Fuzzy data envelopment analysis and its application to location problems

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In this paper, fuzzy DEA (data envelopment analysis) models are proposed for evaluating the efficiencies of objects with fuzzy input and output data. The obtained efficiencies are also fuzzy numbers that reflect the inherent ambiguity in evaluation problems under uncertainty. An aggregation model for integrating fuzzy attribute values is provided in order to rank objects objectively. Using the proposed method, a case study involving a restaurant location problem is analyzed in detail. Rent of establishment, traffic amount, level of security, consumer consumption level and competition level are identified as the primary factors in determining an ideal location for a Japanese-style rotisserie restaurant. Based on field investigation, the uncertain information on primary factors is represented by fuzzy numbers. Using the fuzzy aggregation model, the best location of restaurant is determined. The case study shows that fuzzy DEA models can be quite useful for solving business problems under uncertainty.

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1. Introduction

Data envelopment analysis (DEA) is a non-parametric method in which multiple inputs and outputs could be used to measure an entity’s performance. DEA is a mathematical programming technique aiming at the measurement of decision-making units’ (DMUs) relative efficiencies [4,7,10,23,28,35]. Most of DEA researchers assume that input and output data are crisp and without any variation. In fact, inputs and outputs of DMUs are ever-changing. For example, in evaluating operation efficiencies of airlines, the inputs include seat-kilometers available, cargo-kilometers available, fuel and labor while the output involves passenger-kilometers [5]. It is common knowledge that these inputs and output can easily change because of factors such as the weather, seasons of nature, and operating state. As DEA is a ‘boundary’ method sensitive to outliers, it is very difficult to evaluate the efficiency of a DMU with varying inputs and outputs by conventional DEA models. Some researchers have proposed models such as stochastic frontier models [1,2,13,25] to deal with the variation of data in efficiency evaluation problems.

Uncertainty is an attribute of information, and commonly exists in many decision-making problems. Uncertainty is linked to information through the concept of granular structure characterized by indistinguishability, equivalence, similarity, proximity or functionality. The concept of granularity underlies the concept of linguistic variable [33]. For example, an expert can make a general conclusion that the output capacity of airline A is about 200 passenger-kilometers and mileage is high based on his considerable experience. These linguistic variables are used to characterize the general situation of inputs and outputs and reflect the ambiguity of the experts’ judgment. Few researchers have discussed DEA models under uncertainty [6,8,12,14–21,24,27,29,30,34].

Aggregation operator plays an important role in information integration and decision analysis. It integrates information from higher dimensions to one dimension to facilitate an overall judgment in the decision-making procedure. Several kinds of aggregation operators have been discussed in the literature [3,9,11,22,26,31,32]. In essence, in these methods, aggregation

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is represented as a generalized weighted sum where the weights of attributes are predetermined by decision-makers to represent their preferences or thresholds. Clearly, different weights can lead to different aggregation results. In fact, it is very difficult to choose suitable weights because of the inherent uncertainty and subjectivity in determining them. Sometimes there is no specific authority (decision-maker) who can determine the weights of attributes.

Let us consider an example of a motorcycle manufacturing company. The company has designed several new products and is interested in selecting the most popular products for mass-production. In the process, information about attributes such as price, style, comfort, mileage, etc. are collected through a questionnaire administered to possible customers. In this case, it is highly unlikely that this company can predetermine the weights of attributes because the decision to buy is probably not been reached by many customers. However, it is certain that the company can give some suggestions on attributes such as "the price is the most important attribute for a good selling". Customers also cannot determine the weights of attributes because producing the type of motorcycle is completely decided by the company rather than the individual preference of some customer. However, customers can express their comments on the attributes of motorcycles. Hence, in some situations, there is no clear authority to determine the weights of attributes. This type of evaluation system is called agent–client evaluation (ACE) system.

In ACE systems, the agent (company) can collect some information on the evaluated objects from clients (customers) and decide which action should be taken to meet clients’ preference. The ACE systems greatly differ from conventional multi-criteria decision-making problems in the sense that there is an agent rather than an authority that has right to specify weights of attributes. In such a system, the weights need to reflect the inherent intention of the clients. A self-organizing fuzzy aggregation model is proposed in [15] for ACE.

In this research, a fuzzy DEA model is proposed. This model is an extension of CCR model for evaluating the fuzzy efficiency of DMU with fuzzy input and output data. The crisp efficiency in CCR model is generalized to a fuzzy number that reflect the inherent uncertainty in real evaluation problems. Based on the proposed fuzzy DEA model, a fuzzy aggregation model for integrating fuzzy attribute values of objects is provided to rank objects objectively. Using the proposed method, a case study involving a restaurant location problem is presented. Rent of establishment, traffic amount, level of security, consumer consumption level and competition level are identified as the primary factors in determining an ideal location for a Japanese-style rotisserie restaurant. Based on field investigation, the uncertain information on primary factors is represented by fuzzy numbers. Using the proposed fuzzy aggregation model, the best location of restaurant is determined.

This paper is organized as follows: In Section 2, fuzzy DEA models are proposed for obtaining the fuzzy efficiencies of DMUs with symmetrical $L$–$L$ fuzzy input and output data. In Section 3, the methods for evaluating the objects with fuzzy attribute values are provided. In Section 4, a case study involving a restaurant location problem is presented in detail. Section 5 makes some concluding remarks for this research.

2. Fuzzy data envelopment analysis models

2.1. CCR model

DEA (data envelopment analysis) is a non-parametric technique for measuring and evaluating the relative efficiencies of a set of entities with common crisp inputs and outputs. CCR model is a linear programming (LP) based method proposed by Charnes et al. [4]. In CCR model the efficiency of the entity evaluated is obtained as a ratio of its weighted output to its weighed input subject to the condition that the ratio for each entity is not greater than 1. Mathematically, it is described as follows:

\[
\begin{align*}
\max_{\mu \in \mathbb{R}} & \quad \frac{\mu^T y_o}{v^T x_o} \\
\text{s.t.} & \quad \frac{\mu^T y_j}{v^T x_j} \leq 1 \quad (j = 1, \ldots, n), \\
& \quad \mu \geq 0, \\
& \quad v \geq 0.
\end{align*}
\]

(1)

Here the evaluated entities (DMUs) form a reference set and $n$ is the number of DMUs. $y_j = [y_{j1}, \ldots, y_{jm}]^T$ and $x_j = [x_{j1}, \ldots, x_{jm}]^T$ in (1) are the given positive output and input vectors of the $j$th DMU, respectively, and $m$ and $s$ are the numbers of outputs and inputs of DMU, respectively. $\mu$ and $v$ in (1) are the weight vectors of $y_j$ and $x_j$, respectively and the index $o$ indicates the evaluated DMU. $\mu \geq 0$ represents the vector whose elements are non-negative with at least one element having a positive value.

The model (1) is equivalent to the following LP problem:

\[
\begin{align*}
\max_{\mu \in \mathbb{R}} & \quad \mu^T y_o \\
\text{s.t.} & \quad v^T x_o = 1, \\
& \quad \mu^T y_j \leq v^T x_j \quad (j = 1, \ldots, n), \\
& \quad \mu \geq 0, \\
& \quad v \geq 0.
\end{align*}
\]

(2)
Eq. (2) shows that the DMU evaluated tries to find out its own weight vector to maximize its weighted output with the same constraints for all DMUs.

2.2. Fuzzy DEA models based on CCR model

In this section, we discuss the issue of evaluating the efficiencies of DMUs with fuzzy input and output data. Some basic concepts on fuzzy sets are recalled as follows.

**Definition 1.** A fuzzy number \( A \) is called \( L-R \) fuzzy number and denoted as \((a,c,d)_L\) if its membership function is defined by

\[
\mu_A(x) = \begin{cases} 
L((a-x)/c), & x \leq a, \\
1, & x = a, \\
R((x-a)/d), & x \geq a, 
\end{cases}
\]

where \( c > 0, d > 0 \) and reference functions \( L : [0, +\infty) \to [0, 1] \) and \( R : [0, +\infty) \to [0, 1] \) are strictly decreasing functions with \( L(0) = 1 \) and \( R(0) = 1 \). An \( L-R \) fuzzy number \((a,c,d)_L\) becomes an \( L-L \) fuzzy number, denoted as \((a,c,d)_L\), when \( L(x) = R(x) \).

An \( L-L \) fuzzy number \((a,c,d)_L\) with \( L(x) = \max(0,1-|x|) \) is called triangular fuzzy number, denoted as \((a,c,d)_L\). A symmetrical \( L-L \) fuzzy number is denoted as \((a,c)_L\) for the case of \( c = d \). In this paper, symmetrical \( L-L \) fuzzy numbers are used for representing the uncertainty of information for simplification.

An \( n \)-dimensional vector \( \mathbf{x} = [x_1, \ldots, x_n]^T \) can be fuzzified as a symmetrical \( L-L \) fuzzy vector \( \mathbf{A} \) whose membership function is defined as

\[
\Pi_A(\mathbf{x}) = \Pi_{A_1}(x_1) \land \cdots \land \Pi_{A_n}(x_n),
\]

where \( \Pi_A(\mathbf{x}) \) is the membership function of a symmetrical \( L-L \) fuzzy number \((a,c)_L\). The \( n \)-dimensional \( L-L \) fuzzy vector is denoted as \( \mathbf{A} = (a,c)_L \) with \( a = [a_1, \ldots, a_n]^T \) and \( c = [c_1, \ldots, c_n]^T \).

Consider a fuzzy linear system

\[
Y = A_1x_1 + \cdots + A_nx_n = A^t\mathbf{x},
\]

where \( x_i \) is a real number \( (i = 1, \ldots, n) \) and \( A \) is an \( n \)-dimensional symmetrical \( L-L \) fuzzy vector whose element is \((a,c)_L\). From the extension principle, it is known that \( Y \) is a symmetrical \( L-L \) fuzzy number as follows:

\[
Y = \left( \sum_{i=1}^{n} x_ia_i, \sum_{i=1}^{n} |x_i|c_i \right)_L = (a^t\mathbf{x}, c^t\mathbf{x})_L.
\]

Its \( h \)-level set \( [Y]_h \) is as follows:

\[
[Y]_h = \left[ a^t\mathbf{x} - L^{-1}(h)c^t\mathbf{x}, a^t\mathbf{x} + L^{-1}(h)c^t\mathbf{x} \right],
\]

where \( |\mathbf{x}|_h = [x_1, \ldots, x_n]^T \) and \( 0 < h \leq 1 \).

Considering fuzzy input and output data, CCR model (2) can be naturally generalized as the following fuzzy DEA model:

\[
\max_{\mu, \mathbf{Y}} \mu^t\mathbf{y}_o
\]

s.t. \( v^t\mathbf{X}_o = 1 \),

\[
\mu^t\mathbf{Y}_j \leq v^t\mathbf{X}_j \quad (j = 1, \ldots, n),
\]

\[
\mu \geq 0,
\]

\[
v \geq 0,
\]

where \( \mathbf{X}_j = (x_j, c)_L \) and \( \mathbf{Y}_j = (y_j, d)_L \) are an \( s \)-dimensional \( L-L \) fuzzy input vector and an \( m \)-dimensional \( L-L \) fuzzy output vector of the \( j \)th DMU, respectively, which generalize crisp input and output vectors in (2). “Equal to”, “smaller than” and “maximizing weighted output” in (2) are extended to be “almost equal”, “almost smaller than” and “maximizing a fuzzy number”, respectively. 1 in (2) becomes a fuzzy number \( 1 = (1, e)_L \) where \( e \leq 1 \) is the predefined spread of 1. In what follows, we interpret the concepts of “\( \mu^t\mathbf{Y}_j \leq v^t\mathbf{X}_j \)”, “max \( \mu^t\mathbf{Y}_o \)” and “\( v^t\mathbf{X}_o \approx \mathbf{1} \)” in sequence.

**Definition 2.** Given two \( L-L \) fuzzy numbers \( Z_1 = (z_1, w_1)_L \) and \( Z_2 = (z_2, w_2)_L \), the relation \( Z_1 \leq Z_2 \) \((0 < h \leq 1)\) holds if and only if the following inequalities are true for any possibility level \( k \in [h, 1] \):

\[
z_1 - L^{-1}(k)w_1 \leq z_2 - L^{-1}(k)w_2,
\]

\[
z_1 + L^{-1}(k)w_1 \leq z_2 + L^{-1}(k)w_2.
\]

where \( L^{-1}(\cdot) \) is the inverse function of \( L(\cdot) \).
Theorem 1. The necessary and sufficient conditions that (9) and (10) hold for any \( k \in [h,1] \) are as follows:

\[
\begin{align*}
  z_1 - L^{-1}(h)w_1 & \leq z_2 - L^{-1}(h)w_2, \\
  z_1 + L^{-1}(h)w_1 & \leq z_2 + L^{-1}(h)w_2.
\end{align*}
\]

(11) \hspace{1cm} (12)

Proof. It is trivial to prove the necessity part of the theorem. However, sufficiency component merits some discussion. If \( h = 1 \), (11) and (12) are equivalent to (9) and (10), respectively. The sufficiency obviously holds for \( h = 1 \). Thus, we only consider the case of \( h < 1 \) in what follows. Summation of (11) and (12) leads to

\[
z_2 \geq z_1.
\]

(13)

Eq. (11) is equivalent to

\[
z_2 - z_1 \geq L^{-1}(h)(w_2 - w_1).
\]

(14)

It is straightforward that the relation \( 0 \leq L^{-1}(k)/L^{-1}(h) \leq 1 \) holds for \( 0 < h \leq k \). Thus

\[
z_2 - z_1 \geq L^{-1}(h)(w_2 - w_1) \geq L^{-1}(k)(w_2 - w_1).
\]

(15)

Eq. (15) is equivalent to

\[
z_1 - L^{-1}(k)w_1 \leq z_2 - L^{-1}(k)w_2.
\]

(16)

Likewise, we can prove that

\[
z_1 + L^{-1}(k)w_1 \leq z_2 + L^{-1}(k)w_2.
\]

(17)

Now, let us consider maximizing a fuzzy number. Referring to Definition 2, "Maximizing an \( L-L \) fuzzy number \( Z = (z,w)_s \)" can be explained as simultaneously maximizing \( z - L^{-1}(h)w \) and \( z + L^{-1}(h)w \). Here, the following weighted function:

\[
\lambda_1(z - L^{-1}(h)w) + \lambda_2(z + L^{-1}(h)w)
\]

(18)

is introduced to obtain some compromised solution where \( \lambda_1 \geq 0 \) and \( \lambda_2 \geq 0 \) are the weights of left and right endpoints of the \( h \)-level set of \( Z \), respectively, with \( \lambda_1 + \lambda_2 = 1 \). While \( \lambda_1 = 1 \) is regarded as a pessimistic opinion of maximizing \( Z \) because it represents the worst situation, \( \lambda_2 = 1 \) is regarded as an optimistic opinion because it is concerned with the best situation.

Next, let us consider the relation \( v^\prime X_o \approx 1 \) in (8) which plays the same role as \( v^\prime X_o = 1 \) in (2). The crisp input vector \( x_o \) in CCR model becomes a fuzzy vector \( X_o \) so that \( v^\prime X_o = 1 \) is generalized to be \( v^\prime X_o \approx 1 \) where \( 1 = (1, e)_s \) is a fuzzy unity given by decision-makers. Different from the crisp case, i.e., where \( v^\prime X_o = 1 \), where the vector \( v \) can be found out to satisfy this equality, the vector \( v \) cannot always be sought to make the equality \( v^\prime X_o = 1 \) hold in the sense that \( v^\prime X_o \approx 1 \) have the same membership function. As a result, seeking a vector \( v \) to make \( v^\prime X_o \approx 1 \) is translated into finding out \( v \) to make the fuzzy number \( v^\prime X_o \approx 1 \) as much as possible, simply denoted by \( v^\prime X_o = 1 \). Considering Definition 2, the fuzzy number \( v^\prime X_o \approx 1 \) satisfies \( v^\prime X_o \approx 1 \) can be regarded as an upper bound subject to \( v^\prime X_o \approx 1 \). It means that the left endpoints of the \( h \)-level sets of \( v^\prime X_o \approx 1 \) overlap while the right endpoint of \( v^\prime X_o \approx 1 \) expands rightwards as much as possible but is not larger than that of \( 1 \). Thus, with considering the formulations (5) and (7), the problem for seeking \( v \) such that \( v^\prime X_o \approx 1 \), i.e., \( Z = (v^\prime X_o, v^\prime c_o) \approx 1 \), can be converted into the following optimization problem:

\[
\begin{align*}
\max_v & \quad v^\prime c_o \\
\text{s.t.} & \quad v^\prime x_o - L^{-1}(h)v^\prime c_o = 1 - L^{-1}(h)e, \\
& \quad v^\prime x_o + L^{-1}(h)v^\prime c_o \leq 1 + L^{-1}(h)e, \\
& \quad v \geq 0.
\end{align*}
\]

(19)

Remark 1. The optimization problem (19) is used to seek the maximum \( Z = v^\prime X_o \) constrained by \( v^\prime X_o \approx 1 \) with the same left endpoint as fuzzy number 1 in \( h \)-level sets. This approach can be regarded as a generalization of the procedure in which seeking a value \( x \) such that \( x \approx 1 \) is equivalent to finding out the biggest \( x \) subject to \( x \leq 1 \).

Using (9), (10), (18) and (19) and considering (5) and (7), the fuzzy optimization problem (8) can be transformed into the following LP problem with a primary objective function and a secondary objective function:

\[
\begin{align*}
\max_{\mu} & \quad \lambda_1(\mu^\prime y_o - L^{-1}(h)\mu^\prime d_o) + \lambda_2(\mu^\prime y_o + L^{-1}(h)\mu^\prime d_o) \\
\text{s.t.} & \quad \mu^\prime x_o - L^{-1}(h)\mu^\prime c_o = 1 - L^{-1}(h)e, \\
& \quad \mu^\prime x_o + L^{-1}(h)\mu^\prime c_o \leq 1 + L^{-1}(h)e, \\
& \quad \mu \geq 0,
\end{align*}
\]

(20)
It should be noted that the optimization problem (19) is embedded into (20) to obtain \( \nu \) such that \( \nu'X_o \approx 1 \). The obtained optimal vectors from (20) are denoted as \( \nu' \) and \( \mu' \).

**Remark 2.** It can be seen that when \( \mathbf{c}_i = 0, \mathbf{d}_i = 0 \) and \( \epsilon = 0 \), the fuzzy DEA (8) just becomes CCR model. It means that the model (8) can evaluate the efficiencies of DMUs in a more general way, by which the crisp, fuzzy and hybrid inputs as well as outputs can be handled homogeneously.

**Theorem 2.** If there is an optimal solution of (19), then there exists an optimal solution of (20).

**Proof.** It reasonable to assume that \( x_j - L^{-1}(h)\mathbf{c}_j > 0 \) and \( y_j - L^{-1}(h)\mathbf{d}_j > 0 \) hold for all \( h \in (0, 1] \). Denote the optimal solution of (19) as \( \nu' > 0 \). If we take \( \mu \) as \( [0, \ldots, \mu_k, \ldots, 0]' \) with

\[
\mu_k = \min_{j=1,\ldots,n} \left\{ \frac{(\nu'x_j + L^{-1}(h)\nu'\mathbf{c}_j)}{(y_j + L^{-1}(h)\mathbf{d}_j)} \right\} (21)
\]

for all \( h \in (0, 1], \mu \) and \( \nu' \) satisfy all constraints of (20). Thus, \( \mu \) and \( \nu' \) are feasible solutions of (20). As the constraints of (20) form a bounded closed set (compact set), there exists an optimal solution in (20).

**Theorem 3.** If \( \max\{c_{o1}/x_{o1}, \ldots, c_{o|\mathbf{c}_o|}/x_{o|\mathbf{c}_o|}\} \leq \epsilon, \) then an optimal solution of (19) always exists.

**Proof.** Let us take \( \nu' = [0, \ldots, 0, v_k, 0, \ldots, 0]' \) where \( v_k = (1 - L^{-1}(h)\epsilon) / (x_{o_k} - L^{-1}(h)\epsilon) > 0 \). It is easy to prove that \( \nu' \) satisfies the constraint \( \nu'X_o > L^{-1}(h)\nu'\mathbf{c}_o = 1 - L^{-1}(h)\epsilon \). Furthermore

\[
\nu'X_o + L^{-1}(h)\nu'\mathbf{c}_o - 1 - L^{-1}(h)\epsilon = \frac{x_{o_k} + L^{-1}(h)c_{o_k}}{x_{o_k} - L^{-1}(h)c_{o_k}} (1 - L^{-1}(h)\epsilon) - 1 - L^{-1}(h)\epsilon
\]

\[
= \frac{x_{o_k} + L^{-1}(h)c_{o_k}}{x_{o_k} - L^{-1}(h)c_{o_k}} - \frac{x_{o_k} + L^{-1}(h)c_{o_k}}{x_{o_k} - L^{-1}(h)c_{o_k}} L^{-1}(h)\epsilon - 1 - L^{-1}(h)\epsilon
\]

\[
= \frac{x_{o_k} + L^{-1}(h)c_{o_k}}{x_{o_k} - L^{-1}(h)c_{o_k}} - 1 - \frac{\left(x_{o_k} + L^{-1}(h)c_{o_k}\right)}{x_{o_k} - L^{-1}(h)c_{o_k}} L^{-1}(h)\epsilon
\]

\[
= \frac{2L^{-1}(h)c_{o_k}}{x_{o_k} - L^{-1}(h)c_{o_k}} - \frac{2x_{o_k}L^{-1}(h)\epsilon}{x_{o_k} - L^{-1}(h)c_{o_k}} = \frac{2L^{-1}(h)\epsilon}{x_{o_k} - L^{-1}(h)c_{o_k}}
\]

Since \( \max\{c_{o1}/x_{o1}, \ldots, c_{o|\mathbf{c}_o|}/x_{o|\mathbf{c}_o|}\} \leq \epsilon, c_{o_k}/x_{o_k} \leq \epsilon \) holds. Thus, the value of (22) is not greater than zero. As a result, \( \nu' \) is a feasible solution of (19). As the constraints of (19) form a bounded closed set (compact set), there is an optimal solution of (19).

Considering \( n \) DMUs, \( \epsilon \) is taken as \( \epsilon = \max_{i=1,\ldots,n} \left( \max_{j=1,\ldots,n} \frac{c_{o_j}}{x_{o_j}} \right) \) in the optimization problem (20). Assuming that the optimal value of the objective function of (19) is \( g_{o0} \), the optimization problem (20) can be rewritten as the following LP problem:

\[
\max_{\lambda} \quad \lambda_1 (\mu'y_o - L^{-1}(h)\mu'd_o) + \lambda_2 (\mu'y_o + L^{-1}(h)\mu'd_o)
\]

s.t. \( \nu'X_o \geq g_{o0}, \)

\[
\mu'y_j - L^{-1}(h)\mu'd_j \leq \nu'x_j - L^{-1}(h)\nu'c_j \quad (j = 1, \ldots, n),
\]

\[
\mu'y_j + L^{-1}(h)\mu'd_j \leq \nu'x_j + L^{-1}(h)\nu'c_j \quad (j = 1, \ldots, n),
\]

\[
\mu \geq 0, \quad \nu \geq 0.
\]

**Definition 3.** The fuzzy efficiency of an evaluated DMU with the \( L-L \) fuzzy input vector \( X_o = (x_o, c_o)_L \) and output vector \( Y_o = (y_o, d_o)_L \) is defined as an \( L-L \) fuzzy number \( E = (w_o, \eta, w_o)_L \) as follows:

\[\eta = \frac{\mu'y_o}{\nu'X_o},\]

\[w_o = \frac{\eta - \mu'y_o - L^{-1}(h)}{\nu'X_o + c_o L^{-1}(h)},\]

\[w_o = \frac{\mu'y_o + L^{-1}(h)}{\nu'X_o - c_o L^{-1}(h)} - \eta.\]

Clearly, the uncertainty from the inputs and outputs of DMUs characterized by fuzzy numbers is transferred to the uncertainty of the evaluated efficiency, which is very close to human thinking.
**Definition 4.** The DMU with \( \eta + w_i \geq 1 \) for a given possibility level \( h \) is called an \( h \)-possibilistic D efficient DMU (PD DMU). On the contrary, the DMU with \( \eta + w_i < 1 \) for a given possibility level \( h \) is called an \( h \)-possibilistic D inefficient DMU (PDI DMU). The set of all PD DMUs is called the \( h \)-possibilistic nondominated set, denoted by \( S_h \).

Obviously, the \( h \)-possibilistic D efficient DMUs (PD DMUs) and the \( h \)-possibilistic D inefficient DMUs (PDI DMUs) in the case of \( h = 1 \) become the conventional D efficient DMUs and D inefficient DMUs in CCR model.

**Theorem 4.** The center of the fuzzy efficiency of any DMU obtained from (20) is not greater than 1.

**Proof.** Suppose that \( \mu^0 \) and \( v^0 \) are obtained from (20) for an evaluated DMU. Thus the following inequalities hold:

\[
\mu^0 y_j - L^{-1}(h)\mu^0 d_j \leq v^0 x_j - L^{-1}(h)v^0 c_j \quad (j = 1, \ldots, n),
\]

\[
\mu^0 y_j + L^{-1}(h)\mu^0 d_j \leq v^0 x_j + L^{-1}(h)v^0 c_j \quad (j = 1, \ldots, n).
\]

Taking the sum of (25) and (26), the following inequalities hold:

\[
\mu^0 y_j \leq v^0 x_j \quad (j = 1, \ldots, n).
\]

Then

\[
\eta = \frac{\mu^0 y_0}{v^0 x_0} \leq 1. 
\]

The formulation (28) means that evaluating fuzzy efficiencies of DMUs by the model (20) is similar to evaluating crisp efficiencies of DMUs by CCR model. Both of them seek the one nondominated by other DMUs. As far as the possibility level \( h \) is concerned, if we take a large value for \( h \), it means that we consider a relatively narrow range of input and output data where all of the data considered have high possibility degrees. Conversely, if we take a small value for \( h \), it means that we investigate the input and output data in a relatively wide range.

3. Evaluation of objects with fuzzy attribute values

Let us now consider an evaluation system \( D = (O, A, Y) \), where \( O = \{o_1, \ldots, o_n\} \) is a set of the objects evaluated, \( A = \{A_1, \ldots, A_m\} \) is a set of the attributes of \( o_i \) \( (i = 1, \ldots, n) \) and \( Y \) is a mapping defined as

\[
Y : O \times A \rightarrow V,
\]

where \( V \) is a set of all fuzzy numbers defined on the space \( R^1 \). \( Y_j \) is an \( m \)-dimensional fuzzy vector whose element is a realization of the mapping \( Y \) to represent a fuzzy attribute value of \( o_i \). For the sake of simplicity, the symmetrical \( L-L \) fuzzy vector \( (y_i, d_i) \) is used to represent \( Y_i \). It should be noted that \( Y_i \) is the evaluation vector rather than the original attribute vector. For example, the prices of three motorcycles A, B and C are $10,000, $6000 and $3000, respectively. The evaluations of the prices could be “high”, “middle” and “low”, respectively.

The problem for evaluating objects with multiple attributes can be regarded as a special case of FDEA model (8) with unity input shown as follows [15]:

\[
\max_{y_0} \mu^0 Y_0 \\
\text{s.t.} \quad \mu^0 y_j \leq 1 \quad (j = 1, \ldots, n), \quad \mu_{ai} - \mu_{aj} \geq d(i,j) \geq 0 \quad (i \neq j (i,j) \in B \subset \{1, \ldots, m\}^2), \quad \mu_{ai} \geq \varepsilon \quad (i = 1, \ldots, m),
\]

where \( \varepsilon \) is a positive constant and \( d(i,j) \) represents some suggestion on the minimum difference of importance degrees between the attributes \( A_i \) and \( A_j \). No such suggestion, these constraints are not needed. The constraint \( \mu_{ai} \geq \varepsilon \) makes the weight of the ith attribute at least larger than \( \varepsilon \), which plays a crucial role to prevent the dominance effect of some large-valued attribute. Denote the optimal solution of (30) as \( \mu^*_o \). The value of objective function \( \mu^*_o Y_0 \) is the aggregated evaluation of the object \( o \). The central idea of (30) is that each evaluated object tries to find out its own best weights of attributes under the same constraint conditions. Thus the obtained rank can be regarded as the result of fair competition rather than the one predetermined by an evaluator.

If “\( \leq \)” is explained by Definition 2, the model (30) can be transformed into the following optimization problem with considering (5), (7), (9), (10) and (18):

\[
\max_{y_0} \lambda_1 (\mu^0 y_0 - L^{-1}(h)\mu^0 d_0) + \lambda_2 (\mu^0 y_0 + L^{-1}(h)\mu^0 d_0) \\
\text{s.t.} \quad \mu^0 y_j + L^{-1}(h)\mu^0 d_j \leq 1 \quad (j = 1, \ldots, n), \quad \mu_{ai} - \mu_{aj} \geq d(i,j) \geq 0 \quad (i \neq j (i,j) \in B \subset \{1, \ldots, m\}^2), \quad \mu_{ai} \geq \varepsilon \quad (i = 1, \ldots, m).
\]
If an evaluator can suggest a linearly ordered attribute set \( A_{\text{order}} \) whose \( i \)-th element is the \( i \)-th most important attribute in \( A \), (31) becomes

\[
\begin{align*}
\max_{u_i} & \quad \lambda_1(u_j^3y_j - L^{-1}(h)u_j^3d_j) + \lambda_2(u_j^3y_j + L^{-1}(h)u_j^3d_j) \\
\text{s.t.} & \quad \mu_i^1y_j + L^{-1}(h)\mu_i^1d_j \leq 1 \quad (j = 1, \ldots, n), \\
& \quad \mu_{i-1} - \mu_{i+1} \geq \varepsilon_i \geq 0 \quad (i = 1, \ldots, m - 1), \\
& \quad \mu_{om} \geq \epsilon_m > 0,
\end{align*}
\]

(32)

where \( Y_j \) is reordered to correspond to \( A_{\text{order}} \) and \( \varepsilon_i \) \( (i = 1, \ldots, m - 1) \) are positive constants reflecting the differences of important degrees between two consecutive attributes in \( A_{\text{order}} \) and \( \mu_{om} \) represents the lowest limit of weights.

**Theorem 5.** There exists an optimal solution in (32) if and only if the constants \( \varepsilon_i \) \( (i = 1, \ldots, m) \) satisfy the following inequalities [15]:

\[
r^i(y_j + L^{-1}(h)d_j) \leq 1 \quad (j = 1, \ldots, n),
\]

where

\[
r_i = \sum_{j=1}^{m} \varepsilon_j.
\]

**Proof.** Necessary condition: Suppose there is a feasible solution in (32) and \( \mu_{om} \) satisfies the following relation:

\[
\mu_{om} = \chi \geq \epsilon_m.
\]

Thus, the following relations hold:

\[
\begin{align*}
\mu_{om} & \geq \chi + \epsilon_{m-1}, \\
\mu_{om-1} & \geq \chi + \epsilon_{m-1} + \epsilon_{m-2}, \\
\vdots & \\
\mu_{1} & \geq \chi + \sum_{i=1}^{m-1} \epsilon_i.
\end{align*}
\]

Then

\[
\mu_i^3y_j + L^{-1}(h)\mu_i^3d_j \geq r^i(y_j + L^{-1}(h)d_j),
\]

where \( r \) is defined by (34). Considering the constraint \( u_i^3y_j + L^{-1}(h)u_i^3d_j \leq 1 \) in (32), the following inequality should hold:

\[
r^i(y_j + L^{-1}(h)d_j) \leq 1 \quad (j = 1, \ldots, n).
\]

It proves the necessity condition.

**Sufficient condition:** Suppose (33) holds, it is easy to check that there is a feasible solution in the constraint conditions of (32). That is

\[
\mu_i = \sum_{j=1}^{m} \varepsilon_j \quad (i = 1, \ldots, m).
\]

Moreover, the constraint condition of (32) is a bounded closed set (compact set). Thus, there exists an optimal solution in (32). \( \square \)

4. Case study: A restaurant location problem

Location problem is one where we are interested in determining the best location for the given business requirements. Location decision is very crucial in restaurant business. In this section, we demonstrate a procedure to find an ideal location for opening a Japanese-style rotisserie in a metropolis, Henan Province, China. We consider four possible locations, i.e., Railway Station Area (P1), Government Administrative Zone (P2), University Zone (P3) and Commercial Street (P4).

Establishment rent, traffic amount, security level, consumer consumption level and competition level are mainly considered as the factors for selecting a restaurant location. The investigation results in four locations are shown in Table 1. where \( X_1, X_2, X_3, X_4 \) and \( X_5 \) are the rent of an establishment per square meter a month (Yuan), daily traffic count (10 thousands persons), the number of criminal cases a month, the customer consumption level (Yuan) and the number of similar restaurants, respectively. The daily traffic count is estimated by foot traffic plus vehicle traffic * 2 (assuming that there are two persons in a car). In fact, it may be very time-consuming and costly to investigate the statistical situation of \( X_1, X_2, X_3 \) and \( X_4 \) on each location. \( X_5 \) is based mainly on judgment on how many potential competitors are in each location. It is easy to use fuzzy
numbers to represent a general feeling about $X_1, X_2, X_3, X_4$ and $X_5$. For example, based on our investigation, the approximate monthly rent in P1 is 500 Yuan per square meter with approximately 475 Yuan in the relatively poor sections and approximately 525 Yuan in the rich neighbourhoods. This kind of uncertain information is characterized by the symmetrical triangular fuzzy number (500, 25).

According to the business plan, the restaurant is targeted towards an establishment with the rent less than 100 Yuan per square meters a month, the traffic count flowing past the location more than 150,000 everyday, with little or no crime, a consumption level close to 30 Yuan for each customer, and with no similar restaurant in its proximity. So that the satisfaction functions for establishment rents, traffic counts, security levels, consumption levels and competition levels are given as follows:

\[
y_1 = \begin{cases} 
1; & x_1 \leq 100, \\
(600 - x_1)/500; & 100 < x_1 \leq 600, \\
0; & 600 < x_1,
\end{cases}
\]

where $x_1$ is the rent, and $y_1$ is the satisfaction degree of the rent.

\[
y_2 = \begin{cases} 
1; & 15 \leq x_2, \\
(x_2 - 1)/14; & 1 \leq x_2 < 15, \\
0; & x_2 < 1,
\end{cases}
\]

where $x_2$ is the amount of traffic, and $y_2$ is the satisfaction degree of the traffic count.

\[
y_3 = \begin{cases} 
1; & x_3 = 0, \\
(140 - x_3)/140; & 0 < x_3 \leq 140, \\
0; & 140 < x_3,
\end{cases}
\]

where $x_3$ is the number of criminal cases, $y_3$ is the satisfaction degree of security.

\[
y_4 = \begin{cases} 
0; & 10 < x_4, \\
(x_4 - 10)/20; & 10 \leq x_4 < 30, \\
1; & x_4 = 30, \\
(90 - x_4)/60; & 30 \leq x_4 < 90, \\
0; & 90 < x_4,
\end{cases}
\]

where $x_4$ is the customer consumption level, $y_4$ is the corresponding satisfaction degree.

\[
y_5 = \begin{cases} 
1; & x_5 = 0, \\
(10 - x_5)/10; & 0 < x_5 \leq 10, \\
0; & 10 < x_5,
\end{cases}
\]

where $x_5$ is the number of potential competitors, $y_5$ is the corresponding satisfaction degree.

Based on the data in Table 1 and using (40)–(44), the satisfaction degrees for establishment rents, traffic counts, security levels, consumption levels and competition levels in four locations, denoted as $Y_1, Y_2, Y_3, Y_4$ and $Y_5$, are obtained and listed in Table 2.

We think that the rent and the traffic amount have the same level of importance and are more important than the other factors. The customer consumption level is more important than the competition level, and the competition level is more
important than the security level. Setting $h = 0.7$, $v = 0.05$, $d(i,j) = 0.1$, $i_1 = 1$ and $i_2 = 0$, the weights obtained by (31) is listed in Table 3.

The given possibility degree by a decision-maker reflects his attitude on uncertainty. The lower the given value of $h$, the decision-maker is considered more cautious. The values of the objective function of (31) listed in Table 4 are used to evaluate the four locations for different $h$ and the rank of locations is listed in Table 5. It can be seen from Table 5 that the best location for Japanese-style rotisserie is Commercial Street (P4) for each $h$. On closely reviewing the original data in Table 1, we find that P4 is almost at the middle levels on all factors except the third important factor for which P4 is top ranked. It is the reason why P4 is chosen as the most suitable location. However, P1 is ranked highest on traffic amount but the lowest on rent attribute. Likewise, P3 is the best location for rent but the worst for traffic amount.

5. Conclusions

In this paper, fuzzy DEA models are proposed for evaluating the efficiencies of DMUs with symmetrical L–L fuzzy input and output data. The fuzzy DEA models extend CCR model to more general forms where crisp, fuzzy and hybrid data can be handled easily. Based on the fuzzy DEA model, an aggregation model for integrating fuzzy attribute values is presented in order to rank objects objectively. A case study of restaurant location decision in China is discussed. In this application, the primary factors for selecting an ideal location of Japanese-style rotisserie are identified as establishment rent, traffic amount, security level, consumer consumption level and competition level. Based on field investigation, the uncertain information on the primary factors is represented by fuzzy numbers. According to the business plan, the satisfaction functions are set. The satisfaction degrees of each factor in four locations are obtained as fuzzy numbers. Using the fuzzy aggregation model, the best location is obtained based on total evaluation. The analysis of the result shows that the recommended location decision is quite reasonable. The application demonstrates that fuzzy DEA models are quite powerful in evaluating business problems under uncertainty.

References


