An Intersection-Movement-Based Dynamic User Optimal Route Choice Problem

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In this paper a novel variational inequality (VI) formulation of the dynamic user optimal (DUO) route choice problem is proposed using the concept of approach proportion. An approach proportion represents the proportion of travelers that select a turning or through movement when leaving a node. Approach proportions contain travelers’ route information so that the realistic effects of physical queues can be captured in a formulation when a physical-queue traffic flow model is adopted, and so that route enumeration and path-set generation can be avoided in the solution procedure. In addition, the simple structure of the approach proportion representation allows us to decompose the constraint set for solving large-scale DUO route choice problems. This paper also discusses the existence and uniqueness of the solutions to the VI problem and develops a solution algorithm based on the extragradient method to solve the proposed VI problem. This solution algorithm makes use of the decomposition property of the constraint set and is convergent if the travel time functions are pseudomonotone and Lipschitz continuous. It is not necessary to know the Lipschitz constant of the travel time functions in advance. Finally, numerical examples are given to demonstrate the properties of the proposed model and the performance of the solution algorithm.

Subject classifications: dynamic traffic assignment; dynamic user optimal; approach proportion; variational inequality; extragradient method.

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DTA problems can be analytically formulated as both path-based models (e.g., Friesz et al. 1993; Huang and Lam 2002; Lo and Szeto 2002; Bliemer and Bovy 2003; Szeto and Lo 2004, 2006; Perakis and Roels 2006; Szeto et al. 2011) and link-based models (e.g., Carey 1987, Ran and Boyce 1996, Chen and Hsueh 1998, Wie et al. 2002, Bliemer and Bovy 2003, Ban et al. 2008). Path-based models can obtain path-related information such as path flows, which are the inputs of dynamic network loading (DNL) models. DNL models are used to model traffic flow propagation and determine travel time. Realistic traffic flow models can be used for DNL so that traffic dynamics, such as queue spillback and junction blockage, and the dynamic version of the user equilibrium (UE) principle can be captured in an analytical framework at the same time. Path-based models have the advantage that stochastic assignment using logit-type models can be applied easily, ensuring much faster convergence to an equilibrium than a deterministic approach. However, an important feature of path-based models is that the path set has to be explicitly defined and can be very large for even a medium-sized network. Hence, for large-scale network applications, path enumeration has not been used to obtain the path set. Instead, path-set generation has been used in these applications, which have generated paths when needed. For example, the INtegrated DYNAMIC traffic assignment model (INDY) (e.g., Bliemer et al. 2004) and the DTA model Streamline (e.g., Raadsen et al. 2009, 2010) use path-set generation on large networks (with more than 1,000 zones) with up to one million routes, and can be handled by a regular personal computer. Some smarter ways of using route sets, such as the concept of subpaths, were also developed to improve the efficiency of using computing resources (e.g., Chabini 2001, Raadsen et al. 2009, 2010).

Link-based models can avoid path enumeration in the solution procedure, and hence can be applied to large-scale networks. The outcomes of link-based DTA models are time-dependent link flows that do not contain path-related information. To arrive at a solution, a link-based DNL model is required to depict traffic flow propagation in the DTA models and obtain time-dependent link travel times. However, link-based DNL models cannot capture certain realistic traffic dynamics such as dynamic traffic interactions across multiple links.

This paper proposes a new intersection-movement-based DTA model using the concept of approach proportion. An approach proportion is defined as the proportion of travelers that select a turning or through movement when they leave a node. This concept is very similar to that used in Bar-Gera’s (2002) origin-based static traffic assignment problem. In our model, the traffic flow follows the dynamic extension of the UE principle, referred to as the dynamic user optimal (DUO) route choice principle (Ran and Boyce 1996), and the decision variables are time-dependent approach proportions rather than link or path flows. However, the relationship between approach proportions and link or path flows is defined in the DNL model. Using the concept of approach proportion, the DTA problem can be formulated as an intersection-movement-based model, an extension of the link-based model that can retain the advantages of both the link- and path-based models. First, path enumeration and path-set generation can be avoided in the solution procedure. Second, the realistic effects of physical queues can be captured when a physical queue DNL model is adopted, as the approach proportions contain the travelers’ path information.

In this paper, we provide the DUO conditions in terms of intersection-movement-based link flows (i.e., link flows are disaggregated according to turning and through movements at the downstream intersection of the link in addition to the destination). The concept of approach proportion and the VI approach are used to formulate the intersection-movement-based DTA problem in both the continuous-time space and the discrete-time space, in which the traffic flows are assumed to follow the DUO principle. The proposed VI model is general in the sense that it can incorporate any types of DNL models, including those that rely on travel time functions and hydrodynamic models, to depict traffic flow propagation inside the traffic network and determine travel times. An analysis of solution existence for the proposed model is also given. The analysis shows that although the existence of solutions to the proposed intersection-movement-based DTA problem with point queues is guaranteed, it may not be guaranteed in the case of queue spillback.

The proposed DTA model can be solved by any general computational techniques developed for VI problems, such as the method of successive averages (e.g., Tong and Wong 2000), the day-to-day swapping method (e.g., Huang and Lam 2002, Szeto and Lo 2006, Mounce and Carey 2011), the projection method (e.g., Lo and Szeto 2002, Szeto and Lo 2004), and so on, provided that the convergent requirements are satisfied. In this paper, the intersection-movement-based DUO problem is formulated in terms of approach proportions, which leads to a decomposable structure of the constraint set. The special structure of the constraint set can allow us to apply decomposition techniques to solve large-scale DUO problems in discrete-time space. This paper extends Khobotov’s (1987) extragradient method to solve the proposed VI problem. This method has several merits. It is convergent if the travel time functions are pseudomonotone and Lipschitz continuous. It is not necessary to know the Lipschitz constant of the travel time functions, and the decomposable structure is used to perform the projection efficiently.

The rest of this paper is organized as follows. In the next section, an intersection-movement-based DUO problem under the continuous time setting is formulated in terms of approach proportion. In §3, the intersection-movement-based DUO problem is reformulated in the discrete-time setting, and the existence and uniqueness of the solution to the DUO problem are discussed. In §4, an extragradient
method is developed to solve the discretized DTA model. Numerical examples are given in §5, and conclusions are provided in §6.

2. Continuous-Time Intersection-Movement-Based DUO Problem

2.1. Notations

We consider a multidestination network $G(N, A)$, where $N$ denotes the set of nodes and $A$ denotes the set of links. Link $a = (l_a, h_a)$ is the link with the tail node $l_a$ and the head node $h_a$, $a \in A$; $A(i)$ is the set of links leaving node $i$, and $B(i)$ is the set of links heading to node $i$; $R$ and $S$ denote the sets of origin nodes (with positive traffic demand) and destination nodes, respectively. Although all of the nodes in the network can be viewed as origins, some of them may not have traffic demand. The network is assumed to be empty initially. The following notations are adopted to formulate the intersection-movement-based DUO problem:

- $d_a(t)$: traffic demand rate from origin $i$ to destination $s$ at time $t$;
- $u_a(t)$: initial inflow rate of link $a \in A(i)$ from origin $i$ to destination $s$ at time $t$;
- $u_a'(t)$: inflow rate of link $a$ at time $t$;
- $u_a'(l_a)$: inflow rate of link $a$ through link $b \in A(h_a)$ to destination $s$ at time $t$;
- $\mathbf{u}$: the vector of intersection-movement-based link inflow rates $\mathbf{u} = [u_a(t), u_a'(t)]$;
- $v_a(t)$: outflow rate of link $a$ at time $t$;
- $v_a'(t)$: outflow rate of link $a$ heading to destination $s$ at time $t$;
- $v_{ab}(t)$: outflow rate of link $a$ through link $b \in A(h_a)$ to destination $s$ at time $t$;
- $\tau_a(t)$: travel time for flows entering link $a$ at time $t$;
- $\mathbf{a}$: the vector of link travel time $\mathbf{a} = [\tau_a(t)]$;
- $\alpha_a^s(t)$: approach proportion of flows that depart from node $i$ at time $t$ and pass through link $a \in A(i)$ to destination $s$;
- $\alpha_{ab}^s(t)$: approach proportion of flows that enter link $a$ at time $t$ and pass through link $b \in A(h_a)$ to destination $s$;
- $\mathbf{a}$: the approach proportion vector $\mathbf{a} = [\alpha_a^s(t), \alpha_{ab}^s(t)]$;
- $\pi_u^s(t)$: minimum travel time for any traffic demand from node $i$ to destination $s$ departing at time $t$;
- $\pi_u^s(t)$: minimum travel time for any flows from node $i$ to destination $s$ entering link $a \in A(h_i)$ at time $t$;
- $\mathbf{\pi}_u(t)$: the approach travel time vector $\mathbf{\pi}_u(t) = [\pi_u^s(t), \pi_{ab}^s(t)]$; and
- $\pi_{u_{\min}}(t)$: the minimum approach travel time vector $\mathbf{\pi}_{u_{\min}}(t) = [\pi_{u_{\min}}^s(t), \pi_{u_{\min}}^s(t)]$.

By definition, $\pi_{u_{\min}}(t)$ and $\pi_{ab}^s(t)$, which are referred to as the approach travel times, can be calculated as follows:

$$\pi_{u_{\min}}(t) = \tau_a(t) + \pi_{u_{\min}}^s(t)$$

(1)

$$\pi_{ab}^s(t) = \tau_a(t) + \pi_{b}^s(t)$$

(2)

2.2. Link-Based DUO Conditions

The static UE principle can be extended to describe the optimality conditions of the DUO problem, which can be expressed as follows (Ran and Boyce 1996):

For each origin-destination (OD) pair at each instant of time, if the actual travel times experienced by travelers departing at the same time are equal and minimal, the dynamic traffic flow over the network is in a travel-time-based DUO state.

Both the path- and link-based approaches can be used to formulate DUO conditions. For comparison purposes, we only consider the DUO conditions for the link-based approach, given by

$$\pi_u(t) \begin{cases} = \pi_u^s(t), & \text{if } u_a'(t) > 0, \\ \geq \pi_u^s(t), & \text{if } u_a'(t) = 0. \end{cases}$$

(3)

The link-based DUO conditions (3) imply that if the flows between node $i$ and destination $s$ entering link $a$ at time $t$ are positive, i.e., $u_a'(t) > 0$, then their minimum travel time through link $a$ to destination $s$ equals the minimum travel time for any flows from node $i$ to destination $s$ departing at time $t$. However, if there is no flow between node $i$ and destination $s$, the corresponding minimum travel time from node $i$ through link $a$ to destination $s$ is at least equal to the minimum travel time for any flows from node $i$ to destination $s$ departing at time $t$.

2.3. Intersection-Movement-Based DUO Conditions

A route that is used by travelers from an origin to a destination consists of a sequence of turning or through movements. Moreover, a traveler’s departure from an origin can also be viewed as an intersection movement, as the origin can be considered an intersection and the traveler can select links for entering the network. Before arriving at a decision node (including origins but not destinations), travelers are assumed to choose an optimal intersection movement to minimize their individual travel times to their destinations. At equilibrium, the turning or through movement used at each instant of time for a destination has equal and minimal travel time, i.e., no travelers would be better off by unilaterally changing their decisions (i.e., by selecting alternative...
turning or through movements). The equilibrium conditions, referred to as the intersection-movement-based DUO conditions, can be mathematically expressed as follows:

\[
\pi_{ab}^i(t) = \begin{cases} 
\pi_{ab}^i(t), & \text{if } u_{ab}^i(t) > 0, \\
\geq \pi_{ab}^i(t), & \text{if } u_{ab}^i(t) = 0,
\end{cases}
\]

\[
\pi_{ab}^s(t) = \begin{cases} 
\pi_{ab}^s(t), & \text{if } u_{ab}^s(t) > 0, \\
\geq \pi_{ab}^s(t), & \text{if } u_{ab}^s(t) = 0.
\end{cases}
\]

The DUO conditions for the intersection-movement-based link flow pattern (4) imply that if the number of travelers departing from node \(i\) at time \(t\) and selecting link \(a \in A(i)\) to enter the network to destination \(s\) is positive, i.e., \(u_{ab}^i(t) > 0\), then the corresponding approach travel time is minimal. However, if no travelers depart from node \(i\) at time \(t\) and select link \(a\) to enter the network to destination \(s\), i.e., \(u_{ab}^i(t) = 0\), then the corresponding approach travel time is at least as long as the minimum travel time. Conditions (5) similarly imply that if the number of travelers entering link \(a\) at time \(t\) and passing through link \(b \in A(h_s)\) to destination \(s\) is positive, i.e., \(u_{ab}^c(t) > 0\), then the corresponding approach travel time is minimal. However, if no travelers enter link \(a\) at time \(t\) and pass through link \(b\) to destination \(s\), i.e., \(u_{ab}^d(t) = 0\), then the corresponding approach travel time is at least as long as the minimum travel time.

### 2.4. Requirements of Dynamic Network Loading Models

The DNL model depicts how traffic propagates inside a traffic network and hence governs network performance in terms of travel time. In general, link travel times can be either obtained by a link travel time function or determined through DNL models and the travel time extraction procedure depicted by Lo and Szeto (2002) and Long et al. (2011). In any case, there exists a unique mapping between link flows and travel times, denoted by \(\tau = \tau(u)\). Approach travel times are functions of link travel times, and hence there is a unique mapping between link flows and approach travel times, i.e., \(\pi = \pi(u)\).

DNL models must meet certain requirements to be used in DTA models. Those requirements can be classified into five categories (Ban et al. 2008): mass balance, link first-in-first-out (FIFO), flow conservation, flow propagation, and other requirements. The mass balance requirement defines the relationships among the link flow (i.e., the number of vehicles on the link), the inflow rate, and the outflow rate: the change in link flow at each time instant is equal to the difference between the inflow and outflow rates at that time. The link FIFO requirement ensures that vehicles that enter the link earlier leave it sooner. Most DUO-based DTA models tend to implicitly guarantee link FIFO using a proper link travel time “model” (not restricted to a link travel time function) that satisfies this property. If the link travel time model used for the DUO-based DTA model satisfies the FIFO requirement, then the solution to the DUO-based DTA problem also satisfies the route and OD FIFO requirements (Wu et al. 1998).

The flow conservation requirement ensures that all of the flows entering a node (except the destination node) together with the demand generated at that node must exit from that node. Different from the link-node complementarity model (Ban et al. 2008), the model in this paper is concerned with intersection movements, and link inflow equals the sum of the flows from upstream links plus the flow generated at the node. Thus, the corresponding flow conservation can be formulated as follows:

\[
u_{ab}^i(t) = \nu_{ab}^s(t) + \sum_{bc \in B(l_s)} v_{bc}^i(t), \quad \forall s, a, t.
\]

The flow conservation requirement also ensures that the demand rate of each OD pair at each instant of time is equal to the sum of the inflow rates at that time:

\[
\sum_{bc \in A(l_s)} \nu_{ab}^i(t) = d_{ab}^i(t), \quad \forall i, s, t.
\]

The flow propagation requirement ensures the consistent evolvement of traffic flows in both temporal and spatial domains, which in turn restricts the relationships between the link inflows and outflows and the time-dependent link travel times as follows:

\[
u_{ab}^i(t + \tau_{ab}(t)) = \frac{u_{ab}^s(t)}{1 + d_{ab}(t)/dt}, \quad \forall s, a, b, t.
\]

Following Ban et al. (2008), we use an “inverse link travel time function” \(p_{ab}(t) = \tau_{ab}(t - p_{ab}(t))\), which denotes the travel time of link \(a\) for vehicles exiting the link at time \(t\), to reformulate the link outflow rate in terms of the link inflow rate:

\[
u_{ab}^s(t) = \nu_{ab}^s(t - p_{ab}(t))(1 - dp_{ab}(t)/dt), \quad \forall s, a, b, t.
\]

By definition, we have

\[
u_{ab}^i(t) = \sum_{bc \in A(h_s)} \nu_{ab}^i(t), \quad \forall a, s, t
\]

and thus

\[
\sum_{bc \in A(h_s)} \nu_{ab}^i(t) = \nu_{ab}^i(t) + \sum_{bc \in B(l_s)} u_{bc}^i(t - p_{bc}(t))(1 - dp_{bc}(t)/dt), \quad \forall s, a, t.
\]

The other requirements are used to describe the relationship between the aggregated and disaggregated variables in DTA models and also some of the nonnegativity requirements. In our model, intersection movement-based inflow must be nonnegative at any time:

\[
u_{ab}^i(t) \geq 0, \quad \forall i, s, a, t, \quad \text{and} \quad u_{ab}^i(t) \geq 0, \quad \forall s, a, b, t.
\]
2.5. An Intersection-Movement-Based Dynamic Route Choice Model

DUO conditions (4) and (5) for the intersection-movement-based link flow pattern can be defined alternatively using the concept of approach proportion. The DUO conditions for the approach proportions can be mathematically expressed as follows:

\[
\pi^{i}\alpha_{a}(t) = \pi^{i}(t), \quad \text{if } \alpha^{i}_{a}(t) > 0, \quad \text{and} \quad \pi^{i}_{a}(t) \geq \pi^{i}(t), \quad \text{if } \alpha^{i}_{a}(t) = 0, \quad (14)
\]

\[
\pi^{i}\alpha_{ab}(t) = \pi^{i}_{ab}(t), \quad \text{if } \alpha^{i}_{ab}(t) > 0, \quad \text{and} \quad \pi^{i}_{ab}(t) \geq \pi^{i}_{ab}(t), \quad \text{if } \alpha^{i}_{ab}(t) = 0. \quad (15)
\]

Given the demand at each node and the aggregated inflows to each link, the approach proportions can be used to determine the amount of turning or through flows at each node:

\[
u^{i}_{a}(t) = \alpha^{i}_{a}(t)d^{i}(t), \quad \forall i, s, a, t, \quad \text{and} \quad (16)
\]

\[
u^{i}_{ab}(t) = \alpha^{i}_{ab}(t)u^{i}_{b}(t), \quad \forall s, a, b, t. \quad (17)
\]

The link travel time functions or DNL models can then be used to determine the link and approach travel times.

**Proposition 1.** If the network is empty initially and the road traffic flow satisfies causality, then the link inflow vector \( \mathbf{u} \) that satisfies (6)–(13) is a function of the approach proportion vector \( \alpha \), and the function \( \mathbf{u}(\alpha) \) is surjection. In other words, \( \mathbf{u} \) can be unilaterally determined by the approach proportion vector \( \alpha \).

The proof is given in Appendix EC.1 (available as supplemental material at http://dx.doi.org/10.1287/opre.2013.1202).

According to Proposition 1, link flows are functions of approach proportions. Because approach travel times are functions of link flows, they are also functions of approach proportions, i.e., \( \mathbf{u} = \mathbf{u}(\pi) \).

The following assumption is used to discuss the continuity of the link inflow vector function with respect to the approach proportion vector.

**Assumption 1.** The road traffic flow satisfies the link FIFO and causality requirements, and both the cumulative outflows and link travel times are continuous with respect to the link inflow vector.

**Proposition 2.** Under Assumption 1, the link inflow vector \( \mathbf{u} \) is continuous with respect to the approach proportion vector \( \alpha \).

The proof is given in Appendix EC.2.

Combining Equations (7) and (10) with Equations (16) and (17), we have

\[
\sum_{a \in A(i)} \alpha^{i}_{a}(t) = 1, \quad \forall i, s, a, t, \quad \text{and} \quad (18)
\]

\[
\sum_{b \in A(h)} \alpha^{i}_{ab}(t) = 1, \quad \forall s, a, t. \quad (19)
\]

By definition, the approach proportions must be nonnegative at all times:

\[
\alpha^{i}_{a}(t) \geq 0, \quad \forall i, s, a, t, \quad \text{and} \quad (20)
\]

\[
\alpha^{i}_{ab}(t) \geq 0, \quad \forall s, a, b, t. \quad (21)
\]

We can rewrite DUO conditions (14) and (15) for the approach proportions as an NCP, given by

\[
\begin{align*}
\left( \pi^{i}_{a}(t) - \pi^{i}(t) \right)\alpha^{i}_{a}(t) &= 0, \quad \forall i, s, a, t, \\
\left( \pi^{i}_{ab}(t) - \pi^{i}_{ab}(t) \right)\alpha^{i}_{ab}(t) &= 0, \quad \forall s, a, b, t, \\
\pi^{i}_{a}(t) - \pi^{i}(t) &\geq 0, \quad \forall i, s, a, t, \\
\pi^{i}_{ab}(t) - \pi^{i}_{ab}(t) &\geq 0, \quad \forall s, a, b, t, \\
\alpha^{i}_{a}(t) &\geq 0, \quad \forall i, s, a, t, \\
\alpha^{i}_{ab}(t) &\geq 0, \quad \forall s, a, b, t. \quad (22)
\end{align*}
\]

The NCP (22) can be reformulated as a VI problem, which may be stated as follows.

**Theorem 1.** The dynamic approach proportion vector \( \alpha = [\alpha^{i}_{a}(t), \alpha^{i}_{ab}(t)] \) is in a route choice state in DUO conditions (14) and (15) (or (22)) if and only if the vector satisfies the following VI problem, which involves finding a vector \( \alpha^{*} \) that satisfies constraints (18)–(21) such that for all \( \alpha \) satisfying constraints (18)–(21).

\[
\begin{align*}
\int_{0}^{T} \sum_{i} \sum_{s \in A(i)} \sum_{a \in A(i)} \pi^{i}_{a}(t)\left[ \alpha^{i}_{a}(t) - \alpha^{*}_{a}(t) \right] dt \\
+ \int_{0}^{T} \sum_{s} \sum_{t} \sum_{b \in A(b)} \pi^{i}_{ab}(t)\left[ \alpha^{i}_{ab}(t) - \alpha^{*}_{ab}(t) \right] dt &\geq 0. \quad (23)
\end{align*}
\]

The proof is given in Appendix EC.3.

**Proposition 3.** If \( \alpha^{*} \) is a solution to the VI problem (23), then the corresponding link inflow vector \( \mathbf{u}^{*} = \mathbf{u}(\alpha^{*}) \) also satisfies DUO conditions (4) and (5).

**Proof.** According to Theorem 1, the VI problem (23) is equivalent to DUO conditions (14) and (15). Hence, we have \( \alpha^{i}_{a}(t) = 0 \) if \( \pi^{i}_{a}(t) > \pi^{i}_{a}(t) \) and \( \alpha^{i}_{ab}(t) = 0 \) if \( \pi^{i}_{ab}(t) > \pi^{i}_{ab}(t) \). By combining these DUO conditions with Equations (16) and (17), we have \( u^{i}_{a}(t) = \alpha^{i}_{a}(t) \cdot d^{i}(t) = 0 \) if \( \pi^{i}_{a}(t) > \pi^{i}_{a}(t) \) and \( u^{i}_{ab}(t) = \alpha^{i}_{ab}(t)u^{i}_{b}(t) = 0 \) if \( \pi^{i}_{ab}(t) > \pi^{i}_{ab}(t) \), respectively, which can be further simplified as DUO conditions (4) and (5) (i.e., \( u^{i}_{a}(t) = 0 \) if \( \pi^{i}_{a}(t) > \pi^{i}_{a}(t) \) and \( u^{i}_{ab}(t) = 0 \) if \( \pi^{i}_{ab}(t) > \pi^{i}_{ab}(t) \)). This completes the proof.

**Proposition 4.** If \( \mathbf{u}^{*} = [u^{i}_{a}(t), u^{i}_{ab}(t)] \) satisfies intersection-movement-based DUO conditions (4) and (5), then the corresponding link inflow pattern \( \mathbf{u}^{*}(t) \) satisfies link-based DUO conditions (3).

The proof is given in Appendix EC.4.

**Proposition 5.** If \( \alpha^{*} \) is a solution to the VI problem (23), then the corresponding link inflow pattern \( \mathbf{u}^{*}(t) \) satisfies DUO conditions (3).

**Proof.** This proposition follows directly from Propositions 3 and 4. \( \square \)
3. Discrete-Time DUO Problem

This section reformulates the problem in discrete-time space to solve it numerically or by using numerical methods that involve discretizing time. We discretize the time period $T$ of interest into a finite set of time intervals, $K = \{ k : 1, 2, \ldots, K \}$. Let $\delta$ be the interval length such that $\delta K = T$. The notations defined in §2.1 related to time $t$ are equivalently redefined in the discrete-time setting by changing time instant $t$ into time interval $k$.

3.1. Discretized Model

In terms of approach proportions, the continuous-time DTA model can be reformulated into a discrete-time version: finding a vector $\alpha^* \in \Omega$ such that for all $\alpha \in \Omega$,

$$\sum_{i} \sum_{a} \sum_{s} \pi_{ia}^{ws}(k) [\alpha_{ia}^{ws}(k) - \alpha_{ia}^{ws}(k)] + \sum_{k} \sum_{a} \sum_{s} \sum_{b \in A(h_a)} \pi_{ab}^{ts}(k) [\alpha_{ab}^{ts}(k) - \alpha_{ab}^{ts}(k)] \geq 0, \quad (24)$$

where $\Omega$ is a closed convex set

$$\Omega = \left\{ \alpha \geq 0 : \sum_{a \in A(i)} \alpha_{ai}(k) = 1, \forall i, s, k, \sum_{b \in A(h_a)} \alpha_{ab}(k) = 1, \forall s, a, k \right\}. \quad (25)$$

The feasible approach proportion set has a decomposable structure and can be reformulated as follows:

$$\Omega = \prod_{k} \prod_{i} \prod_{s} \Omega_{i}^{ws}(k) \times \prod_{k} \prod_{a} \prod_{s} \Omega_{a}^{ts}(k), \quad (26)$$

where

$$\Omega_{i}^{ws}(k) = \left\{ \alpha \in R_{+}^{[A(i)]} : \sum_{a \in A(i)} \alpha_{ai}(k) = 1 \right\}, \quad (27)$$

$$\Omega_{a}^{ts}(k) = \left\{ \alpha \in R_{+}^{[A(h_a)]} : \sum_{b \in A(h_a)} \alpha_{ab}(k) = 1 \right\}. \quad (28)$$

The DUO problem (24) can then be reformulated as the following VI problem, which determines a vector $\alpha^* \in \Omega$ such that for all $\alpha \in \Omega$,

$$\langle \pi(\alpha^*), \alpha - \alpha^* \rangle \geq 0, \quad (29)$$

where $\pi(\alpha)$ is both the approach travel time vector and a function of the approach proportion vector $\alpha$.

3.2. Discretized Travel Time Model

A great many DNL models can be used to depict how traffic propagates inside a traffic network along assigned routes (see Mun 2007 for a comprehensive review). They can also estimate link travel times via a well-defined link travel time “model” (e.g., Daganzo 1995, Carey and Ge 2005, Long et al. 2011), in which very often the travel time of a link is modeled as a function of the flow on that link. The proposed DTA model is formulated in a general form, and any type of DNL model can be used to estimate the traffic pattern over time. The link travel times are then calculated by link travel time models. With the link travel times obtained from a travel time model, the travel time $t_p(k)$ required to traverse a path $p = \{a_1, a_2, \ldots, a_n\}$ for vehicles using this path during interval $k$ can be computed using the following nested function (Ran and Boyce 1996):

$$t_p(k) = t_{a_1}(k) + t_{a_2}(k + t_{a_1}(k)) + \cdots + t_{a_n}(k + t_{a_{n-1}}(k)), \quad (29)$$

where $t_{a_1}(k), t_{a_2}(k) = t_{a_1}(k + t_{a_2}(k)), \ldots$, for short and $t_{a_n}(k)$ is the link travel time for flows entering at interval $k$.

**Proposition 6.** Under Assumption 1, link travel times are continuous with respect to the approach proportion vector $\alpha$.

**Proof.** Under Assumption 1, the link travel times are continuous with respect to the link inflow vector $u$. Proposition 3 shows that the link inflow vector $u$ is continuous with respect to the approach proportion vector $\alpha$, and therefore the link travel times are continuous with respect to the approach proportion vector $\alpha$. □

**Proposition 7.** Under Assumption 1, path travel times are continuous with respect to the approach proportion vector $\alpha$.

**Proof.** The nested function $t_p(k)$ in Equation (29) is continuous with respect to the link travel times (see Huang and Lam 2002, Szeto and Lo 2006). According to Proposition 6, the link travel times are continuous with respect to the approach proportion vector $\alpha$, and therefore the route travel times are continuous with respect to the approach proportion vector $\alpha$. □

**Proposition 8.** Under Assumption 1, the approach travel time vector $\pi(\alpha)$ is a continuous function of the approach proportion vector $\alpha$.

**Proof.** Under Assumption 1, the minimum travel time from node $i$ to destination $s$ during interval $k$, i.e., $\pi^{ws}(k)$, is continuous with respect to $\alpha$ because $\pi^{ws}(k)$ is the minimum travel time of all of the routes connecting $i$ and $s$ and all of the route travel times are continuous with respect to $\alpha$ (see Proposition 7). The approach travel time vector $\pi(\alpha)$ is defined by the nested functions in Equations (1) and (2), and is a continuous function with respect to the approach proportion vector $\alpha$ because both the link travel times and $\pi^{ws}(k)$ are continuous with respect to $\alpha$. This completes the proof.

The feasible approach proportion set $\Omega$ in Equation (25) is obviously a compact convex set, and Proposition 8 proves...
that \( \pi(\alpha) \) is continuous on \( \Omega \) under Assumption 1. Therefore, the existence of the solutions to the VI problem (28) can be guaranteed under Assumption 1. However, some discretized link travel time functions are only monotone in general (e.g., Long et al. 2011), and hence the approach travel time functions are not strictly monotone. Thus, the uniqueness of the solutions to the intersection-movement-based DTA problem may not be guaranteed.

4. Solution Algorithm

4.1. Gap Functions

Gap functions are developed to provide a convenient measure of convergence for a solution algorithm. In this paper, the following gap function is developed:

\[
G_1(\alpha) = \max \left\{ \frac{\delta^{iv}(k)(\pi^{iv}(k) - \pi^{iv}(k))}{\pi^{iv}(k)}, \frac{\delta^{iv}(k)(\pi^{iv}(k) - \pi^{iv}(k))}{\pi^{iv}(k)} \right\},
\]

where \( \delta^{iv}(k) = 1 \) if \( \alpha^{iv}(k) > 0 \); otherwise, \( \delta^{iv}(k) = 0 \). Further, \( \delta^{iv}(k) = 1 \) if \( \alpha^{iv}(k) > 0 \); otherwise, \( \delta^{iv}(k) = 0 \).

Equation (30) gives the maximum relative gap (i.e., the largest relative difference between the approach travel times of all of the approaches used and the corresponding minimum approach travel times). According to DUO conditions (14) and (15) for approach proportions, the gap function (30) is equal to zero if the DUO conditions for approach proportions hold. Otherwise, the gap is positive.

If the link-based DUO conditions and DUO conditions for intersection-movement-based link flows are satisfied, then the following two gap functions are equal to zero, respectively,

\[
G_2(\alpha) = \frac{\sum_k \sum_{a} u_a(k) (\pi^{iv}(k) - \pi^{iv}(k))}{\sum_k \sum_{a} u_a(k) t_a(k)}
\]

\[
G_3(\alpha) = \frac{(\pi^{iv}(\alpha) - \pi^{iv}(\alpha)) u(\alpha)}{\sum_k \sum_{a} u_a(k) t_a(k)}
\]

where the denominators in both Equations (31) and (32) represent the total system travel time, and are used to determine relative gaps of the solutions. These two gaps are greater than zero if the corresponding DUO conditions are not satisfied. These two gaps are used in the numerical study to demonstrate that if \( G_1(\alpha) \) is close to zero (i.e., the current solution is close to the DUO solution in terms of approach proportions), then \( G_k(\alpha) \) and \( G_g(\alpha) \) are also close to zero (i.e., the corresponding link flows and intersection-movement-based link flows in terms of approach proportions are close to their optimal solutions).

The following relative gap function is adopted to compare the proposed method with the path-based method:

\[
G_4(\alpha) = 1 - \frac{\sum_k \sum_{a} d^{iv}(k) (\pi^{iv}(k))}{\sum_k \sum_{a} u_a(k) t_a(k)}
\]

4.2. The Extragradient Method

The extragradient method for solving the VI problem (28) is outlined as follows:

**Step 0. Initialization.** Determine a feasible approach proportion vector \( \alpha_0 \). Set the parameters \( \beta, \xi \in (0, 1), \lambda > 0 \), the convergence tolerance \( \varepsilon > 0 \), \( \lambda_i = \lambda \), and iteration index \( \iota = 0 \).

**Step 1. Check the stopping criterion.** Terminates it if

\[
G_1(\alpha_\lambda) < \xi
\]

**Step 2. Update approach proportions.**

**Step 2.1. \( \alpha_\lambda \) computation.** Compute \( \alpha_\lambda = \text{Proj}_\Omega(\alpha_i - \lambda_i \pi(\alpha_i)) \).

**Step 2.2. Determination of the stepsize.** If \( \lambda_i > \beta(\|\alpha_i - \alpha_\lambda\|/\|\pi(\alpha_i) - \pi(\alpha_\lambda)\|) \), then decrease \( \lambda_i \) using

\[
\lambda_i = \min \left\{ \xi \lambda_i, \beta(\|\alpha_i - \alpha_\lambda\|/\|\pi(\alpha_i) - \pi(\alpha_\lambda)\|) \right\}
\]

and return to Step 2.1. Otherwise, go to Step 2.3.

**Step 2.3. Computation of \( \alpha_{i+1} \) and \( \lambda_{i+1} \).** Update the approach proportion vector by

\[
\alpha_{i+1} = \text{Proj}_\Omega(\alpha_i - \lambda_i \pi(\alpha_i)),
\]

set

\[
\lambda_{i+1} = \min \left\{ \lambda_i, \beta(\|\alpha_i - \alpha_\lambda\|/\|\pi(\alpha_i) - \pi(\alpha_\lambda)\|) \right\}, \quad \iota = \iota + 1,
\]

and return to Step 1.

In this study, an all-or-nothing assignment under the free flow condition was used to determine the feasible approach proportion vector \( \alpha_0 \) in Step 0. Hence, in this step, the approach proportions were the same across the study period. In Step 2.1, \( \text{Proj}_\Omega(\cdot) \) was the Euclidean projection map onto \( \Omega \). The notations \( \alpha_\lambda \) and \( \lambda_\iota \) were the approach proportion vector and the stepsize at iteration \( \iota \), respectively. The projection on \( \Omega \) in Step 2.1 was decomposed into many projections on simplices \( \Omega^{iv}(k) \) and \( \Omega^{iv}(k) \), each of which was performed efficiently by the linear projection method proposed by Panicucci et al. (2007). A reinitialization of the stepsize was carried out in Step 2.3 to avoid the problem of a convergence rate reduction because of the stepsizes being too small at some iterations.

This algorithm was proved to converge to an equilibrium flow under some mild assumptions on the approach travel time function. To improve its precision, we introduce the following definitions (Panicucci et al. 2007):

**Definition 1 (Lipschitz Continuous).** The function \( \pi(\alpha) \) is Lipschitz continuous on \( \Omega \) if there exists a positive constant \( L \) such that

\[
\|\pi(\alpha_1) - \pi(\alpha_2)\| \leq L \|\alpha_1 - \alpha_2\|, \quad \forall \alpha_1, \alpha_2 \in \Omega.
\]

**Definition 2 (Pseudomonotone).** The function \( \pi(\alpha) \) is pseudomonotone on \( \Omega \) if for all \( \alpha_1, \alpha_2 \in \Omega, \lambda_1, \lambda_2 \in \Omega, \)

\[
\langle \pi(\alpha_1), \alpha_1 - \alpha_2 \rangle, \langle \pi(\alpha_2), \alpha_2 - \alpha_1 \rangle \geq 0 \implies \langle \pi(\alpha_1), \alpha_1 - \alpha_2 \rangle \geq 0.
\]
Theorem 2. If \( \pi(\alpha) \) is pseudomonotone and Lipschitz continuous on \( \Omega \), then any cluster point of the sequence \( \{\alpha_i\}_{i \in \mathbb{N}} \) is a DUO flow.

The proof follows Khobotov (1987) and Panicucci et al. (2007).

The requirement of pseudomonotone is weaker than that of monotone. However, the mapping function of the VI problem (28) may not be pseudomonotone, and the convexity of the solution set may not hold.

5. Numerical Examples

This section presents the results of five numerical experiments to illustrate the properties of the proposed DUO problem and the performance of the solution algorithm. All of the experiments were run on a computer with an Intel (R) Core(TM) i7-2600 3.40 GHz CPU and 6.00 GB of RAM.

Example 1. The equivalency between link-based solutions and intersection-movement-based solutions.

The test network shown in Figure 1 was taken from Chen and Hsheh (1998), and has five nodes, six links, two origins (nodes 1 and 2), one destination (node 3), and two origin-destination (OD) pairs. The length of each time interval was set at 0.25 min (15 s). The traffic demands lasted for 120 intervals. The link travel time function and time-dependent demand rate function adopted in this example are given as follows (Ban et al. 2008):

\[
\tau_a(k) = r^a_0 [1 + \beta^x_a x_a(k + 1)] \quad \text{and} \quad d^e(k) = 40 + 120 \left(1 - \frac{2k}{K} \right)^2, \quad \forall 1 \leq k \leq K, \; i \in R, \; s \in S, \tag{34}
\]

where \( \beta^x_a > 0 \) is a constant. The free flow travel time \( r^a_0 \) (in minutes) and the parameter \( \beta^x_a \) are given in Figure 1.1 The values of the parameters for the solution algorithm are as follows: \( \varepsilon = 0.01, \; \beta = 0.5, \; \xi = 0.6, \) and \( \lambda = 5.0. \)

We applied the solution algorithm presented in §4.2 to solve the proposed DUO problem. The inflow rates of each link are shown in Figure 2. The link inflow rates are basically the same as those presented in Figure 6 in Ban et al. (2008). This implies that the solution of the proposed model in this paper agrees with the solution of the model developed by Ban et al. (2008).

Example 2. Discontinuity of approach travel time under the physical queue consideration.

In this example, a four-node test network (see Figure 3) was adopted to demonstrate that approach travel times may be discontinuous functions of approach proportions. The network consisted of five links and one OD pair. The number of lanes and the length (in meters) of each link are shown in Figure 3. Origin node 1 and link 1–3 had two downstream links (see Table 1), and other links had only one or no downstream link. We assumed all of the links to be empty initially. To capture the physical queue phenomenon, we adopted the link transmission model (LTM) (Yperman 2007) for DNL. Further, we adopted the discretized link travel time model proposed by Long et al. (2011) to calculate the link travel times. The input parameters of the LTM for all of the links were the same and are given as follows:

- Jam density: 133 veh/km (i.e., 7.5 m for every vehicle);
- Free flow speed: 54 km/h (i.e., 15 m/s);
- Backward shock wave speed: 18 km/h (i.e., 5 m/s);
- Flow capacity: 1,800 veh/h/lane (i.e., 0.5 veh/s/lane);
- Length of each time interval \( \delta \): 10 s.

The OD demands lasted for 20 intervals with a demand rate of nine veh/interval. A traffic signal was located at node 4, with a cycle time of 10 intervals and green time of five intervals for both links 1–4 and 3–4. The signal indicated green for link 1–4 and red for link 3–4 at the start

Figure 1. Test network for Example 1.

Figure 2. Link inflow rates in Example 1.

Figure 3. Test network for Examples 2 and 3.
of the modeling horizon. In this example, the following approach proportions were fixed:

\[
\begin{align*}
\alpha_1(k) &= \alpha_4(k) = 0.5, \quad \alpha_3(k) = 0.0, \\
\alpha_2(k) &= 1.0, \quad \text{if } k < 5, \\
\alpha_3(k) &= 0.0, \quad \alpha_4(k) = 1.0, \quad \text{if } k = 5, \quad \text{and} \\
\alpha_1(k) &= \alpha_2(k) = 0.5, \quad \alpha_3(k) = 1.0, \\
\alpha_4(k) &= 0.0, \quad \text{if } k > 5.
\end{align*}
\]

Figure 4 plots the approach travel time for the vehicles departing from origin 1 and traveling through link 1–3 (i.e., approach 1) for intervals 6–10 (i.e., \(\pi_1(6), \pi_1(7), \pi_1(8), \pi_1(9), \pi_1(10)\)) against the corresponding approach proportion of interval 5 (i.e., \(\alpha_1(5)\)). From this figure, we can clearly observe that the discontinuity occurs at \(\alpha_1(5) = 0.222\) and 0.778, respectively, because of the queue spillback. As the free flow travel times for the vehicles traveling through links 1–3 and 3–4 were four intervals and one interval, respectively, the free flow travel times for the vehicles departing from origin 1 through links 1–3 and 3–4 to node 4 were five intervals. Hence, all of the vehicles departing before the first five intervals had to stop on link 3–4 because of the red signal time from the sixth interval to the 10th interval. Link 3–4 held a maximum of 20 vehicles, and 18 vehicles on approach 4 entered the link after departing from origin 1 during the first four intervals. If more than two vehicles entered link 3–4, a queue spillback occurred and node 3 was blocked. This condition is equivalent to \(\alpha_1(5) > 0.222\) (i.e., 2/9). Both the inflow capacity and outflow capacity of link 3–4 were five veh/interval. If \(\alpha_1(5) > 0.778\) (i.e., 7/9), then more than 25 vehicles were using approach 4, and the vehicles on approach 3 had an additional delay of one time interval due to the queue spillback, resulting in an increase of one time interval on the travel time of this approach (see Figure 4).

Example 3. A test of solution quality and computation time using a four-node test network with physical queues.

In this example, the four-node test network in Example 2 was also adopted to demonstrate that the solutions of the proposed model with the physical queue consideration satisfy the DUO conditions. The values of the parameters for the solution algorithm were the same as those in Example 1. The input parameters for the LTM for each link and the signal setting were the same as those in Example 2. We solved the VI problem (28) using the proposed solution algorithm, and present the approach proportions and travel times of the solution in Table 2. The results show that the approaches with positive proportions had minimum approach travel times. This implies that the solution satisfies DUO conditions (14) and (15).

Table 3 illustrates the CPU time and the number of iterations required for different convergence criteria, and the corresponding values of the gap functions. We can observe that the values of all of the gap functions decrease if a smaller convergence tolerance is adopted. In particular, when the convergence tolerance approaches zero, both of the values of the gap functions \(G_3\) and \(G_1\) tend to approach zero. This observation is consistent with the results presented in Propositions 4 and 5 in the sense that if the DUO conditions for approach proportions are satisfied, then the link-based DUO conditions and the DUO conditions for intersection-movement-based link flows are also satisfied.

Example 4. A test of solution quality and computation time using the Nguyen and Dupuis network under point-and physical-queue considerations.

We adopted the Nguyen and Dupuis (1984) network (see Figure 5) to illustrate the solution quality of the proposed model and the performance of the proposed algorithm. The network had 13 nodes, 19 links, and four OD pairs. The length of each link is given in Table 4. The modeling horizon was set at 200 intervals, and all of the OD demands lasted for the first 30 intervals. Following Lo and Szeto (2002), the OD demand rates were five veh/interval for OD pair 1–2, 10 veh/interval for OD pair 1–3, and 7.5 veh/interval for OD pairs 4–2 and 4–3. The numbers of lanes were one for links 8–2, 12–8, and 13–3; two for links 7–8 and 9–13, and three for the other 14 links. The outflow capacities of the links heading to nodes 2, 3,
Table 2. Approach proportion and approach travel time under the DUO conditions in Example 3.

<table>
<thead>
<tr>
<th>Interval</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_4$</th>
<th>$\pi_1$</th>
<th>$\pi_2$</th>
<th>$\pi_3$</th>
<th>$\pi_4$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>—</td>
<td>1.000</td>
<td>1.000</td>
<td>—</td>
<td>8.000</td>
<td>5.850</td>
<td>8.000</td>
<td>9.500</td>
</tr>
<tr>
<td>2</td>
<td>0.500</td>
<td>0.950</td>
<td>1.000</td>
<td>—</td>
<td>8.000</td>
<td>8.000</td>
<td>8.000</td>
<td>8.500</td>
</tr>
<tr>
<td>3</td>
<td>1.000</td>
<td>—</td>
<td>0.750</td>
<td>0.250</td>
<td>8.175</td>
<td>12.765</td>
<td>8.175</td>
<td>8.175</td>
</tr>
<tr>
<td>4</td>
<td>1.000</td>
<td>—</td>
<td>0.750</td>
<td>0.250</td>
<td>8.525</td>
<td>11.765</td>
<td>8.525</td>
<td>8.525</td>
</tr>
<tr>
<td>5</td>
<td>1.000</td>
<td>—</td>
<td>0.811</td>
<td>0.189</td>
<td>8.875</td>
<td>10.765</td>
<td>8.875</td>
<td>8.875</td>
</tr>
<tr>
<td>6</td>
<td>1.000</td>
<td>—</td>
<td>0.811</td>
<td>0.189</td>
<td>9.291</td>
<td>9.765</td>
<td>9.291</td>
<td>9.291</td>
</tr>
<tr>
<td>7</td>
<td>0.485</td>
<td>0.515</td>
<td>1.000</td>
<td>—</td>
<td>9.461</td>
<td>9.461</td>
<td>9.461</td>
<td>11.296</td>
</tr>
<tr>
<td>8</td>
<td>0.722</td>
<td>0.278</td>
<td>1.000</td>
<td>—</td>
<td>9.532</td>
<td>9.532</td>
<td>9.532</td>
<td>12.568</td>
</tr>
<tr>
<td>9</td>
<td>0.478</td>
<td>0.522</td>
<td>1.000</td>
<td>—</td>
<td>9.612</td>
<td>9.612</td>
<td>9.612</td>
<td>11.568</td>
</tr>
<tr>
<td>10</td>
<td>0.726</td>
<td>0.274</td>
<td>1.000</td>
<td>—</td>
<td>9.688</td>
<td>9.688</td>
<td>9.688</td>
<td>10.600</td>
</tr>
<tr>
<td>11</td>
<td>0.812</td>
<td>0.188</td>
<td>0.830</td>
<td>0.170</td>
<td>9.953</td>
<td>9.953</td>
<td>9.953</td>
<td>9.953</td>
</tr>
<tr>
<td>12</td>
<td>1.000</td>
<td>—</td>
<td>0.674</td>
<td>0.326</td>
<td>10.168</td>
<td>13.565</td>
<td>10.168</td>
<td>10.168</td>
</tr>
<tr>
<td>13</td>
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<td>—</td>
<td>0.830</td>
<td>0.170</td>
<td>10.524</td>
<td>12.565</td>
<td>10.524</td>
<td>10.524</td>
</tr>
<tr>
<td>14</td>
<td>1.000</td>
<td>—</td>
<td>0.708</td>
<td>0.292</td>
<td>10.902</td>
<td>11.565</td>
<td>10.902</td>
<td>10.902</td>
</tr>
<tr>
<td>15</td>
<td>0.607</td>
<td>0.393</td>
<td>1.000</td>
<td>—</td>
<td>11.095</td>
<td>11.095</td>
<td>11.095</td>
<td>15.554</td>
</tr>
<tr>
<td>16</td>
<td>0.596</td>
<td>0.404</td>
<td>1.000</td>
<td>—</td>
<td>11.171</td>
<td>11.171</td>
<td>11.171</td>
<td>14.577</td>
</tr>
<tr>
<td>17</td>
<td>0.604</td>
<td>0.396</td>
<td>1.000</td>
<td>—</td>
<td>11.251</td>
<td>11.251</td>
<td>11.251</td>
<td>13.577</td>
</tr>
<tr>
<td>18</td>
<td>0.596</td>
<td>0.404</td>
<td>1.000</td>
<td>—</td>
<td>11.331</td>
<td>11.331</td>
<td>11.331</td>
<td>12.577</td>
</tr>
<tr>
<td>19</td>
<td>0.754</td>
<td>0.246</td>
<td>1.000</td>
<td>—</td>
<td>11.513</td>
<td>11.513</td>
<td>11.513</td>
<td>11.736</td>
</tr>
<tr>
<td>20</td>
<td>1.000</td>
<td>—</td>
<td>0.572</td>
<td>0.428</td>
<td>11.722</td>
<td>15.540</td>
<td>11.722</td>
<td>11.722</td>
</tr>
</tbody>
</table>

Table 3. Performance of the solution algorithms for Example 3.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>CPU time (s)</th>
<th>Number of iterations</th>
<th>$G_1$</th>
<th>$G_2$</th>
<th>$G_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00E−01</td>
<td>0.88</td>
<td>521</td>
<td>1.41E−02</td>
<td>1.25E−02</td>
<td></td>
</tr>
<tr>
<td>1.00E−02</td>
<td>2.05</td>
<td>1,489</td>
<td>1.23E−03</td>
<td>3.16E−04</td>
<td></td>
</tr>
<tr>
<td>1.00E−03</td>
<td>2.78</td>
<td>2,069</td>
<td>1.17E−05</td>
<td>6.13E−06</td>
<td></td>
</tr>
<tr>
<td>1.00E−04</td>
<td>7.35</td>
<td>5,799</td>
<td>6.10E−06</td>
<td>3.19E−06</td>
<td></td>
</tr>
<tr>
<td>1.00E−05</td>
<td>7.38</td>
<td>5,815</td>
<td>6.18E−07</td>
<td>3.24E−07</td>
<td></td>
</tr>
</tbody>
</table>

5, 6, 9, 10, and 11 were 3.0 veh/interval/lane. The values of the solution algorithm parameters were the same as those in Examples 1 and 3.

First we used the point queue model (see Huang and Lam 2002 for details) to implement the DNL procedure during the solution process. Both the link cumulative outflows and link travel times generated by the point queue model were continuous of the link inflows (Szeto and Lo 2006). According to Proposition 8, the approach travel times are continuous of the approach proportions, and thus the proposed DTA model has at least one solution. We used the proposed algorithm to obtain the solution of the DTA problem in the Nguyen and Dupuis network with point queues, and the convergence performance of the algorithm is shown in Figure 6. Figure 6 also shows that the values of all three gap functions converge to zero when the number of iterations grows.

It is also interesting to investigate the convergence of the solution algorithm for the proposed DTA model when considering physical queues. Szeto and Lo (2006) pointed out that link (route) travel times may be not continuous with respect to link flows, and that the existence of solutions to DTA problems may be not guaranteed if queue spillback is considered. The results of Example 2 also illustrate that the approach travel time functions may not be continuous with respect to approach proportions, and hence that the existence of solutions to the proposed DTA model may be not guaranteed. However, once the existence of solutions to DTA problems with physical queues is guaranteed, the

Table 4. Length of each link in the Nguyen and Dupuis network.

<table>
<thead>
<tr>
<th>Link</th>
<th>Length (m)</th>
<th>450</th>
<th>600</th>
<th>600</th>
<th>750</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2</td>
<td>1-5</td>
<td>4-5</td>
<td>12-8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7-8</td>
<td>4-9</td>
<td>5-6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7-11</td>
<td>5-9</td>
<td>6-7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9-13</td>
<td>6-10</td>
<td>9-10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8-2</td>
<td>11-2</td>
<td>11-3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10-11</td>
<td>12-6</td>
<td>13-3</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The proposed algorithm can be used to obtain their solutions. We set the same LTM input parameters as those in Examples 2 and 3 and used the proposed algorithm to obtain the solution of the DTA problem in the Nguyen and Dupuis network with physical queues. Figure 7 demonstrates the convergence performance of the algorithm. Compared with the DTA problems with point queues, the solutions to DTA problems with physical queues are reached with a slower convergence rate (see Figures 6 and 7). Table 5 illustrates the CPU time and the number of iterations required for different convergence criteria under two queue representations. The results presented in Table 5 show that the time required for solving the physical queue DTA problem is much longer than that for the point queues under the same convergence criterion.

**Example 5. Applications to larger networks.**

The networks presented in Table 6 were adopted to illustrate that the proposed model formulation can handle large networks. The relevant network data were taken from the transportation network data sets maintained by Bar-Gera (2012) (http://www.bgu.ac.il/~barga/tntp/). We slightly modified the free flow travel time of each link. In this example, the point queue model was adopted to implement the DNL. The length of each time interval was set at one min, the modeling horizon was set at 300 intervals (i.e., 30 min). The traffic demand of the Anaheim network was doubled. The values of the solution algorithm parameters were the same as those in previous examples.

We can observe from Figure 8 that the value of the convergence indicator generally decreases as the iteration proceeds. The value of gap function $G_1$ can reach 0.01 after 250 iterations for all of the test networks. We also found

**Table 5.** Performance of the solution algorithm under two queue representations in Example 4.

<table>
<thead>
<tr>
<th>Queue type</th>
<th>$\varepsilon$</th>
<th>CPU time (s)</th>
<th>Number of iterations</th>
<th>$G_2$</th>
<th>$G_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point queue</td>
<td>1.00E-01</td>
<td>1.57</td>
<td>75</td>
<td>2.07E-02</td>
<td>1.05E-02</td>
</tr>
<tr>
<td>Physical queue</td>
<td>1.00E-02</td>
<td>3.14</td>
<td>171</td>
<td>2.78E-03</td>
<td>1.39E-03</td>
</tr>
<tr>
<td></td>
<td>1.00E-03</td>
<td>12.99</td>
<td>762</td>
<td>2.13E-05</td>
<td>7.80E-06</td>
</tr>
<tr>
<td></td>
<td>1.00E-04</td>
<td>15.23</td>
<td>883</td>
<td>3.16E-06</td>
<td>1.75E-06</td>
</tr>
<tr>
<td></td>
<td>1.00E-05</td>
<td>18.29</td>
<td>955</td>
<td>2.90E-07</td>
<td>1.38E-07</td>
</tr>
<tr>
<td>Physical queue</td>
<td>1.00E-01</td>
<td>2.43</td>
<td>119</td>
<td>1.88E-02</td>
<td>1.40E-02</td>
</tr>
<tr>
<td></td>
<td>1.00E-02</td>
<td>7.73</td>
<td>428</td>
<td>2.07E-03</td>
<td>1.21E-03</td>
</tr>
<tr>
<td></td>
<td>1.00E-03</td>
<td>20.23</td>
<td>1,158</td>
<td>6.43E-05</td>
<td>3.04E-05</td>
</tr>
<tr>
<td></td>
<td>1.00E-04</td>
<td>131.59</td>
<td>7,623</td>
<td>1.69E-05</td>
<td>1.52E-06</td>
</tr>
<tr>
<td></td>
<td>1.00E-05</td>
<td>398.92</td>
<td>23,026</td>
<td>1.51E-05</td>
<td>6.80E-08</td>
</tr>
</tbody>
</table>

**Table 6.** Detail of the test networks in Example 5.

<table>
<thead>
<tr>
<th>Network</th>
<th>Sioux Falls</th>
<th>Anaheim</th>
<th>Barcelona</th>
<th>Chicago sketch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node</td>
<td>24</td>
<td>416</td>
<td>1,020</td>
<td>933</td>
</tr>
<tr>
<td>Link</td>
<td>76</td>
<td>914</td>
<td>2,522</td>
<td>2,950</td>
</tr>
<tr>
<td>Zone</td>
<td>24</td>
<td>38</td>
<td>110</td>
<td>386</td>
</tr>
<tr>
<td>OD pairs</td>
<td>528</td>
<td>1,406</td>
<td>7,922</td>
<td>93,135</td>
</tr>
</tbody>
</table>

**Figure 6.** Convergence of the algorithm for the Nguyen and Dupuis network with point queues.

**Figure 7.** Convergence of the algorithm for the Nguyen and Dupuis network with physical queues.

**Figure 8.** Convergence of the algorithm for the test networks in Example 5.
Table 7. Memory required to store approach proportions for different test networks in Example 5.

<table>
<thead>
<tr>
<th>Network</th>
<th>Sioux Falls</th>
<th>Anaheim</th>
<th>Barcelona</th>
<th>Chicago sketch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full vector (MB)</td>
<td>5.10</td>
<td>84.02</td>
<td>828.79</td>
<td>4,507.80</td>
</tr>
<tr>
<td>Sparse vector (MB)</td>
<td>1.73</td>
<td>22.64</td>
<td>128.62</td>
<td>694.17</td>
</tr>
<tr>
<td>Saving (%)</td>
<td>66.08</td>
<td>73.06</td>
<td>84.48</td>
<td>84.61</td>
</tr>
</tbody>
</table>

that the vector \( \mathbf{\alpha} - \mathbf{\alpha}_0 \) is a sparse vector, with most of the elements equal to zero. This property enables the realization of substantial memory requirement reductions by storing only the nonzero entries in \( \mathbf{\alpha} - \mathbf{\alpha}_0 \). The results presented in Table 7 show that the memory required to store the full vector of approach proportions becomes huge as the network size grows. However, the memory can be significantly reduced by storing the approach proportions as a sparse vector.

We also compared the performance of the proposed method with the path-based method in terms of efficiency and memory usage. The DUO-based dynamic route choice problem can be formulated as a path-based VI problem (Friesz et al. 1993, Lo and Szeto 2002). Hence, the extragradient method presented in §4.2 can be directly extended to solve the path-based model, and is more efficient than the MSA used in INDY (Bliemer et al. 2004). The parameters of the extragradient method for the path-based model are the same as those of the intersection-movement-based model. We adopted a column-generation step to update the path set required for solving the path-based VI problem (Panicucci et al. 2007). We focused on the number of DNL required rather than the CPU time to evaluate the efficiency of the solution algorithms. Performing DNL is the most time-consuming step in solving DTA models, and CPU time is generally affected by the compiler, coding skills, computer model, and so on. Both solution algorithms stopped when the value of \( G_1 \) was less than \( 1.0 \times 10^{-4} \).

Table 8 provides the number of DNL required for both the path-based method and the proposed intersection-movement-based method to achieve a value of \( G_1 \) less than \( 1.0 \times 10^{-4} \). The proposed method outperforms the path-based method in terms of efficiency, especially for large networks. Performing DNL is also the most memory-consuming step in solving DTA models. Table 8 also provides the additional memory required for both methods, where the additional memory is defined as the total memory required for DNL minus the memory required for the DNL inputs (e.g., route flows, approach proportions). The additional memory required for the proposed method is much less than that required for the path-based method, especially for large networks.

Similar to INDY, the proposed DTA model can use a simulation time step of just a few seconds and a route choice time step from 5 to 15 min. In aforementioned studies, both the simulation time step and route choice time step were one min. If we decreased the simulation time step from one min to 10 sec, the additional memory required for DNL would increase five times, and the memory required for the inputs of DNL would not change. If we enlarged the route choice time steps from one min to 10 min, the memory required for storing approach proportions would decrease by about 90%. According to the preceding analysis, we estimated the memory usage for both the path-based method and proposed method under the dual time step strategy, and the results are presented in Table 9. The memory usage for DNL inputs is very small, and the additional memory required for DNL plays the dominant role. The results presented in Table 9 also show that the total memory required for the proposed method is much less than that required for the path-based method. Therefore, the

Table 8. Performance of the proposed method and path-based method.

<table>
<thead>
<tr>
<th>Network</th>
<th>The number of DNL required</th>
<th>Additional memory required for DNL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Path-based method</td>
<td>Proposed method</td>
</tr>
<tr>
<td>Sioux Falls</td>
<td>504</td>
<td>401</td>
</tr>
<tr>
<td>Anaheim</td>
<td>288</td>
<td>82</td>
</tr>
<tr>
<td>Barcelona</td>
<td>850</td>
<td>189</td>
</tr>
<tr>
<td>Chicago sketch</td>
<td>1,601</td>
<td>297</td>
</tr>
</tbody>
</table>

Table 9. Estimated memory usage for DNL under a dual time step strategy.

<table>
<thead>
<tr>
<th>Network</th>
<th>Sioux Falls</th>
<th>Anaheim</th>
<th>Barcelona</th>
<th>Chicago sketch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Memory required for DNL inputs</td>
<td>Path-based method (MB)</td>
<td>0.05</td>
<td>0.12</td>
<td>1.04</td>
</tr>
<tr>
<td>Additional memory required for DNL</td>
<td>Path-based method (MB)</td>
<td>17.23</td>
<td>148.11</td>
<td>584.66</td>
</tr>
<tr>
<td>Total memory usage</td>
<td>Path-based method (MB)</td>
<td>17.27</td>
<td>148.23</td>
<td>585.70</td>
</tr>
<tr>
<td>Reduction (%)</td>
<td>54.66</td>
<td>48.79</td>
<td>52.91</td>
<td>56.01</td>
</tr>
</tbody>
</table>
proposed method can outperform the path-based method in terms of memory usage when a dual time step strategy is adopted.

6. Conclusions

In this paper, a novel VI formulation of the DUO route choice problem is proposed using the concept of approach proportion. The DUO conditions of the proposed model are proved to be equivalent to the existing DUO conditions, and the existence and uniqueness of the solutions to the VI problem are clarified. If a traffic flow model that satisfies the link FIFO requirements is adopted, the solution of the proposed model will satisfy the route and OD FIFO requirements (Wu et al. 1998). A solution algorithm based on the extragradient method was developed to solve the proposed DTA problem. The decomposition property of the constraint set of the proposed model is used to compute the projections in the solution algorithm. The numerical results show that the proposed DTA model is consistent with some other models (e.g., Ban et al. 2008), and that the proposed method can outperform the path-based method in terms of both efficiency and memory usage. The results also show that the approach travel time may not be continuous with respect to the approach proportions due to queue spillback. However, the proposed algorithm is capable of obtaining a DUO solution under the point queue and physical queue settings if the existence of solutions is guaranteed. In the future, we will extend the proposed method to model travelers’ travel behavior under different travel choice principles and develop more efficient solution algorithms.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/opre.2013.1202.

Endnote

1. There is a typo in Table 1 in Ban et al. (2008). The constant parameter $\beta^*_i$ should be reduced by 10 times. This was confirmed by Dr. Ban.

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References


References


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EC.1. The proof of Proposition 1

Proof. In Eq. (16), \( d^{s*}(t) \) is the traffic demand, which is independent in terms of \( \alpha \), and hence \([u^{s*}(t)]\) can be unilaterally determined by the approach proportion vector \( \alpha \). The method of mathematical induction is used to prove that \([u^{s*}_{ab}(t)]\) can also be unilaterally determined by the approach proportion vector \( \alpha \). We assume that for every \( t' < t \), the link inflows \( u^{s*}_{ab}(t') \) can be unilaterally determined by \( \alpha \). Substituting Eq. (6) into Eq. (17), we have

\[
u^{s*}_{ab}(t) = \alpha^{s*}_{ab}(t)u^{s*}_a(t) = \alpha^{s*}_{ab}(t)\left\{ u^{s*}_a(t) + \sum_{b \in B(l_a)} v^{s*}_{ba}(t) \right\}, \forall a, b, t. \tag{EC.1}
\]

Substituting Eqs. (9) and (16) into Eq. (EC.1), we have

\[
u^{s*}_{ab}(t) = \alpha^{s*}_{ab}(t)\left\{ \alpha^{s*}_{a}(t)d^{s*}(t) + \sum_{b' \in B(l_a)} w^{s*}_{ba}(t-p_{b'}(t))(1-dp_{b'}(t))/dt \right\}, \forall a, b, t. \tag{EC.2}
\]

In Eq. (EC.2), \( d^{s*}(t) \) is the traffic demand, which is independent in terms of \( \alpha \). The definition of \( p_{b'}(t) \) implies that \( p_{b'}(t) \geq \tau^0_{b'} > 0 \), where \( \tau^0_{b'} \) is the free flow travel time of link \( b' \), and that

\[
dp_{b'}(t)/dt = \frac{d\tau_{b'}(t-p_{b'}(t))/dt}{1 + \hat{\tau}_{b'}(t-p_{b'}(t))}. \tag{EC.3}
\]

According to Eq. (EC.3), we have \( dp_{b'}(t)/dt = \tau_{b'}(t-p_{b'}(t))/[1 + \hat{\tau}_{b'}(t-p_{b'}(t))] \). Because the road traffic flow satisfies causality and \( t-p_{b'}(t) < t \), both \( \tau_{b'}(t-p_{b'}(t)) \) and \( \hat{\tau}_{b'}(t-p_{b'}(t)) \) are determined by the traffic flows that enter into link \( b' \) not later than \( t-p_{b'}(t) \). These imply that both \( u^{s*}_{ba}(t-p_{b'}(t)) \) and \( dp_{b'}(t)/dt \) can be unilaterally determined by \( \alpha \). Therefore, if the link inflows \( u^{s*}_{ab}(t') \) for every \( t' < t \) are unilaterally determined by \( \alpha \), then \( u^{s*}_{ab}(t) \) is also unilaterally determined by \( \alpha \). Because the network is empty initially, \( v^{s*}_{ba}(t) = 0 \). According to Eqs. (EC.1) and (EC.2), we have

\[
u^{s*}_{ab}(0) = \alpha^{s*}_{a}(0)\alpha^{s*}_a(0)d^{s*}(0). \tag{EC.4}
\]

This implies that the link inflow \( u^{s*}_{ab}(0) \) is unilaterally determined by \( \alpha \). Therefore, \([u^{s*}_{ab}(t)]\) can be unilaterally determined by the approach proportion vector \( \alpha \). This completes the proof. \( \square \)

EC.2. The proof of Proposition 2

Proof. Because \( d^{s*}(t) \) is the traffic demand, which is independent in terms of \( \alpha \), Eq. (16) implies that \([u^{s*}_a(t)]\) is continuous with respect to the approach proportion vector \( \alpha \). The method of mathematical induction can also be used to prove that \([u^{s*}_{ab}(t)]\) is continuous with respect to the approach proportion vector \( \alpha \).
proportion vector $\alpha$. We assume that for every $t' < t$, the link inflows $u_{ab}(t')$ are continuous with respect to $\alpha$. Under Assumption 1, the road traffic flow satisfies link FIFO and causality, and the link outflow $v_{ab}^s(t)$ only depends on link inflows at time $t - \tau^0_a$ or earlier than $t - \tau^0_a$ with $\tau^0_a$ as the free flow travel time of link $a$. Because the cumulative outflows are continuous with respect to link inflows, $v_{ab}^s(t)$ is continuous with respect to $u_{ab}(t')$, $t' \leq t - \tau^0_a < t$, and thus $v_{ab}^s(t)$ is continuous with respect to $\alpha$. Therefore, Eq. (EC.1) implies that $u_{ab}^s(t)$ is also continuous with respect to $\alpha$. The network is empty initially, and therefore $v_{ab}^s(0) = 0$. According to Eqs. (EC.1) and (EC.2), we have $u_{ab}^s(t) = \alpha_{ab}^s(0)\alpha_a^{is}(0)d^s(0)$. This implies that the link inflow $u_{ab}^s(0)$ is continuous with respect to $\alpha$. This completes the proof. □

**EC.3. The proof of Theorem 1**

*Proof.* Because all of the feasible approach proportions must satisfy Eq. (18), we have

$$\sum_{a \in A(i)} \alpha_{is}^a(t) = \sum_{a \in A(i)} \alpha_{is}^a(t) = 1, \forall i, s, t.$$ 

From the first term on the left-hand side of Eq. (23), we have

$$\int_0^T \sum_{i} \sum_{a \in A(i)} \sum_{s \in A(i)} \pi_{is}^a(t)[\alpha_{is}^a(t) - \alpha_{is}^a(t)]dt$$

$$= \int_0^T \sum_{i} \sum_{a \in A(i)} \sum_{s \in A(i)} \pi_{is}^a(t)[\alpha_{is}^a(t) - \alpha_{is}^a(t)]dt - \int_0^T \sum_{i} \sum_{a \in A(i)} \sum_{s \in A(i)} \pi_{is}^a(t)
\left\{ \sum_{a \in A(i)} \alpha_{is}^a(t) - \sum_{a \in A(i)} \alpha_{is}^a(t) \right\}$$

$$= \int_0^T \sum_{i} \sum_{a \in A(i)} \sum_{s \in A(i)} \left[ \pi_{is}^a(t) - \pi_{is}^a(t) \right][\alpha_{is}^a(t) - \alpha_{is}^a(t)]dt$$

where $\psi_{is}^a(t) = \pi_{is}^a(t) - \pi_{is}^a(t)$.

From the second term on the left-hand side of Eq. (23), we can similarly obtain the following equation:

$$\int_0^T \sum_{a} \sum_{s \in A(h_a)} \sum_{b \in A(h_a)} \pi_{ab}^s(t)[\alpha_{ab}^s(t) - \alpha_{ab}^s(t)]dt = \int_0^T \sum_{a} \sum_{s \in A(h_a)} \sum_{b \in A(h_a)} \psi_{ab}^s(t)[\alpha_{ab}^s(t) - \alpha_{ab}^s(t)]dt, \text{(EC.5)}$$

where $\psi_{ab}^s(t) = \pi_{ab}^s(t) - \pi_{ab}^s(t)$. 
we have considered two cases: \( \tau \) obtain. We then have the following VI problem:

\[
\int_0^T \sum_{i} \sum_{s} \sum_{a \in A(i)} \psi_{is}^a(t) [\alpha_{is}^a(t) - \alpha_{is}^a(t)] dt + \\
\int_0^T \sum_{a} \sum_{b \in A(b_a)} \psi_{ab}^a(t) [\alpha_{ab}^a(t) - \alpha_{ab}^a(t)] dt \geq 0. 
\]

(EC.6)

Hence, to prove this theorem, we only need to prove that the dynamic approach proportion vector satisfying constraints (18)-(21) is in a DUO route choice state if and only if the vector satisfies the VI problem (EC.6). The rest of the proof is similar to that of Theorem 5.2 in Ran and Boyce (1996). □

**EC.4. The proof of Proposition 4**

*Proof. *According to DUO conditions (4) and (5), we have \( \pi_{is}^{l_a} \geq \pi_{is}^{l_a} \), where \( i = l_a \). Consider two cases: \( \pi_{is}^{l_a} = \pi_{is}^{l_a} \) and \( \pi_{is}^{l_a} > \pi_{is}^{l_a} \). For the first case, because \( \pi_{is}^{l_a} = \pi_{is}^{l_a} \), we have \( u_{is}^{l_a} \geq 0 \). Hence, we have \( u_{is}^{l_a} \geq 0 \) according to Eq. (10). For the second case, we need to prove that \( u_{is}^{l_a} = 0 \). DUO conditions (4) and (5) also imply that \( u_{is}^{l_a} = 0 \) if \( \pi_{is}^{l_a} = \pi_{is}^{l_a} \), and that \( u_{is}^{l_a} = 0 \) if \( \pi_{is}^{l_a} > \pi_{is}^{l_a} \). In the second case, \( \pi_{is}^{l_a} > \pi_{is}^{l_a} \), i.e.,

\[
\pi_{is}^{l_a}(t) > \pi_{is}^{l_a}(t). 
\]

(EC.7)

Therefore, we can add \( \tau_{ib}^a(t - p_{ib}^a(t)) \) for a given \( b' \in B(l_a) \) on both sides of inequality (EC.7) to obtain \( \tau_{ib}^a(t - p_{ib}^a(t)) + \pi_{is}^{l_a}(t) > \tau_{ib}^a(t - p_{ib}^a(t)) + \pi_{is}^{l_a}(t) \), which can be expressed as

\[
\tau_{ib}^a(t - p_{ib}^a(t)) + \pi_{is}^{l_a}(t - p_{ib}^a(t) + \tau_{ib}(t - p_{ib}^a(t)) \geq 0 \right) \\
\tau_{ib}^a(t - p_{ib}^a(t)) + \pi_{is}^{l_a}(t - p_{ib}^a(t) + \tau_{ib}(t - p_{ib}^a(t)) \right)
\]

(EC.8)

due to the definition of \( p_{ib}^a(t) \) (i.e., \( p_{ib}^a(t) = \tau_{ib}(t - p_{ib}^a(t)) \)). According to Eqs. (1) and (2), the left-hand and right-hand sides of the inequality (EC.8) are, respectively, \( \pi_{is}^{l_a}(t - p_{ib}^a(t)) \) and \( \pi_{is}^{l_a}(t - p_{ib}^a(t)) \).

We then have \( \pi_{is}^{l_a}(t - p_{ib}^a(t)) > \pi_{is}^{l_a}(t - p_{ib}^a(t)) \), which implies that \( u_{is}^{l_a}(t - p_{ib}^a(t)) = 0 \) for every \( b' \in B(l_a) \). Using Eqs. (10) and (11), we have \( u_{is}^{l_a}(t) = 0 \) if \( \pi_{is}^{l_a}(t) > \pi_{is}^{l_a}(t) \). This completes the proof. □