|\Sigma|}\} \subset V(G)$ satisfies that $G - \Sigma$ is a planar graph and $|\Sigma| = vd(G)$ if and only if the sequence $G_0, G_1, G_2, G_3, \ldots, G_{|\Sigma|}$, where $G_0 = G$ and $G_k = G - \{v_1, v_2, v_3, \ldots, v_k\}$, $k \in \{1, 2, 3, \ldots, |\Sigma|\}$ satisfies $vd(G_k) = |\Sigma| - k$. 

Proof. We prove by induction on $k$ that $vd(G_k) = |\Sigma| - k$, for $0 \leq k \leq |\Sigma|$. The basis $k=0$ follows from the definition of $\Sigma$. Suppose $vd(G_k) = |\Sigma| - k$, for $0 \leq k < |\Sigma|$. The definitions of $\Sigma$ and $G_{k+1}$ say that $G_{k+1} - \{v_{k+2}, v_{k+3}, v_{k+4}, \ldots, v_{|\Sigma|}\}$
is a planar graph, which implies $vd(G_{k+1}) \leq |\Sigma| - (k + 1)$. Since a planar graph is obtained from $G_k$ by removal of $vd(G_{k+1})$ vertices together with vertex $v_{k+1}$ we have that $vd(G_k) \leq vd(G_{k+1}) + 1$. On the other hand, the induction hypothesis says $vd(G_k) = |\Sigma| - k$. Hence, $vd(G_{k+1}) \geq vd(G_k) - 1 = |\Sigma| - (k + 1)$. The other direction follows from the definition of $G_k$ and the assumption that $vd(G_k) = |\Sigma| - k$. □

4. Relating vertex deletion to edge deletion

We define a special algorithm $f$ that relates the computation of edge deletion to the computation of vertex deletion. Algorithm $f$ is used in Sections 5 and 6.

We construct a polynomial-time algorithm $f$ which, given a cubic graph $G$, produces a cubic graph $H$ satisfying $Opt_{MINED}(G) = Opt_{MINVD}(H)$, where $Opt_{MINED}(G)$ and $Opt_{MINVD}(H)$ denote, respectively, the optimum values of MINED for $G$ and of MINVD for $H$. We show how this last equality and our previous results in [7] and [6,8] allow us to define the NP-completeness and Max SNP-hardness reductions.

Given a cubic graph $G$, algorithm $f$ described next produces, in polynomial time on the size of $G$, a cubic graph $f(G) = H = (V(H), E(H))$. The construction of $f$ is simple: given a cubic graph $G$, which is an instance of MINED, let $H$ be the graph obtained from $G$ by substituting each vertex by a triangle (with each vertex adjacent to one of the neighbours of the original vertex, so that the resulting graph is still cubic). Note that algorithm $f$ can be designed to run in time $O(n)$, where $n = |V(G)|$, since $G$ is cubic and so $|E(G)| = 3n/2$.

In Fig. 1 we give two examples of the application of the algorithm $f$. One with a planar cubic graph $G_1$ which is the 3-cube graph, and one with a nonplanar cubic graph $G_2$ which is $K_{3,3}$. These graphs are depicted on the left of Figs. 1(a) and (b) and their corresponding outputs from algorithm $f$ are depicted on the right.

Next we state the fundamental property for the complexity result.

**Theorem 7 (fundamental property).** Let $G$ be a cubic graph and $H$ be the image of $G$ by algorithm $f$. Then,

1. For a given feasible solution $R$ of MINED for $G$, we can construct in polynomial time a feasible solution $S$ of MINVD for $H$ satisfying $|S| \leq |R|$.

2. For a given feasible solution $S$ of MINVD for $H$, we can construct in polynomial time a feasible solution $R$ of MINED for $G$ satisfying $|R| \leq |S|$.

**Proof.** Let $R$ be an edge set, such that $G - R$ is planar. We show that $R$ can be easily converted into a set $S$ of vertices from $H$ whose removal makes $H$ planar. It is enough to include in $S$ one of the endpoints of each edge in $R$. Note that $|S| \leq |R|$. To argue that $H - S$ is planar, it is enough to observe that, if $H - S$ is not planar, then the graph $H'$ obtained from $H - S$ by contracting the remaining edges in the inserted triangles is also not planar. But this graph is exactly $G - R$.

Given a set $S$ of vertices of $H$ such that $H - S$ is planar, let $R$ be the set of edges of $G$ that are edges of $H$ (not in the triangles) incident to some vertex in $S$. Note that there is one such edge for each vertex in $S$, by the construction of $H$. Thus $|R| \leq |S|$. Now, $G - R$ is planar because it is the result of contracting all edges remaining in $H - S$ in the inserted triangles. As $H - S$ is planar, $G - R$ has to be planar as well. □

**Corollary 8.** Given a cubic graph $G$ and $f(G) = H$ the cubic graph obtained by algorithm $f$ from $G$, let $Opt_{MINED}(G)$ and $Opt_{MINVD}(H)$ denote, respectively, the optimum values of MINED for $G$ and of MINVD for $H$. Then, $Opt_{MINED}(G) = Opt_{MINVD}(H)$.

![Fig. 1. Example for application of algorithm $f$ in a planar graph (a) and in a nonplanar graph (b).](image-url)
Proof. The first part of Theorem 7 says that $\text{Opt}_{\text{MINED}}(G) \geq \text{Opt}_{\text{MINVD}}(H)$ and the second part of Theorem 7 says that $\text{Opt}_{\text{MINVD}}(H) \geq \text{Opt}_{\text{MINED}}(G)$. □

5. Vertex deletion is NP-complete for cubic graphs

In this section, we use the NP-complete decision problem $\text{ED}$ restricted to cubic graphs [7] to show that the decision problem $\text{VD}$ is NP-complete even when restricted for cubic graphs.

Corollary 9. $\text{VD}$ restricted to cubic graphs is NP-complete.

Proof. Let a cubic graph $G$ and a nonnegative integer $k$ be an instance for the NP-complete problem $\text{ED}$ to cubic graphs. Let $f(G) = H$ be the cubic graph obtained from $G$ by algorithm $f$. Let $H, k$ be the corresponding instance for $\text{VD}$. By Corollary 8, $\text{Opt}_{\text{MINED}}(G) \leq k$ if and only if $\text{Opt}_{\text{MINVD}}(H) \leq k$. □

6. Vertex deletion is Max SNP-hard for cubic graphs

In order to establish that $\text{MINVD}$ is Max SNP-hard even for cubic graphs, we use the concept of $L$-reductions of Papadimitriou and Yannakakis [12], a special kind of reduction that preserves approximability. Let $A$ and $B$ be two optimization problems. We say that $A$ $L$-reduces to $B$ if there are two polynomial-time algorithms $f$ and $g$ and positive constants $\alpha$ and $\beta$, such that for each instance $I$ of $A$,

(1) Algorithm $f$ produces an instance $I' = f(I)$ of $B$ such that the optima of $I$ and $I'$, satisfy $\text{Opt}_B(I') \leq \alpha \cdot \text{Opt}_A(I)$;
(2) Given any feasible solution of $I'$ with cost $c'$, algorithm $g$ produces a solution of $I$ with cost $c$ such that $|c - \text{Opt}_A(I)| \leq \beta |c' - \text{Opt}_B(I')|$. 

We constructed [6,8] an $L$-reduction from the Max SNP-complete problem [12] MAX3-SAT with at most three occurrences of each literal to $\text{MINED}$ restricted to cubic graphs, establishing that $\text{MINED}$ is Max SNP-hard even for cubic graphs. In this section, we prove that $\text{MINVD}$ is Max SNP-hard even for cubic graphs. For, we $L$-reduce the Max SNP-hard problem $\text{MINED}$ restricted to cubic graphs to $\text{MINVD}$ restricted to cubic graphs.

Theorem 10. $\text{MINVD}$ restricted to cubic graphs is Max-SNP-hard.

Proof. We $L$-reduce $\text{MINED}$ restricted to cubic graphs to $\text{MINVD}$ restricted to cubic graphs. We start by setting algorithm $f$ defined in Section 4 to be the algorithm $f$ of the $L$-reduction. By Corollary 8, we can set $\alpha = 1$. Given a feasible solution $S$ of $\text{VD}$ for $H$, Theorem 7 says that a feasible solution $R$ of $\text{ED}$ for $G$ can be produced from $S$ in polynomial time in the size of $G$, such that, $|R| \leq |S|$. Hence, $|R| - \text{Opt}_{\text{MINED}}(G) \leq |S| - \text{Opt}_{\text{MINED}}(G) = |S| - \text{Opt}_{\text{MINVD}}(H)$, showing that $\beta = 1$ suffices. □

7. Approximation algorithms for MAXPIS in general graphs

In 1992, Arora and Safra [3] proved that unless $P = NP$, MAXIMUM INDEPENDENT SET does not allow a polynomial-time approximation algorithm with a fixed ratio. We use this result to prove that unless $P = NP$, MAXPIS does not allow a polynomial-time approximation algorithm with a fixed ratio either.

Theorem 11. If there exists a polynomial-time approximation algorithm with a fixed ratio for MAXPIS, then $P = NP$.

Proof. Suppose that $A$ is a polynomial-time $\varepsilon$-approximation algorithm for MAXPIS with $0 < \varepsilon < 1$. Let $G = (V, E)$ be a graph and $H$ the planar induced subgraph of $G$ obtained from $A$. We have that, $|V(H)|/\text{Opt}_{\text{MAXPIS}}(G) \geq \varepsilon$. As $H$ is a planar graph, we know from the result due to Chiba et al. [5] that there is an algorithm $B$ which runs in $O(n \log n)$ time and defines an independent set $I$ of $H$ such that $|I|/\alpha(H) \geq 1/4$, where $\alpha(H)$ is the size of a maximum independent set of $H$. Since Appel and Haken [1] proved that $H$ is 4-colourable we have that $\alpha(H) \geq |V(H)|/4$. So, $8|I| \geq |V(H)| \geq \varepsilon \text{Opt}_{\text{MAXPIS}}(G)$. As every independent set of $G$ is also a planar induced subgraph of $G$, it is valid that
Opt_{MAXPIS}(G) \geq \omega(G)$. Hence, $|I|/\omega(G) \geq |I|/Opt_{MAXPIS}(G) \geq \epsilon/8$. Observe that as $I$ is an independent set of $H$ and $H$ is an induced subgraph of $G$, then $I$ is also an independent set of $G$. Thus, the composition of the algorithm $B$ with algorithm $A$ defines a polynomial-time $\epsilon/8$-approximation algorithm for MAXIMUM INDEPENDENT SET, which together with Arora and Safra’s [3] result show that $P = NP$. □

8. An approximation algorithm for finding maximum planar induced subgraphs in maximum degree 3 graphs

We propose a simple greedy $\frac{3}{4}$-approximation algorithm for MAXPIS in a maximum degree 3 graph. Let $x$ be a vertex of degree 2 adjacent to vertices $a$ and $b$ in a graph $G$. We smoothen $x$, when we remove $x$ from $G$ and we add edge $ab$. Note that the obtained graph may have multiple edges. When we smoothen a degree 2 vertex, we preserve the property of being planar.

Algorithm: planar induced subgraph
Input: Connected graph $G = (V, E)$ with maximum degree 3
Output: Subset $X$ of $V$, such that $G[X]$ is a planar graph

1. $X \leftarrow V$
2. $i \leftarrow 0$
3. $G_i \leftarrow G$
4. While $G_i$ is nonplanar do
   (a) While $G_i$ is not a cubic simple graph update $G_i$ by
      Removing vertices of degree 1 from $G_i$
      Smoothening vertices of degree 2 of $G_i$
      Replacing multiple edges by ordinary edges in $G_i$
      Removing the loops from $G_i$
   (b) Select a noncut vertex $u_i$ of $G_i$; $N_{G_i}(u_i) = \{a_i, b_i, c_i\}$
   (c) $X \leftarrow X\{u_i\}$
   (d) $G_{i+1} \leftarrow G_i - u_i$
   (e) $i \leftarrow i + 1$
5. Return $X$

8.1. An example of application

For the convenience of the reader we run the proposed algorithm through an example. In this example we have as input the graph $G = (V, E)$ in Fig. 2(a) defined from graph $K_{3,3}$ in Fig. 2(b) by replacing each vertex of $K_{3,3}$ by a subdivision of $K_{3,3}$.

Observe that the optimum of vertex deletion for $G = (V, E)$ must be greater than or equal to 6, because there are 6 vertex-disjoint subdivisions of $K_{3,3}$ as subgraphs of $G$. In Fig. 3(a) we select an optimum solution $S \subset V$, with $|S| = 6$, for vertex deletion in $G$ and in Fig. 3(b) we exhibit the corresponding planar subgraph obtained by the removal of $S$ from $G$. So the existence of $S$ proves that $Opt_{MINVD}(G) = 6$ and, since $|V| = 54$, that $Opt_{MAXPIS}(G) = 48$.

Now note that our algorithm can output a solution of vertex deletion for $G$ with cost no greater than 13, because for each vertex removed in subsequent steps (4)(b) of the algorithm, three additional vertices are smoothened. Since $G$ has 54 vertices, the removal of a set with 14 or more vertices would require at least $4 \times 14 = 56$ vertices in $V$, which is more than the number of vertices of $G$. In Fig. 4 we show how the algorithm may select two vertices in step (4)(b) from a subdivision of $K_{3,3}$ of $G$ producing a vertex of degree 3.

Hence, the removal of the corresponding sets with two vertices from each one of the six subdivisions of $K_{3,3}$, subgraphs of $G$, yields the $K_{3,3}$ in Fig. 5(a) as the remaining graph from $G$. This $K_{3,3}$ requires additionally one more deletion in order to obtain a planar graph as the one in Fig. 5(b), which shows that our algorithm can find a feasible solution by removing a subset $S$ of $V$ with 13 vertices. And in this particular case the size for the output $X$ of the algorithm is $|X| = 41 = 54 - 13 > 54 - \frac{54}{4} = \frac{3}{4}54 = \frac{3}{4}|V|$. 


8.2. The performance analysis of the algorithm

**Theorem 12.** The performance ratio of algorithm planar induced subgraph is at least $\frac{3}{4}$.

**Proof.** By the condition at the while of line 4 of the algorithm, the set $X$ returned by the algorithm induces a planar subgraph.

Consider the vertex $u_i$ selected at iteration $i$ to be discarded from set $X$. Let $a_i, b_i, c_i$ be the neighbourhood of $u_i$ in $G_i$. Either $G_{i+1} = G_i - u_i$ is planar or $a_i, b_i, c_i$ will be smoothened at iteration $i + 1$. In either case, those three vertices will not be discarded from set $X$ in a further iteration $j$, with $j > i$. 

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Fig. 2. (a) Graph $G = (V, E)$ obtained from (b) $K_{3,3}$.

Fig. 3. An optimum solution of vertex deletion for $G = (V, E)$.

Fig. 4. Two vertices removed from one of the six disjoint subdivisions of $K_{3,3}$ in $G$—(a) removing a vertex $u$, (b) and (c) smoothening the three neighbours of $u$, (d) removing a vertex $v$ and (e) and (f) smoothening the three neighbours of $v$. 

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Fig. 5. (a) $K_{3,3}$ obtained from $G = (V, E)$ by the removal of a set of vertices $X \subseteq V$ and (b) a planar graph obtained by removing one vertex from $K_{3,3}$.

Since each vertex $u_i \in V \setminus X$ corresponds to a distinct set of three vertices $a_i, b_i, c_i \in X$, we have that $|X| \geq 3n/4$, where $n = |V(G)|$. Thus, $|X|/\text{Opt}_{\text{MAXPIS}}(G) \geq |X|/n \geq 3n/4/n = 3/4$. □

Theorem 12 has as consequence a lower bound for the size of a maximum planar induced subgraph in a maximum degree 3 graph.

Corollary 13. Every graph $G = (V, E)$ with $|V| = n$ and maximum degree 3 satisfies $\text{Opt}_{\text{MAXPIS}}(G) \geq 3n/4$.

9. Final remarks and open problems

With respect to MAXPS for cubic graphs, any spanning tree plus one edge gives a planar subgraph with $2/3$-approximation bound, which is better than the performance ratio of $4/9$ obtained in [4], the best known result for graphs in general. In spite of being naive, the best known algorithm to MAXPS for cubic graphs is the algorithm for finding spanning trees.

In the present paper, we propose a polynomial-time $3/4$-approximation algorithm for MAXPIS in a maximum degree 3 graph. The worst case of the proposed algorithm is reached when the selected vertex set $X$ has size $|X| = 3n/4$. Note that the vertices discarded by the algorithm from $V$ define an independent set. Given a cubic graph $G = (V, E)$, one can define a subgraph $H$ of $G$, a feasible solution to MAXPS for $G$, by adding to $G[X]$, the subgraph of $G$ induced by $X$, each one of the discarded $n/4$ vertices plus one incident edge to each. The number of edges of the subgraph $H$ is the number of edges of $G$ minus twice the number of vertices in $V \setminus X$ which is $3n/2 - 2n/4 = n$. Hence, our algorithm can be modified to produce a solution to MAXPS for cubic graphs, such that the worst performance of our algorithm is not worse than the best existing algorithm for MAXPS in cubic graphs.

Although one could consider that knowing in advance the maximum degree of a graph would facilitate the finding of its associated maximum planar induced subgraph, the first contribution of this work is showing that both $\text{VD}$ and $\text{PIS}$ are NP-complete even for cubic graphs. Moreover, it is proved that $\text{MINVD}$ is Max SNP-hard for cubic graphs and left as an open problem whether $\text{MAXPIS}$ is Max SNP-hard for cubic graphs.

This paper has also introduced a $3/4$-approximation algorithm for MAXPIS in graphs with maximum degree 3. It is a property of the Max SNP-hard class that there are no polynomial time approximation schemes (PTASs) for problems in this class unless $P = NP$ [2,12]. This property means that for some $0 < \varepsilon_0$, there exists a polynomial-time $(1 + \varepsilon_0)$-approximation algorithm, and that the existence of a polynomial-time $(1 + \varepsilon)$-approximation for all $0 < \varepsilon < \varepsilon_0$ would imply $P = NP$. Hence, although the existence of a PTAS to MAXPS for graphs with maximum degree 3 is plausible, there was no approximation stated in the literature for this.

The analysis of the proposed algorithm considers an input $G = (V, E)$ with maximum degree 3, an output $X \subseteq V$ and the inequality $|X|/\text{Opt}_{\text{MAXPIS}}(G) \geq |X|/|V(G)|$. The analysis is concluded by evaluating a lower bound for the right-hand side, which is $3/4$. Observe that the size $|V(G)|$ is used as an upper bound to the value $\text{Opt}_{\text{MAXPIS}}(G)$. In Fig. 6 we show two graphs: $P_1$ and $P_2$, where $P_1$ is the Petersen graph and $P_2$ is the graph obtained from Petersen graph with 4 additional vertices. Both graphs are vertex-transitive, i.e., each one of them has the property that once
fixed a pair of vertices $u$ and $v$ in the graph there is an automorphism such that the image of $u$ is $v$. The fact that both are vertex transitive immediately yields the ratios $\frac{\text{Opt}_{\text{MAXPIS}}(P_1)}{|V(P_1)|} = \frac{4}{5}$ and $\frac{\text{Opt}_{\text{MAXPIS}}(P_2)}{|V(P_2)|} = \frac{11}{14}$, respectively, because $vd(P_1) = 2$ and $vd(P_2) = 3$. Note that $\frac{2}{3} < \frac{11}{14} < \frac{4}{5}$. Hence, an analysis of an algorithm to MAXPIS for cubic graphs which uses the size $|V(G)|$ as an upper bound for $\text{Opt}_{\text{MAXPIS}}(G)$ must have a fixed ratio no greater than $\frac{11}{14}$. Therefore, a natural problem left open is whether there is a family of cubic graphs $P_n, n \in \mathbb{N}$, such that the ratio $\frac{\text{Opt}_{\text{MAXPIS}}(P_n)}{|V(P_n)|}$ is asymptotically close to $\frac{3}{4}$.

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