GALLAI’S INEQUALITY FOR CRITICAL GRAPHS
OF REDUCIBLE HEREDITARY PROPERTIES

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Abstract

In this paper Gallai’s inequality on the number of edges in critical graphs is generalized for reducible additive induced-hereditary properties of graphs in the following way. Let \( P_1, P_2, \ldots, P_k (k \geq 2) \) be additive induced-hereditary properties, \( R = P_1 \circ P_2 \circ \cdots \circ P_k \) and \( \delta = \sum_{i=1}^{k} \delta(P_i) \). Suppose that \( G \) is an \( R \)-critical graph with \( n \) vertices and \( m \) edges. Then \( 2m \geq \delta n + \frac{\delta^2}{2n} + \frac{\delta}{n} + \frac{2\delta}{n^2} \) unless \( R = C^2 \) or \( G = K_{\delta+1} \). The generalization of Gallai’s inequality for \( P \)-choice critical graphs is also presented.

Keywords: additive induced-hereditary property of graphs, reducible property of graphs, critical graph, Gallai’s Theorem.

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1. Introduction and Notation

A convenient language that may be used in formulating problems of graph colouring in a general setting is the language of reducible properties of graphs. Let us denote by $\mathcal{I}$ the class of all finite simple graphs. A property $P$ of graphs is any nonempty proper isomorphism closed subclass of $\mathcal{I}$. Let $P_1, P_2, \ldots, P_n$ be properties of graphs. A graph $G$ is vertex-($P_1, P_2, \ldots, P_n$)-colorable (denoted $G \in P_1 \circ P_2 \circ \cdots \circ P_n$) if the vertex set $V(G)$ of $G$ can be partitioned into $n$ sets $V_1, V_2, \ldots, V_n$ such that the subgraph $G[V_i]$ of $G$ induced by $V_i$ belongs to $P_i$, $i = 1, 2, \ldots, n$. The corresponding vertex coloring $f$ is defined by $f(v) = i$ whenever $v \in V_i$, $i = 1, 2, \ldots, n$. In the case $P_1 = P_2 = \cdots = P_n = P$ we write $P_1 \circ P_2 \circ \cdots \circ P_n = P^n$ and we say that $G \in P^n$ is ($P, n$)-colorable. Let us denote by $\mathcal{O}$ the class of all edgeless graphs. The classical graph coloring problems deals with proper coloring where $P_1 = P_2 = \cdots = P_n = \mathcal{O}$ so that a graph $G$ is $n$-colorable if and only if $G \in \mathcal{O}^n$. The basic property of the proper coloring is that every induced subgraph of a $n$-colorable graph is $n$-colorable and if every connected component of a graph $G$ is $n$-colorable, then $G$ is $n$-colorable, too. In this paper we consider as the generalizations of the proper coloring only such vertex-($P_1, P_2, \ldots, P_n$)-colorings where the properties $P_1, P_2, \ldots, P_n$ preserve the above mentioned requirements i.e., they are closed to induced subgraphs and disjoint union of graphs. Such properties of graphs are called induced-hereditary and additive. The set of all (additive) induced-hereditary properties will be denoted by $(\mathcal{M}^n) \mathcal{M}$.

An additive induced-hereditary property $\mathcal{R}$ is said to be reducible if there exist additive induced-hereditary properties $P_1$ and $P_2$ such that $\mathcal{R} = P_1 \circ P_2$, otherwise the property $\mathcal{R}$ is irreducible.

If $\mathcal{P}$ is an induced-hereditary property, then the set of minimal forbidden subgraphs of $\mathcal{P}$, called $\mathcal{P}$-critical graphs, is defined as follows:

$$C(\mathcal{P}) = \{ G \in \mathcal{I} : G \notin \mathcal{P} \text{ but for each proper induced subgraph } H \text{ of } G, \}$$

Every additive induced-hereditary property $\mathcal{P}$ is uniquely determined by the set of connected minimal forbidden subgraphs. For the class $\mathcal{O}^k$ of all $k$-colorable graphs the set $C(\mathcal{O}^k)$ consists of vertex-$(k + 1)$-critical graphs.

To investigate the structure of $\mathcal{R}$-critical graphs the following invariants of properties are useful. For an arbitrary graph theoretical invariant $\rho$ and
an induced-hereditary property \( \mathcal{P} \) let us define:

\[
\rho(\mathcal{P}) = \min\{\rho(F) : F \in \mathcal{C}(\mathcal{P})\}.
\]

E.g. the invariant \( \chi(\mathcal{P}) \) is used in extremal graph theory. It is quite easy to prove that for every \( G \in \mathcal{C}(\mathcal{R}) \), \( \mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n \), the minimum degree \( \delta(G) \) of \( G \) is at least \( \delta = \delta(\mathcal{P}_1) + \delta(\mathcal{P}_2) + \cdots + \delta(\mathcal{P}_n) \) i.e., \( \delta(\mathcal{R}) \geq \delta \). Let us call the vertices of degree \( \delta = \delta(\mathcal{P}_1) + \delta(\mathcal{P}_2) + \cdots + \delta(\mathcal{P}_n) \) in the graph \( G \in \mathcal{C}(\mathcal{R}) \) minor.

Analogously as for \( \mathcal{O}^n \)-critical graphs, using the classical recoloring method of Gallai [6], generalizations of the well-known Gallai’s theorem can be obtained.

**Theorem 1** [4]. Let \( \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n \) be additive induced-hereditary properties, \( \mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n \) and \( G \in \mathcal{C}(\mathcal{R}) \). Then every block \( B \) of the subgraph induced by the set of minor vertices of \( G \) is one of the following types:

- (a) \( B \) is a complete graph of order \( \leq \delta + 1 \),
- (b) \( B \) is a \( \delta(\mathcal{P}_i) \)-regular graph and \( B \in \mathcal{C}(\mathcal{P}_i) \) for some \( i \),
- (c) \( \Delta(B) \leq \delta(\mathcal{P}_i) \) and \( B \in \mathcal{P}_i \),
- (d) \( B \) is an odd cycle.

An analogous result for \( \mathcal{P} \)-choice critical graphs have been obtained in [3]. The presented results can be considered as generalizations of Gallai’s and Brooks’ Theorems (see [2, 5, 8, 15, 16]).

Let \( G \) be a graph and let \( L(v) \) be a list of colours prescribed for the vertex \( v \), and \( \mathcal{P} \in \mathbb{M} \). A \( (\mathcal{P}, L) \)-colouring is a graph \( \mathcal{P} \)-colouring \( f \) with the additional requirement that for all \( v \in V(G) \), \( f(v) \in L(v) \). If \( G \) admits a \( (\mathcal{P}, L) \)-colouring, then \( G \) is said to be \( (\mathcal{P}, L) \)-colourable. The graph \( G \) is \( (\mathcal{P}, k) \)-choosable if it is \( (\mathcal{P}, L) \)-colourable for every list \( L \) of \( G \) satisfying \( |L(v)| \geq k \) for every \( v \in V(G) \). The \( \mathcal{P} \)-choice number \( \chi_{\mathcal{P}}(G) \) of the graph \( G \) is the smallest natural number \( k \) such that \( G \) is \( (\mathcal{P}, k) \)-choosable.

For a property \( \mathcal{P} \in \mathbb{M} \) a graph \( G \) is said to be \( (\mathcal{P}, L) \)-critical if \( G \) has no \( (\mathcal{P}, L) \)-colouring but \( G - v \) is \( (\mathcal{P}, L) \)-colourable for all \( v \in V(G) \). The following statement is easy to prove: If \( \mathcal{P} \in \mathbb{M} \) and \( G \) is \( (\mathcal{P}, L) \)-critical, then \( d_G(v) \geq \delta(\mathcal{P})|L(v)| \) for any vertex \( v \) of \( G \). Let us denote by \( S(G) = \{ v : v \in V(G), \quad d_G(v) = \delta(\mathcal{P})|L(v)| \} \). For a nontrivial property \( \mathcal{P} \in \mathbb{M} \), a graph \( G \) is said to be \( (\text{vertex}) \) \( (\mathcal{P}, k) \)-choice critical if \( \chi_{\mathcal{P}}(G) = k \geq 2 \) but \( \chi_{\mathcal{P}}(G - v) < k \) for all vertices \( v \) of \( G \). According to the previous definitions,
it follows immediately that if $G$ is $(\mathcal{P}, k+1)$-choice critical, then $G$ is $(\mathcal{P}, L)$-critical with some list $|L(v)| = k$ for all $v \in V(G)$.

**Theorem 2** [3]. Let $\mathcal{P}$ be an additive induced-hereditary property and $G$ be a $(\mathcal{P}^k, L)$-critical graph (i.e., a $(\mathcal{P}, k+1)$-choice critical graph). Then every block $B$ of the subgraph of $G$ induced by the set $S(G) = \{v : v \in V(G), \deg_G(v) = \delta(\mathcal{P})|L(v)|\}$ of minor vertices is one of the following types:

(a) $B$ is a complete graph,
(b) $B$ is a $\delta(\mathcal{P})$-regular graph and $B \in C(\mathcal{P})$,
(c) $\Delta(B) \leq \delta(\mathcal{P})$ and $B \in \mathcal{P}$,
(d) $B$ is an odd cycle.

As for chromatically critical graphs, the structure of the subgraph induced by minor vertices of a critical graphs $G$ implies a lower bound on the number of edges of $G$, which will be considered in Section 3.

## 2. $\delta$-Graphs

Denote by $K_n^+$ the graph comprising of two blocks where the first one is isomorphic to $K_n$ and the second is isomorphic to $K_2$. The graph $K_n^+$ has only one vertex of degree 1, we call it a *pendant-vertex* of $K_n^+$, the subgraph $K_n$ in $K_n^+$ is the *head* of $K_n^+$.

A connected graph $G$ is a *$\delta$-graph* ($\delta \geq 1$) if all cut-vertices of $G$ are of degree $\delta$ and all other vertices are of degree $\delta - 1$. Thus $K_\delta$ is a $\delta$-graph. Let $G$ be a $\delta$-graph and let $B$ be an endblock of $G$ isomorphic to $K_\delta$. Note that if $B \neq G$, then $B$ is a head of a subgraph, say $H$, isomorphic to $K_\delta^+$. We will use to say that $H$ is a *pendant $K_\delta^+$ of $G$*. The graph $H$ is *redundant*, if by deleting the head of $H$ in $G$, the remaining graph is also a $\delta$-graph. If $G$ is a $\delta$-graph with no redundant pendant $K_\delta^+$ subgraphs, then $G$ is a *compact $\delta$-graph*.

**Theorem 3.** Let $G^*$ be a $\delta$-graph with $n$ vertices and $c$ cut-vertices. Then,

\[
\frac{c}{2} \leq \frac{n}{\delta} - 1.
\]

**Proof.** For the sake of simplicity in this proof, for an arbitrary $\delta$-graph $H$ on $n_h$ vertices and with $c_h$ cut-vertices, we define $\varphi(H) = n_h/\delta - c_h/2$. So, we should prove that $\varphi(G^*) \geq 1$. 

Suppose that the claim is false and $G^*$ is a counterexample with $n$ minimum. Thus, $\varphi(G^*) < 1$. It is easy to see that $G^*$ is not 2-connected, since in this case $c = 0$ and $n \geq \delta$. Note that the claim is valid for $\delta = 1$. So, we assume that $\delta > 1$. Let $B$ be the set of blocks of $G^*$.

**Claim 1.** $G^*$ is a compact $\delta$-graph.

Suppose that it is false. Then there is a redundant pendant $K^+_\delta$ subgraph $H$ in $G^*$. Let $\hat{G}$ be the graph constructed from $G^*$ by deleting the head of $H$. Then $\hat{G}$ is a $\delta$-graph. Let $\hat{n}$ and $\hat{c}$ be the number of vertices and the number of cut-vertices of $\hat{G}$. Obviously, $\hat{n} = n - \delta$. Observe that the pendant-vertex $v$ of $H$ is incident with precisely two blocks of $G^*$. In the first block (that is the bridge of $H$) $v$ has degree 1 and in the second block it has degree $\delta - 1$. So after deleting the head of $H$, in the remaining graph $\hat{G}$ the vertex $v$ is not a cut-vertex any more. Therefore, $\hat{c} = c - 2$. By the minimality of $G^*$, $\varphi(\hat{G}) \geq 1$. So,

$$\varphi(G^*) = \frac{n}{\delta} - \frac{c}{2} = \frac{\hat{n}}{\delta} - \frac{\hat{c}}{2} = \varphi(\hat{G}) \geq 1.$$ 

But it is a contradiction.

**Claim 2.** *Every bridge of $G^*$ is an edge of a pendant $K^+_\delta$.*

Suppose that it is false. Let $e = u_1u_2$ be an bridge that is not an edge of a pendant $K^+_\delta$ subgraph of $G^*$. Denote by $G_1$ and $G_2$ the components of the graph $G^* - e$ and let us assume that $u_i$ be a vertex in $G_i$. Let $G^*_i$ be the $\delta$-graph constructed from $G_i$ by gluing at $u_i$ the pendant vertex of a $K^+_\delta$. Since, $e$ is not a part of a pendant $K^+_\delta$ subgraph of $G^*$, it follows that $G^*_1$ and $G^*_2$ have smaller number of vertices than $G^*$. Now, by the minimality, we infer that $\varphi(G^*_1) \geq 1$ and $\varphi(G^*_2) \geq 1$. Denote by $n_i$ and $c_i$ the number of vertices and the number of cut-vertices in $G^*_i$. Then,

$$n = n_1 + n_2 - 2\delta \quad \text{and} \quad c = c_1 + c_2 - 2.$$ 

Now, we obtain a contradiction in the following way

$$\varphi(G^*) = \frac{n}{\delta} - \frac{c}{2} = \frac{n_1 + n_2}{\delta} - \frac{c_1 + c_2}{2} - 1 = \varphi(G^*_1) + \varphi(G^*_2) - 1 \geq 1.$$ 

Thus Claim 2 is proved.
Let $B^*$ be a block of $G^*$. Since $G^*$ is a compact $\delta$-graph, we may assume that $B^*$ is not a head of a pendant $K^+_\delta$ subgraph of $G^*$. We consider the block structure of $G^*$ as a kind of rooted tree, whose root is $B^*$. In other words, we define a function depth : $\mathcal{B} \rightarrow \mathbb{N}$, as it follows. First, set depth($B$) = $\infty$ for every $B \in \mathcal{B}$. Now, apply the following steps until every block gets finite depth:

**Step 0.** depth($B^*$) = 0.

**Step i.** ($i \geq 1$) If $B$ is a block incident with a block whose depth is $i - 1$, then depth($B$) := min (depth($B$), $i$).

If blocks $B_1$ and $B_2$ have common cut-vertex and depth($B_1$) = depth($B_2$) − 1, then we will say that $B_1$ is a parent of $B_2$ and $B_2$ is a son of $B_1$. Note that every block different from $B^*$ has precisely one parent and it may have many sons. In the sequel, we will denote by $n_B$ and $c_B$ the number of vertices and the number of cut-vertices of a block $B$.

We assign a charge $\varphi(B)$ to every block $B \in \mathcal{B}$ in the next way:

$$
\begin{align*}
\varphi(B) &= \begin{cases} 
n_B - \frac{1}{\delta} - \frac{c_B - 1}{2}, & B \neq B^*; 
\frac{n_B}{\delta} - \frac{c_B}{2}, & B = B^*.
\end{cases}
\end{align*}
$$

In fact, we assign charge $\frac{1}{\delta} - \frac{1}{2}$ to every cut-vertex of $G^*$ and $\frac{1}{2}$ to every other vertex of $G^*$. Then, $\varphi(B)$ is the sum of the charges of all of its vertices except the cut-vertex incident with its parent. Note that the total sum of the charges of all blocks (or all vertices) is equal to $\varphi(G^*)$.

Now, we apply to every block the following rule. First, we apply it on the blocks with highest depth, then on the blocks with depth smaller for one, and so on.

**Rule R.** Suppose that $B_1$ is a son of $B_2$ attached at a vertex $v$. Then, $B_1$ sends (through $v$) its charge and the charge received from its sons to $B_2$.

Note that the redistribution will stop at block $B^*$ since it has no parent. The total charge $\varphi(G^*)$ will be accumulated in $B^*$. Denote by $\hat{c}(B, v)$ the charge that a block $B$ receives from its sons attached at the cut-vertex $v$ by Rule R.
Claim 3. Suppose that \( v \) is a cut-vertex of \( G^* \) incident with a block \( B \) and incident also with some sons of \( B \). Then, \( \hat{c}(B, v) \geq \frac{\delta - 1}{\delta} \).

Suppose that the claim is false and the pair \((B, v)\) is a counterexample. We may assume that depth\((B)\) is as large as possible. Suppose also that \( \hat{B} \) is an arbitrary son of \( B \) attached at \( v \). Let us consider the minimal possible value of charge that \( \hat{B} \) could send to \( B \) by Rule R.

Note that every end-block of \( G^* \) has \( \geq \delta \) vertices. So, if \( \hat{B} \) is an end-block, then it sends \( \varphi(\hat{B}) \geq \frac{\delta - 1}{\delta} \) charge to \( B \).

Suppose now that \( \hat{B} \) is a bridge. By Claim 2, \( \hat{B} \) is an edge of \( K^*_\delta \) whose pendant-vertex is \( v \). So, in this case \( \hat{B} \) sends

\[
\frac{\delta - 1}{\delta} + \left( \frac{1}{\delta} - \frac{1}{2} \right) = \frac{1}{2}
\]

charge to \( B \).

Finally, we may assume that \( \hat{B} \) is neither an end-block nor a bridge of \( G^* \). Note that \( c_{\hat{B}} \geq 2 \). By the maximality of the depth of \( B \), we infer that \( \hat{B} \) sends at least

\[
\frac{n_{\hat{B}} - 1}{\delta} - \frac{c_{\hat{B}} - 1}{2} + (c_{\hat{B}} - 1) \frac{\delta - 1}{\delta}
\]

charge to \( B \). If \( c_{\hat{B}} \geq 3 \), then by (3) and by \( n_{\hat{B}} \geq c_{\hat{B}} \), we infer \( \hat{c}(B, v) \geq (c_{\hat{B}} - 1) \frac{\delta - 1}{\delta} \geq 1 \). So, let \( c_{\hat{B}} = 2 \). Since \( \hat{B} \) is not a bridge, there is a vertex \( \hat{v} \) of \( \hat{B} \) which is not a cut-vertex of \( G^* \). Since \( \hat{v} \) is of degree \( \delta - 1 \) in \( \hat{B} \), it follows that \( \hat{B} \) has at least \( \delta \) vertices, i.e., \( n_{\hat{B}} \geq \delta \). Thus, by (3) and by \( \delta \geq 2 \), we obtain that \( \hat{B} \) sends charge to \( B \) at least

\[
2 \left( \frac{\delta - 1}{\delta} \right) - \frac{1}{2} \geq \frac{\delta - 1}{\delta}.
\]

By above, if \( B \) has a son which is not a bridge attached at \( v \) then \( \hat{c}(B, v) \geq \frac{\delta - 1}{\delta} \). So assume that all sons of \( B \) attached at vertex \( v \) are bridges. Then, by Claims 1 and 2 and by the choice of \( B^* \), it follows that \( k \geq 2 \), and hence \( \hat{c}(B, v) \geq 1 \). Thus Claim 3 is proved.

Using Claim 3, we will prove that \( \varphi(G^*) \geq 1 \) in a similar way as we argue above. Note that

\[
\varphi(G^*) \geq \varphi(B^*) + c_{B^*} \cdot \frac{\delta - 1}{\delta} \geq \frac{n_{B^*}}{\delta} - \frac{c_{B^*}}{2} + c_{B^*} \cdot \frac{\delta - 1}{\delta}.
\]
If $c_{B^*} \geq 2$, then by (4), we obtain $\varphi(G^*) \geq \frac{c_{B^*}}{2} \geq 1$. Thus let us assume that $c_{B^*} = 1$. In this case, $B^*$ is an end-block and so $n_{B^*} \geq \delta$. Thus we infer that $\varphi(G^*) \geq \delta - \frac{1}{2} + \frac{\delta - 1}{\delta} \geq 1$. This completes the proof of the theorem. ■

The following result is a generalization of Gallai’s technical lemma [6, Lemma 4.5].

**Corollary 4.** Let $G$ be a graph with $n$ vertices, $m$ edges and let $\delta \geq 1$. Suppose that $\Delta(G) \leq \delta$ and each block $B$ of $G$ has maximum degree $\Delta(B) \leq \delta$. Then,

$$m \leq \left(\frac{\delta - 1}{2} + \frac{1}{\delta}\right)n - 1.$$  

**Proof.** Let us remark, that if $G$ is 2-connected, then $\Delta(G) \leq \delta - 1$. Let $G^*$ be the graph constructed from $G$ in the following way: at every cut-vertex $v \in V(G)$ glue $\delta - d(v)$ copies of pendant $K^+_\delta$ and at every other vertex of degree $\delta - 1$ glue also $\delta - d(v)$ copies of pendant $K^+_\delta$. Note that $G^*$ is a $\delta$-graph.

Suppose that we have added $k$ copies of $K^+_\delta$ in $G$ in order to construct $G^*$. Denote by $n^*$, $c^*$, and $m^*$ the number of vertices, the number of cut-vertices, and the number of edges of $G^*$, respectively. By Theorem 3, $\frac{c^*}{2} \leq \frac{n^*}{\delta} - 1$. Then,

$$m^* = \frac{(\delta - 1)}{2}n^* - c^* = \frac{(\delta - 1)n^*}{2} + \frac{c^*}{2} \leq \frac{(\delta - 1)n^*}{2} + \frac{n^*}{\delta} - 1 = \left(\frac{\delta - 1}{2} + \frac{1}{\delta}\right)n^* - 1.$$

Thus we have proved the claim for $G^*$. Since,

$$n^* = n + \delta k \quad \text{and} \quad m^* = m + k\left(\frac{\delta}{2}\right) + k,$$

we have

$$m = m^* - k\left(\frac{\delta}{2}\right) - k \leq \left(\frac{\delta - 1}{2} + \frac{1}{\delta}\right)n^* - 1 - k\left(\frac{\delta}{2}\right) - k \leq \left(\frac{\delta - 1}{2} + \frac{1}{\delta}\right)n - 1. \quad \square$$
3. Gallai’s Inequality

Gallai [6] proved that a $k$-critical graph ($k \geq 4$) on $n$ vertices and $m$ edges, different from $K_k$ satisfies the following inequality

$$2m \geq (k-1)n + \frac{k-3}{k^2-3}n.$$ 

This classical result was later improved by Krivelevich [11, 12] and Kostochka and Stiebitz [9, 10]. See also the book of Jansen and Toft [7] for critical graphs with few edges.

**Theorem 5.** Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ ($k \geq 2$) be additive induced-hereditary properties, $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \mathcal{P}_k$ and $\delta = \sum_{i=1}^{k} \delta(\mathcal{P}_i)$. Suppose that $G$ is an $\mathcal{R}$-critical graph with $n$ vertices and $m$ edges. Then

$$2m \geq \delta n + \frac{\delta - 2}{\delta^2 + 2\delta - 2} n + \frac{2\delta}{\delta^2 + 2\delta - 2}$$

unless $\mathcal{R} = \mathcal{O}^2$ or $G = K_{\delta+1}$.

**Proof.** Obviously, if $G = K_{\delta+1}$ then (6) is not satisfied. It is easy to see that if $\mathcal{R} = \mathcal{O}^2$, then $G$ is an odd cycle. In this case inequality (6) is also not satisfied. So, assume that $\mathcal{R} \neq \mathcal{O}^2$ and $G \neq K_{\delta+1}$. It is easy to see, that inequality (6) is satisfied for $\delta = 2$, since a cycle can be critical only for $\mathcal{R} = \mathcal{O}^2$. Hence we infer that $\delta \geq 3$. Denote by $H$ the subgraph of $G$ induced by the minor vertices i.e., vertices of degree $\delta$. Let $n'$ and $m'$ be the number of vertices and the number of edges of $H$. It is not hard to see that

$$m \geq \delta n' - m'.$$

Since $\mathcal{R} \neq \mathcal{O}^2$, by Theorem 1 it follows that $\Delta(H) \leq \delta$ and each block $B$ of $H$ has $\Delta(B) < \delta$.

$$m \geq \delta n' - \left(\frac{\delta - 1}{2} + \frac{1}{\delta}\right)n' + 1 \geq \left(\frac{\delta + 1}{2} - \frac{1}{\delta}\right)n' + 1.$$ 

Similarly, the following is satisfied

$$2m \geq \delta n' + (\delta + 1)(n - n') = (\delta + 1)n - n'.$$
After multiplying (9) by \((\frac{\delta+1}{2} - \frac{1}{\delta})\) and adding it to (8), we obtain:

\[
(\delta + 2 - \frac{2}{\delta})m \geq (\delta + 1)(\frac{\delta + 1}{2} - \frac{1}{\delta})n + 1.
\]

Finally, from (10) by some calculations, we easily obtain (6).

Remark that a special case of the above theorem was proved in [14]. Also remark, that Corollary 4 is a generalization of the Gallai’s technical lemma [6, Lemma 4.5].

Using Theorem 2, the same arguments give us the \(P\)-choice version of Gallai’s inequality (as it is mentioned for \(P = O\) in [8]):

**Theorem 6.** Let \(P\) be additive induced-hereditary property and let \(k \geq 2\). Let \(G\) be a \((P, k+1)\)-choice critical graph, with \(n\) vertices and \(m\) edges and \(\delta = \delta(P)k\). Then

\[
2m \geq \delta n + \frac{\delta - 2}{\delta^2 + 2\delta - 2}n + \frac{2\delta}{\delta^2 + 2\delta - 2}
\]

unless \(R = O^2\) or \(G = K_{\delta+1}\).

Let us finish the paper with the following problem. Dirac [1] proved that for every \(k\)-critical graph \(G \neq K_k\) \((k \geq 3)\) on \(n\) vertices for the number of edges \(m\) the following inequality holds:

\[
2m \geq (k - 1)n + (k - 3).
\]

So an interesting problem is to generalize the above inequality for reducible additive induced-hereditary properties of graphs.

**References**


Gallai’s Inequality for Critical Graphs ...


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