Many of the formulations of current research interest, including isogeometric methods [1] and the extended finite element method [2], use nontraditional basis functions. Developing new software for each new class of basis functions is a large research burden, especially if the problems involve large deformations, nonlinear materials, and contact. This short exposition summarizes an element formulation [3] that separates as much as possible the generation and evaluation of the basis functions from the analysis, resulting in a formulation that can be implemented within the traditional structure of a finite element program but that permits the use of arbitrary sets of basis functions that are defined only through the input file. It is applied to linear fracture analysis using a higher-order element combining X-FEM and isogeometric analysis in LS-DYNA.

This work was performed in collaboration with Ted Belytschko and others. The full details may be found in Ref. [3] and Ref. [9], which has just appeared on-line.

Introduction

New elements have traditionally been added to FE programs by writing subroutines and linking them to the existing code. Each new set of basis functions had to be independently implemented along with an appropriate integration rule. A generalized element capability [3] has been added to LS-DYNA to eliminate all programming by reading in all the required information to define a new element. The implementation is restricted to the classical formulation of the finite element method, and users interested in 1-point elements with hourglass control [4] or B-bar methods [5] will still have to implement them in LS-DYNA with the user element subroutines.

The original motivation for developing the generalized element formulation was to facilitate research on isogeometric elements [1] when Benson was on sabbatical with Hughes at ICES at the University of Texas at Austin. Isogeometric analysis uses the same basis functions as CAD, and therefore allows for an exact representation of the geometry in the analysis. It shows a great deal of promise for explicit methods since it does not have the heavy time step size penalty associated with higher order Lagrange polynomials [3]. The higher order elements have higher order continuity, permitting the implementation of thin shell formulations [6] in addition to the traditional shear deformable ones [7].

In this formulation, the basis functions are not assumed to be interpolatory, a departure from the traditional FEA formulations. The variables associated with them are therefore generalized coordinates corresponding to the amplitudes of the basis functions. In fact, for most of the basis functions of interest, e.g., NURBS [8], the generalized coordinates do not even correspond to the locations of material points.

The basis functions are used in the same manner in the generalized elements as in traditional finite elements, but to emphasize that they are not necessarily interpolatory, the generalized coordinates are denoted by $q_A$. The formulation is isoparametric, with the parametric coordinates $s$.

\[
x(s, t) = \sum_{A=1,N} N_A(s)q_A(t)
\]

\[
X(s) = x(s, 0) = \sum_{A=1,N} N_A(s)q_A(0)
\]

\[
u(s, t) = x(s, t) - X(s)
\]
The term *node* is retained for the lack of a better one, but there is no requirement that the generalized coordinates associated with a node represent a physical location on the body.

On substitution into the weak form, these functions immediately lead to a set of semi-discrete equations that have the same structure as found in traditional element formulations,

\[
\sum_B M_{AB} \ddot{q}_B + \int_{\Omega} B^T \sigma d\Omega = \int_{\Gamma_b} h N_A d\Gamma + \int_{\Omega} b N_A d\Omega, \quad \forall A.
\]

where \(M\) and \(B\) are the usual definitions of the mass matrix and the discrete gradient operator, respectively.

Generalized elements as defined in LS-DYNA by their integration rule, and the values of the basis functions and their derivatives at the integration points.

**Combining X-FEM and Isogeometric Analysis**

Fracture analysis has always been a difficult problem for numerical solution methods because the singularity at the crack tip is not adequately captured by traditional basis functions unless the meshes are extremely fine. Beltyschko and his students developed the extended finite element method (X-FEM) [2] to address this class of problem and others containing discontinuities of various types. Additional basis functions that are typically obtained from classical solutions of related problems enrich the solution space to permit accurate solutions on coarse meshes.

The combination of X-FEM and isogeometric analysis plays to the strengths of both methods. For the current exploratory research, the classical crack tip enrichment field

\[
\mathbf{u}_K = \begin{cases} u_{K1} & \text{if } \theta = 0, \\ u_{K2} & \text{if } \theta = \pi, \\ u_{K3} & \text{if } \theta = \pi, \end{cases}
\]

\[
\mathbf{u}_K = \sum A N_A \frac{\sqrt{r}}{2\sqrt{2\pi}\mu} \begin{pmatrix} (-1/2 + k) \cos(\theta) - \cos(3\theta/2) \\ (1/2 + k) \sin(\theta) - \sin(3\theta/2) \end{pmatrix} K_I
\]

\[
k = 3 - 4\nu
\]

was added to the solution along with the Heaviside function for the crack away from the tip where \(r\) and \(\theta\) are a cylindrical coordinate system with its origin at the crack tip.

A linear fracture problem, using the extended finite element enrichment consists of a square domain 6 inches on a side. A crack runs from the left edge to the center of the square. The material is linearly elastic with a modulus of elasticity of \(10^7\) psi and a Poisson's ratio of 0.3. Fifth degree NURBS basis functions were used to solve it on meshes ranging from 3 by 3 to 11 by 11 elements, with the center element containing the crack tip. Since the problem is two-dimensional, linear basis functions are used in the z direction. Gaussian quadrature with 10 points in each direction was used to evaluate the integrals. This integration rule is not optimal, but it was chosen to accurately integrate the products of the fifth degree NURBS polynomials with the high gradient crack tip enrichment function without introducing special purpose integration rules. Displacement boundary conditions were imposed corresponding to an exact solution of \(K_I=100\). On the finer mesh, which is still very coarse by the standards of traditional linear fracture analysis, a value of \(K_I=99.49\) was obtained. Figure 1 shows the results.
Figure 1: The $x$ and $y$ strains calculated using the extended finite element method with fifth degree NURBS basis functions. The top and bottom rows show the results calculated with 3 x 3 and 11 x 11 element meshes, respectively.

Conclusions

The combination of X-FEM with isogeometric analysis is very promising. High levels of accuracy have been obtained on coarse meshes. Additional details on this research are described in Ref. [3] and Ref. [9], and a paper devoted entirely to the topic is currently being written. The generalized element formulation [3] has proven its value during the course of this research by permitting many X-FEM alternatives to be explored without any programming in LS-DYNA.
References


