In this paper we propose clustering methods based on weighted quasiarithmetic means of $T$-transitive fuzzy relations. We first generate a $T$-transitive closure $R^T$ from a proximity relation $R$ based on a max-$T$ composition and produce a $T$-transitive lower approximation or opening $R_T$ from the proximity relation $R$ through the residuation operator. We then aggregate a new $T$-indistinguishability fuzzy relation by using a weighted quasiarithmetic mean of $R^T$ and $R_T$. A clustering algorithm based on the proposed $T$-indistinguishability is thus created. We compare clustering results from three critical $t_i$-indistinguishabilities: minimum ($t_3$), product ($t_2$), and Lukasiewicz ($t_1$). A weighted quasiarithmetic mean of a $t_1$-transitive closure $R^{t_1}$ and a $t_1$-transitive lower approximation or opening $R_{t_1}$ with the weight $p = 0.5$, demonstrates the superiority and usefulness of clustering begun by using a proximity relation $R$ based on the proposed clustering algorithm. The algorithm is then applied to the practical evaluation of the performance of higher education in Taiwan.

Keywords: Clustering; fuzzy relation; $T$-transitive closure; max-$T$ composition; representation theorem; $T$-indistinguishability; performance evaluation.
1. Introduction

Clustering, one of the major tools in pattern recognition, has become an effective technique used in data analysis. Cluster analysis is a method for finding clusters within a data set. Data points within a cluster bear the most similarity with each other and the most dissimilarity to those of different clusters. Since Zadeh introduced fuzzy sets, which involves the concept of partial memberships of multiple clusters described according to membership functions, fuzzy clustering has been widely studied and applied in various substantive areas (see Baraldi and Blonda, Bezdek, Hoppner et al., Yang). Fuzzy clustering methods can be divided into two categories: methods based on an objective function of dissimilarities/similarities, and methods based on $T$-indistinguishabilities by using fuzzy matrix compositions based on a fuzzy relation matrix. Objective-function-based clustering methods, such as the fuzzy c-means (FCM) clustering algorithm and its variations (relational FCM, possibilistic c-means (PCM), alternative FCM, partial supervision FCM and generalized FCM), are well-known and widely used. $T$-indistinguishability-based fuzzy clustering has been comparatively less studied. This paper presents the findings of investigating $T$-indistinguishability fuzzy clustering methods.

Zadeh defined a fuzzy relation $R$ between two sets $X$ and $Y$ as a fuzzy subset of $X \times Y$ by extension to allow a membership function to assume values in the interval $[0,1]$. For any given $t$-norm $T$, Valverde constructed $T$-indistinguishability operators exhibiting properties for fuzzy relations. Valverde also proposed the Representation Theorem, which states that any $T$-indistinguishability operator on a set $X$ can be generated by a family of fuzzy subsets of $X$. Boixader and Recasens considered indistinguishability operators with respect to different $t$-norms. Tamura et al. constructed an $n$-step procedure by using the max-min composition of fuzzy relations and beginning with a proximity relation. A max-min similarity relation can be obtained after the $n$-step max-min composition is performed, enabling multilevel hierarchical clustering based on this
max-min similarity relation. Yang and Shih\textsuperscript{23} extended the procedure used by Tamura to all types of max-$T$ compositions to obtain max-$T$ similarity relations. Yang and Shih then proposed a clustering algorithm based on this max-$T$ similarity relation. Recently, Garmendia and Recasens\textsuperscript{8} introduced a method to design $T$-transitives based on proximity relations.

In this paper, we advance these techniques and propose clustering methods based on a weighted quasiarithmetic mean of $T$-transitive fuzzy relations. We first generate a $T$-transitive fuzzy relation $R^T$ from a proximity relation $R$ based on a max-$T$ composition, and then induce a $T$-transitive lower approximation or opening $R_T$ from the proximity relation $R$ through the residuation operator. A new $T$-indistinguishability is derived by aggregating the weighted average of $R^T$ and $R_T$. A clustering algorithm based on that $T$-indistinguishability is then proposed. The remainder of this paper is organized as follows. In Section 2, $T$-indistinguishability fuzzy relations and their properties are considered. Section 3, proposes a new $T$-indistinguishability-based clustering algorithm. Section 4 presents two examples comparing clustering results from the three critical $t$-indistinguishabilities: minimum ($t_3$), product ($t_2$), and Lukasiewicz ($t_1$). In Section 5, we apply the proposed method to evaluate the performance of higher education in Taiwan. Finally, the conclusion is stated in Section 6.

2. $T$-indistinguishability fuzzy relations and properties

A fuzzy (binary) relation $R$ between two sets $X$ and $Y$, denoted by $R(X,Y)$, is defined as a fuzzy subset of $X \times Y$ (see Zadeh\textsuperscript{26}). The relation $R(X,Y)$ is associated with a membership function $\mu_R(x,y)$, assuming values in the interval $[0,1]$ for all $(x,y)$ in $X \times Y$. The value of $\mu_R(x,y)$ represents the strength of the relationship between $x$ and $y$. In cluster analysis, we are interested only in relations on a single set $X$, i.e. $R(X) = R(X,X)$, a map from $X \times X$ into $[0,1]$.

The $t$-norm is defined as a general form of the fuzzy intersection where Zimmermann\textsuperscript{27} listed some specified $t$-norms as follows:
1. \( t_\omega(x, y) = \begin{cases} 
\min\{x, y\} & \text{if } \max\{x, y\} = 1, \\
0 & \text{otherwise.} 
\end{cases} \) (drastic product);

2. \( t_1(x, y) = \max\{0, x + y - 1\} \) (bounded difference);

3. \( t_{1.5}(x, y) = \frac{xy}{2 - (x + y - xy)} \) (Einstein product);

4. \( t_2(x, y) = xy \) (algebraic product);

5. \( t_{2.5}(x, y) = \frac{xy}{x + y - xy} \) (Hamacher product);

6. \( t_3(x, y) = \min\{x, y\} \) (minimum).

In the above list, the t-norms \( t_1 \), \( t_2 \), and \( t_3 \) are most commonly used. The \( t_1 \)-norm is the Lukasiewicz t-norm, the \( t_2 \)-norm is the product t-norm, and the \( t_3 \)-norm is the minimum t-norm. A fuzzy relation \( R \) in a finite universe \( X \) generally has reflexive and symmetric properties defined as follows.

**Definition 1.** (Proximity relation) A fuzzy relation \( R \) on a set \( X \) is called a proximity relation if it satisfies

1. (reflexivity) \( \mu_R(x, x) = 1 \) \( \forall x \in X \), and
2. (symmetry) \( \mu_R(x, y) = \mu_R(y, x) \) \( \forall x, y \in X \).

In real cases, a proximity relation is not applicable. A \( T \)-transitivity property for a proximity relation \( R \) is required for real applications.

**Definition 2.** (\( T \)-transitivity) A fuzzy relation \( R \) on a set \( X \) is called \( T \)-transitivity if

\[
\mu_R(x, z) \geq T(\mu_R(x, y), \mu_R(y, z))
\]

for all \( x, y, z \in X \), where \( T \) stands for a \( t \)-norm.

**Definition 3.** (Similarity, Zadeh\textsuperscript{26}) A similarity is a reflexive, symmetric and minimum-transitive fuzzy relation.
Definition 4. (*T*-indistinguishability, Trillas and Valverde\(^{19}\)) Given a \(t\)-norm \(T\), a fuzzy relation \(E\) on a set \(X\) is called a \(T\)-indistinguishability if it satisfies

1. (reflexivity) \(\mu_E(x, x) = 1\) \(\forall x \in X\);
2. (symmetry) \(\mu_E(x, y) = \mu_E(y, x)\) \(\forall x, y \in X\);
3. (\(T\)-transitivity) \(\mu_E(x, z) \geq T(\mu_E(x, y), \mu_E(y, z))\) \(\forall x, y, z \in X\).

Obviously, \(T\)-indistinguishability \(E\) is a similarity relation when \(T\) is the minimum \(t\)-norm.

Definition 5. (*\(T\*-transitive closure, Naessens et al.\(^{16}\)*) Given a \(t\)-norm \(T\) and a fuzzy relation \(R\) on a set \(X\). The \(T\)-transitive closure of \(R\) is the smallest \(T\)-transitive fuzzy relation \(R^T\) on \(X\) satisfying \(R \leq R^T\).

Definition 6. (*max-\(T\) composition, Yang and Shih\(^{23}\)*) Given a \(t\)-norm \(T\) and an initial fuzzy relation \(R^{(0)} = [r_{ij}^{(0)}]\) defined on a finite set \(X\) with values \(r_{ij}^{(0)} = \mu_{R^{(0)}}(x_i, x_j)\). Then \(R^{(n)} = [r_{ij}^{(n)}]\) with \(r_{ij}^{(n)} = \max_k \{T(r_{ik}^{(n-1)}, r_{kj}^{(n-1)})\}\), \(n = 1, 2, 3, \ldots\) is called a max-\(T\) composition.

Theorem 1. (*An n-step procedure, Yang and Shih\(^{23}\)*) Suppose that \(R^{(0)}\) is a proximity relation on a finite set \(X\). Then, by a max-\(T\) composition, one has

\[
I < R^{(0)} < R^{(1)} < \ldots < R^{(n)} = R^{(n+1)} = \ldots,
\]

where \(R^{(n)}\) is a \(T\)-indistinguishability. If \(n\) is not finite, then \(\lim_{n \to \infty} R^{(n)} = R^{(\infty)}\) with \(R^{(\infty)}\) a \(T\)-indistinguishability, i.e.

\[
I < R^{(0)} < R^{(1)} < \ldots < R^{(n)} < R^{(n+1)} < \ldots < R^{(\infty)}.
\]

In Theorem 1, Yang and Shih claimed that a \(T\)-indistinguishability can be obtained after using an \(n\)-step procedure of a max-\(T\) composition for a proximity relation. However, Yang and Shih did not claim that \(n\) is finite when the initial proximity relation is defined
on a finite set $X$. We next claim that given an initial proximity relation $R^{(0)}$ defined on a finite set $X$ of cardinality $m$, after an $n$-step procedure of a max-$T$ composition is used for $R^{(0)}$, it is possible to obtain a $T$-indistinguishability at most the $(m - 1)$ finite step, and the obtained $T$-indistinguishability should be a $T$-transitive closure.

**Theorem 2.** Let $R^{(0)} = [r_{ij}^{(0)}]_{m \times m}$ be a proximity relation defined on a finite set $X$ of cardinality $m$. Then, by a max-$T$ composition, one has $I < R^{(0)} < R^{(1)} < \cdots < R^{(n)} = R^{(n+1)} = \cdots$, where $n \leq m - 1$. Furthermore, the $T$-indistinguishability $R^{(n)}$ is also a $T$-transitive closure of $R^{(0)}$.

**Proof.** Recall that $R^{(n)} = [r_{ij}^{(n)}]$ with $r_{ij}^{(n)} = \max_k \{T(r_{ik}^{(n-1)}, r_{kj}^{(n-1)})\}$, $n = 1, 2, 3, \ldots$ Now, we define a new term $R^{(n)} = [r_{ij}^{(n)}]$ with $r_{ij}^{(n)} = \max_k \{T(r_{ik}^{(n-1)}, r_{kj}^{(n)})\}$, $n = 1, 2, 3, \ldots$, where $r_{ij}^{(0)} = r_{ij}^{(0)}$. Due to the associativity of $T$, we have that $R^{(0)} = R^{(0)}, R^{(1)} = R^{(1)}, R^{(2)} = R^{(3)}, R^{(3)} = R^{(7)}, \ldots, R^{(n)} = R^{(2n-1)}$ and so forth. For $R^{(0)}, R^{(1)}, \ldots, R^{(n)}$, Naessens et al.$^{16}$ and Campo et al.$^5$ had given the results that, if $R^{(0)}$ is a proximity relation on a finite set $X$ of cardinality $m$, we have $R^{(0)} \leq R^{(1)} \leq \ldots \leq R^{(n-1)} \leq R^{(n)} \leq \ldots$, and $\sup_{n \in \{0, 1, 2, \ldots\}} \{R^{(n)}\} = R^{(m-1)}$. Thus, $\forall n \geq m - 1, R^{(m-1)} \leq R^{(n)} \leq R^{(n)} = R^{(m-1)}$, i.e. we have that $R^{(n)} = R^{(m-1)}$, $\forall n \geq m - 1$. Since $R^{(n)} = R^{(2n-1)}$, we have that $R^{(n)} = R^{(n+1)} = \ldots$, as $2^n - 1 \geq m - 1$, i.e. $2^n \geq m$. Because of $2^{(m-1)} \geq m$, $\forall m \in N$, we can obtain that $n \leq m - 1$. On the other hand, by Naessens et al.$^{16}$, $R^{(m-1)} = R^{(m-1)} = R^{(n)} = R^{(n)}$ should be a $T$-transitive closure of $R^{(0)}$. That is, $R^{(n)}$ is a $T$-transitive closure of $R^{(0)}$, and the proof is completed. $\square$

From Theorem 2, we know that there exists a finite $n$-step procedure for a finite set $X$ using a max-$T$ composition to produce a $T$-transitive closure by starting from a proximity relation $R$. The algorithm of the $n$-step procedure for obtaining a $T$-transitive closure of $R$ is as follows.

**Algorithm 1**
Let $T$ be a $t$-norm and $R = [r_{ij}]$ be a proximity relation defined on a set $X = \{x_1, x_2, \ldots\}$ with values $r_{ij} = \mu_R(x_i, x_j)$.

**Input:** a proximity relation $R = [r_{ij}]$.

**Output:** a $T$-indistinguishability $E = [e_{ij}]$ satisfying $E \geq R$.

The algorithm is the following:

s1. Let all the diagonal elements of $E$ are equal to 1, i.e. $e_{ii} = 1$ for all $i$.

s2. Let $e_{ij} = e_{ji} = \max_k T(r_{ik}, r_{kj})$ for all $i < j$.

s3. If $E = R$, then stop.

Else let $R = E$ and GOTO s2.

**Proposition 1.** Let $R$ be a fuzzy relation on $X$ and $R^{t_i}$ is the $t_i$-transitive closure of $R$, $i = 1, 2, 3$. Then $R \leq R^{t_1} \leq R^{t_2} \leq R^{t_3}$.

**Proof.** Let $\Omega_{t_i}$ be the set of $t_i$-transitive relations which are greater than or equal to $R$, $i = 1, 2, 3$. Because $R^{t_i}$ be the $t_i$-transitive closure of $R$, then $R^{t_i}(x, y) = \inf_{Q \in \Omega_{t_i}} \{Q(x, y)\}$ $\forall x, y \in X$, $i = 1, 2, 3$. By $t_3 \geq t_2 \geq t_1$, one implies $\Omega_{t_3} \subseteq \Omega_{t_2} \subseteq \Omega_{t_1}$ and $R(x, y) \leq \inf_{Q \in \Omega_{t_1}} \{Q(x, y)\} \leq \inf_{U \in \Omega_{t_2}} \{U(x, y)\} \leq \inf_{V \in \Omega_{t_3}} \{V(x, y)\} \forall x, y \in X$. Thus, we prove that $R \leq R^{t_1} \leq R^{t_2} \leq R^{t_3}$.

**Definition 7.** *(T-transitive lower approximation, Garmendia et al.*) Given a $t$-norm $T$ and a fuzzy relation $R$ on a set $X$, a $T$-transitive lower approximation of $R$ is a $T$-transitive fuzzy relation $A$ satisfying $R \geq A$.

**Definition 8.** *(T-transitive opening, Garmendia et al.*) Given a $t$-norm $T$ and a fuzzy relation $R$ on a set $X$, a $T$-transitive opening of $R$ is a relation $R_T$ satisfying:

1. $R_T$ is a $T$-transitive fuzzy relation.
2. $R_T \leq R$.
3. If $\exists H$ $T$-transitive; $R_T \leq H \leq R$ then $H = R_T$.  

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According to Definitions 7 and 8, a $T$-transitive opening of $R$ is the largest $T$-transitive lower approximation, and a fuzzy relation $R$ can have an infinite number of $T$-transitive openings even in a finite universe. In Garmendia et al.$^{7,9}$, they also proposed some algorithms to compute a $T$-transitive opening of a proximity relation.

**Definition 9.** (The residuation of $T$, Trillas and Valverde$^{19}$, Garmendia et al.$^{6}$) The residuation (or quasi-inverse) $\overrightarrow{T}$ of a $t$-norm $T$ is a function from $[0, 1] \times [0, 1]$ into $[0, 1]$ defined as follows:

$$\overrightarrow{T}(x|y) = \sup\{\alpha \in [0, 1] \mid T(x, \alpha) \leq y\}.$$ 

According to Definition 9, one has

1. $\overrightarrow{t_1}(x|y) = \begin{cases} 1 & \text{if } x \leq y, \\ 1 - x + y & \text{if } x > y, \end{cases}$

2. $\overrightarrow{t_2}(x|y) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{y}{x} & \text{if } x > y, \end{cases}$

3. $\overrightarrow{t_3}(x|y) = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x > y. \end{cases}$

**Theorem 3.** (Representation Theorem, Valverde$^{20}$) Let $R$ be a fuzzy relation on a set $X$ and let $T$ be a $t$-norm. Then $R$ is a $T$-indistinguishability on the set $X$ if and only if there exists a family \{\(h_j\)\}_{j \in J} of fuzzy subsets of $X$ such that for all $x, y \in X$

$$\mu_R(x, y) = \inf_{j \in J} T\left(\overrightarrow{T}(h_j(x)|h_j(y)), \overrightarrow{T}(h_j(y)|h_j(x))\right).$$

**Corollary 1.** Let $R$ be a proximity relation on a set $X$ and \{\(h_j = \mu_R(\cdot, j)\)\}_{j \in X} the $j$th column of $R$. The fuzzy relation $E$ defined for all $x, y \in X$ by

$$\mu_E(x, y) = \inf_{j \in X} T\left(\overrightarrow{T}(h_j(x)|h_j(y)), \overrightarrow{T}(h_j(y)|h_j(x))\right)$$

is a $T$-indistinguishability smaller or equal than $R$ (i.e. $E \leq R$).
Based on the corollary above, given a $t$-norm $T$ and a proximity relation $R$ on $X$, we can calculate a $T$-transitive lower approximation of $R$, but this method does not always yield a $T$-transitive opening. Garmendia et al.\textsuperscript{9} proposed an algorithm to compute a $T$-transitive lower approximation or opening from a proximity relation $R$, and the computed approximation is expected to be a $T$-transitive opening while preserving the properties of reflexivity and symmetry. We state the algorithm as follows.

**Algorithm 2** (Garmendia et al.\textsuperscript{9})

Let $R = [r_{ij}]_{n \times n}$ be a proximity relation defined on a universe $X = \{x_1, x_2, \ldots, x_n\}$ with values $r_{ij} = \mu_R(x_i, x_j)$, let $T$ be a left-continuous $t$-norm, and let $\overrightarrow{T}$ be the corresponding residuation.

**Input:** a proximity relation $R = [r_{ij}]_{n \times n}$.

**Output:** a $T$-indistinguishability $B = [b_{ij}]_{n \times n}$ satisfying $B \leq R$.

The algorithm is the following:

s1. Set $B$ initially blank.

s2. Let $U(R)$ be the list of elements of the upper triangular matrix of $R$ sorted in decreasing order.

s3. Set $b_{ii} = 1$ for all $1 \leq i \leq n$.

s4. While there is a blank in $B$ do

\{ Let $r_{st}$ be the first element in $U(R)$.

If $b_{st}$ is blank do,

\{ Let $I = \{j \mid b_{sj}$ is not blank in $B$\} and $I' = \{i \mid b_{it}$ is not blank in $B$\}.

Let $H(R)$ be the list of elements $r_{ij}, i \in I, j \in I'$ sorted in increasing order.

While $H(R)$ is not empty do

\{ Let $r_{ij}$ be the first element in $H(R)$ such that $b_{ij}$ is blank.

Set $b_{ij} = b_{ji} = \min \{ r_{ij}, \min_k \{ \overrightarrow{T}(b_{ik}, b_{kj}), \overrightarrow{T}(b_{jk}, b_{ki}) \} \}$.

Delete the first element from $H(R)$.

9
Delete the first element from $U(R)$.

Garmendia et al.\textsuperscript{9} claimed that lists of $U(R)$ and $H(R)$ are not always unique. Occasionally, the highest (lowest) values are repeated, and one edge must be arbitrarily chosen as the highest (lowest). However, for a given list, the Garmendia algorithm provides a unique $T$-transitive lower approximation. Therefore, in some cases, the algorithm can be used to find several $T$-transitive lower approximations or openings.

\textbf{Definition 10.} (Weighted quasi-arithmetic mean $m_{f}^{p,q}$, Aczéľ\textsuperscript{1}, Klement\textsuperscript{14}) Given a continuous monotonic map $f : [0, 1] \rightarrow [-\infty, \infty]$ and $p, q$ positive values with $p + q = 1$, the weighted quasi-arithmetic mean $m_{f}^{p,q}$ generated by $f$ and weights $p$ and $q$ is defined for all $x, y \in [0, 1]$ by

$$m_{f}^{p,q}(x, y) = f^{-1}(p \cdot f(x) + q \cdot f(y))$$

where $m_{f}^{p,q}$ is continuous if and only if $\text{Range}(f) \neq [-\infty, \infty]$.

\textbf{Definition 11.} (Archimedean t-norm) Let $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is an Archimedean $t$-norm iff there exists an additive generator $f$, which is a decreasing bijection $f : [0, 1] \rightarrow [0, B]$ ($B \in (0, \infty]$) such that

$$T(x, y) = \begin{cases} f^{-1}(f(x) + f(y)) & \text{if } f(x) + f(y) \leq B, \\ 0 & \text{otherwise.} \end{cases}$$

The $t_1$-norm and $t_2$-norm are both Archimedean $t$-norms, and

1. If $T$ is the $t_1$-norm, then $f(x) = 1 - x$.

2. If $T$ is the $t_2$-norm, then $f(x) = -\ln x$.

We mention that Jacas and Recasens\textsuperscript{13} considered the properties of the aggregation of $T$-transitive fuzzy relations. In Garmendia and Recasens\textsuperscript{8}, they gave the proposition
(i.e. Proposition 7 in Garmendia and Recasens\textsuperscript{8}) that, for an Archimedean $T$-norm $T$, any weighted quasiarithmetic mean $m_f^{p,1-p}$ of two $T$-indistinguishabilities $A$ and $B$ is still a $T$-indistinguishability. Garmendia and Recasens\textsuperscript{8} did not provide the proof of the proposition and cited the paper of Jacas and Recasens\textsuperscript{13}. However, we cannot find enough information and the proof from Jacas and Recasens\textsuperscript{13}. In this sense, we provide the proof of the proposition as follows.

**Theorem 4.** (Garmendia and Recasens\textsuperscript{8}) Let $T$ be an Archimedean $t$-norm with additive generator $f$, $p \in [0,1]$ and let $A$ and $B$ be two $T$-indistinguishabilities on $X$ of cardinality $n$. The weighted quasiarithmetic mean $m_f^{p,1-p}$ with weights $p$ and $1-p$ of $A$ and $B$ is a $T$-indistinguishability.

**Proof.** Let $A = [a_{ij}]_{n \times n}$, $B = [b_{ij}]_{n \times n}$. Because $f$ is an additive generator with respect to $T$ in which $A$ and $B$ are two $T$-indistinguishabilities, $f$ should be a decreasing function with $f(1) = 0$ and $f^{-1}(0) = 1$. Furthermore, by the property of $T$-indistinguishability, we have that

$(1^*) a_{ii} = b_{ii} = 1, \forall i = 1, 2, \ldots, n$;

$(2^*) a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}, \forall i \neq j$;

$(3^*)$ Since $a_{ij} \geq f^{-1}(f(a_{ik}) + f(a_{kj}))$ and $b_{ij} \geq f^{-1}(f(b_{ik}) + f(b_{kj})) \forall i \neq j \neq k$

and $f$ is decreasing, we have that $f(a_{ij}) \leq f(a_{ik}) + f(a_{kj})$ and $f(b_{ij}) \leq f(b_{ik}) + f(b_{kj})$.

We next prove the reflexivity, symmetry and $T$-transitivity properties as follows.

(reflexivity): $m_f^{p,1-p}(a_{ii}, b_{ii}) = f^{-1}(pf(a_{ii}) + (1-p)f(b_{ii}))$

$= f^{-1}(pf(1) + (1-p)f(1)) = f^{-1}(0) = 1.$

(symmetry): $m_f^{p,1-p}(a_{ij}, b_{ij}) = f^{-1}(pf(a_{ij}) + (1-p)f(b_{ij}))$

$= f^{-1}(pf(a_{ji}) + (1-p)f(b_{ji})) = m_f^{p,1-p}(a_{ji}, b_{ji}).$

($T$-transitivity): $T(m_f^{p,1-p}(a_{ik}, b_{ik}), m_f^{p,1-p}(a_{kj}, b_{kj}))$

$= T(f^{-1}(pf(a_{ik}) + (1-p)f(b_{ik})), f^{-1}(pf(a_{kj}) + (1-p)f(b_{kj})))$

$= f^{-1}(f(f^{-1}(pf(a_{ik}) + (1-p)f(b_{ik}))) + f(f^{-1}(pf(a_{kj}) + (1-p)f(b_{kj}))))$
\[
= f^{-1}(pf(a_{ik}) + (1 - p)f(b_{ik}) + pf(a_{kj}) + (1 - p)f(b_{kj})) \\
= f^{-1}(p(f(a_{ik}) + f(a_{kj})) + (1 - p)(f(b_{ik}) + f(b_{kj}))) \\
\leq f^{-1}(pf(a_{ij}) + (1 - p)f(b_{ij})) \quad \text{(by } f \text{ is decreasing and } 3^*) \\
= m_p^{1-p}(a_{ij}, b_{ij}).
\]

Given an Archimedean \( t \)-norm \( T \) with additive generator \( f \), and a proximity relation \( R \) on a set \( X \), we can calculate its \( T \)-transitive closure \( R^T \) and a \( T \)-transitive lower approximation or opening \( R_T \) based on Algorithm 1 and Algorithm 2, respectively. Because \( R^T \) and \( R_T \) are both \( T \)-indistinguishabilities, according to Theorem 4, it is possible to construct a new \( T \)-indistinguishability \( E \) between \( R^T \) and \( R_T \) by using the weighted quasiarithmetic mean \( m_p^{1-p} \) with weights \( p \) and \( 1 - p \) for \( R^T \) and \( R_T \), and \( E = R^T \) as \( p = 1 \) and \( E = R_T \) as \( p = 0 \). However, the weighted quasiarithmetic mean method used for Archimedean \( t \)-norms cannot be applied for the minimum \( t \)-norm. Garmendia et al.\(^7\) provided a simple algorithm to obtain the min-transitive closure, a min-transitive opening and a “closer” min-transitive approximation (using the distance between fuzzy relations) of a proximity simultaneously, and also proved that the outputs of the algorithm are similarities. We state the algorithm as follows.

**Algorithm 3** (Garmendia et al.\(^7\))

Let \( R = [r_{ij}]_{n \times n} \) be a proximity relation defined on a universe \( X = \{x_1, x_2, \ldots, x_n\} \) with values \( r_{ij} = \mu_R(x_i, x_j) \). Let us call node to a subset of \( X \). In order to make an easier notation, we consider the elements of \( X \) by their natural number of their position.

**Input:** a proximity relation \( R = [r_{ij}]_{n \times n} \).

**Output:** three similarities: the min-transitive closure \( A = [a_{ij}]_{n \times n} \), a min-transitive opening \( B = [b_{ij}]_{n \times n} \) and a min-transitive approximation \( C = [c_{ij}]_{n \times n} \) of \( R \).

The algorithm is the following:

s1. Create a set of nodes \( N \) initially with a set of singletons \( N_i = \{x_i\} \) for each element \( x_i \) in \( X \).
s2. Set $a_{ii} = 1, b_{ii} = 1$ and $c_{ii} = 1$ for all $1 \leq i \leq n$.

s3. While $N$ is not the universe $X$ do

\begin{verbatim}
Compute $m(N_i, N_j) = \max_{i \in N_i, j \in N_j} r_{ij}$ for all pair of nodes $N \times N$ with $i \neq j$.

Record $(i, j)$ where $m(N_i, N_j)$ is maximal.

Set $a_{st} = a_{ts} = \max_{i \in N_i, j \in N_j} r_{ij}$ for all $s \in N_i$ and $t \in N_j$.

Set $b_{st} = b_{ts} = \min\{\min_{i \in N_i, j \in N_j} r_{ij}, \min_{k,l \in N_i} b_{kl}, \min_{k,l \in N_j} b_{kl}\}$ for all $s \in N_i$ and $t \in N_j$.

Set $c_{st} = c_{ts} = \min\{\bigoplus_{i \in N_i, j \in N_j} r_{ij}, \min_{k,l \in N_i} c_{kl}, \min_{k,l \in N_j} c_{kl}\}$ for all $s \in N_i$ and $t \in N_j$,

where $\bigoplus$ is an aggregation operator, for example, the arithmetic mean.

Delete nodes $N_i$ and $N_j$ from $N$.

Insert $N_i \cup N_j$ into $N$.
\end{verbatim}

The given algorithm offers a tool for approximating fuzzy proximities exhibiting similarities, because the user can determine which type of min-transitive approximation (closure, opening or just the closest approximation) fits different applications based on computational cost (see Garmendia et al.\textsuperscript{7}).

3. Clustering based on $T$-indistinguishabilities

Subjective human experiences often provides essential information in real applications, such as visual images, smells, and pictures. This subjective information may be represented by a proximity relation $R$ (i.e., a reflexive and symmetric fuzzy relation) rather than a distance measure on feature vectors. In such cases, clustering must begin with a proximity relation $R$. Direct clustering on the proximity relation $R$, results in a partition assembled using only a maximum relation. This is not an ideal result, because a proximity relation $R$ does not have transitivity.

Given a proximity relation $R$ on $X$, it is necessary to replace it with a new fuzzy relation $E$ that satisfies the transitivity property for clustering. Of course, such a fuzzy relation $E$ is a $T$-indistinguishability, where $T$ stands for a $t$-norm. This procedure can be
easily performed using the methods discussed in Section 2. After a $T$-indistinguishability $E$ is obtained, hierarchical clustering based on $E$ is explored. The following discusses the resolution forms of different types of $T$-indistinguishability.

**Definition 12.** For $0 < \alpha \leq 1$, $R_{\alpha} = \{(x, y) \mid \mu_{R}(x, y) \geq \alpha\}$ is called a $\alpha$-cut of the fuzzy relation $R$. Thus, one has the proposition that if $\alpha_1 \geq \alpha_2$, then $R_{\alpha_1} \subseteq R_{\alpha_2}$.

**Proposition 2.** (Zadeh\textsuperscript{26}) For any fuzzy relation $R$ on $X \times Y$, one has the resolution form $R = \bigcup_{\alpha} \alpha R_{\alpha}$, $0 < \alpha \leq 1$, where $\alpha R_{\alpha}$ is a fuzzy relation on $X \times Y$ defined as

$$
\mu_{\alpha R_{\alpha}}(x, y) = \begin{cases} 
\alpha & \text{if } (x, y) \in R_{\alpha}, \\
0 & \text{otherwise.}
\end{cases}
$$

**Definition 13.** (Tolerance relation) A crisp relation $A$ on $X$ is called a tolerance relation if it satisfies the following conditions: for all $x, y$ in $X$,

1. (Reflexivity): $\mu_{A}(x, x) = 1$.
2. (Symmetry): $\mu_{A}(x, y) = 1$ implies $\mu_{A}(y, x) = 1$.

**Definition 14.** (Equivalence relation) A crisp relation $A$ on $X$ is called an equivalence relation if it satisfies the following conditions: for all $x, y, z$ in $X$,

1. (Reflexivity): $\mu_{A}(x, x) = 1$.
2. (Symmetry): $\mu_{A}(x, y) = 1$ implies $\mu_{A}(y, x) = 1$.
3. (Transitivity): $\mu_{A}(x, y) = 1$ and $\mu_{A}(y, z) = 1$ implies $\mu_{A}(x, z) = 1$.

**Proposition 3.** (Zadeh\textsuperscript{26}) If $E$ is a $t_3$-indistinguishability (or a similarity relation), then for any $0 < \alpha \leq 1$, $E_{\alpha}$ should be an equivalence relation.

**Proposition 4.** (Yang and Shih\textsuperscript{23}) If $E$ is a $t_2$ or $t_1$-indistinguishability, then for any $0 < \alpha \leq 1$, we cannot assure $E_{\alpha}$ be an equivalence relation, but it can be a tolerance relation.
According to Proposition 3, a $t_3$-indistinguishability $E$ on $X$ is clearly partitioned into disjoint classes by an equivalence relation $E_\alpha$ which is created by a chosen value $\alpha$. Therefore, a $t_3$-indistinguishability on $X$ can be transformed into a diagram of a multilevel partition tree. However, $t_2$ and $t_1$-indistinguishabilities cannot be partitioned by an equivalence relation into a resolution form, and consequently cannot have partition trees from that resolution form. To solve this problem, Yang and Shih\textsuperscript{23} proposed a clustering algorithm that can generate a cluster resolution for any $t_i$-indistinguishability, $i = 1, 2, 3$. However, this proposed algorithm fails to consider the fact that “multiplication” is a more reasonable and, steadier operator than “addition” when linking the maximum similarity element to a suitable cluster. We next propose an improved version of the algorithm used by Yang and Shih and extend it to all $T$-indistinguishabilities obtained using Algorithms 1, 2, and 3.

**Clustering algorithm based on $T$-indistinguishability**

Let $T$ be a left-continuous $t$-norm and $E = [e_{ij}]_{n \times n}$ be a $T$-indistinguishability defined on a universe $X = \{x_1, x_2, \ldots, x_n\}$ with values $e_{ij} = \mu_E(x_i, x_j)$. In order to make an easier notation, we consider the elements of $X$ by their natural number of their position. Let $\alpha$ be a given cut with $0 < \alpha \leq 1$.

**Input:** a $T$-indistinguishability $E = [e_{ij}]_{n \times n}$, a cut $\alpha$ with $0 < \alpha \leq 1$.

**Output:** a partition of $X$, denoted by $C$.

The algorithm is the following:

s1. Create an index set $N = \{1, 2, \ldots, n\}$ and let $C = \{\}$.

s2. Set $e_{ii} = 0$ for all $1 \leq i \leq n$ and set $e_{ij} = 0$ for all $e_{ij} < \alpha$.

s3. Let $e_{st} = \max_{i,j \in N, i < j} e_{ij}$. Note that a tie is broken randomly.

s4. If $e_{st} \neq 0$ do

{ Let cluster $A$ be an empty set.
Put $s$ and $t$ to the cluster $A$, i.e. $A = \{s, t\}$.

While $I = \{j \mid j \in N \setminus A \text{ and } e_{ij} \neq 0 \text{ for all } i \in A\}$ is not empty do

\{ Let $\prod_{i \in A} e_{iu} = \max_{j \in I} \prod_{i \in A} e_{ij}$. A tie is broken randomly.

Link $u$ into $A$, i.e. $A = \{s, t, u\}$.

\}

Insert cluster $A$ into $C$. Let $N = N \setminus A$ and GOTO s3.

\}

Transfer all indices in $N$ into separated clusters and insert them into $C$.

Previous studies have reported that the values of a $T$-transitive lower approximation or opening $R_T$ of a proximity relation $R$ are all smaller than or equal to the corresponding values of $R$, whereas, the components of the $T$-transitive closure $R^T$ are greater than or equal to the corresponding components of $R$ (see Definitions 5, 7, 8), although, there may be different partitions based on $R_T$ or $R^T$. In the next section, the proposed clustering algorithm is used for two practical examples. Clustering results based on different $t$-norms offer a choice of $T$-indistinguishabilities. For the computation of $R_T$ and $R^T$, $R_{t_1}$ and $R_{t_2}$ were both computed using Algorithm 1, $R_{t_1}$ and $R_{t_2}$ were both derived using Algorithm 2, and, $t_3$-indistinguishabilities were calculated using Algorithm 3.

4. Examples

The first example involves considering the characteristic of Chinese characters.

Example 1. We considered seven Chinese characters, all of which have the basic pattern “口”, but have different crossing lines added: “—”, “|”, “\|”, “ı”. Subjective similarity was assigned using the following values: same=1, very similar=0.75, similar=0.5, different=0.25, and quite different=0. According to this assignment of subjective similarities, we obtained the proximity relation $R$ of these seven Chinese characters.
(1) All seven characters yielded the same clustering results when the proposed clustering algorithm was used on the $t_1$-transitive closure $R^{t_1}$ derived from Algorithm 1, a $t_1$-transitive lower approximation or opening $R_{t_1}$ derived from Algorithm 2, or a weighted quasiarithmetic mean of $R^{t_1}$ and $R_{t_1}$ with weight $p = 0.5$

$$0 < \alpha \leq 0.25 \Rightarrow \{2, 3, 4, 5, 7\}, \{1, 6\} \text{ etc.,}$$

$$0.25 < \alpha \leq 0.5 \Rightarrow \{3, 4, 5, 7\}, \{1, 2\}, \{6\} \text{ etc.,}$$

$$0.5 < \alpha \leq 0.75 \Rightarrow \{1, 2\}, \{3, 4\}, \{5, 7\}, \{6\} \text{ etc.,}$$

$$0.75 < \alpha \leq 1 \Rightarrow \{1\}, \{2\}, \{3\}, \ldots, \{7\}.$$ 

(2) The clustering results were obtained when the proposed clustering algorithm was used on (a), (b), and (c), as follows:

(a) the $t_2$-transitive closure $R^{t_2}$ derived from Algorithm 1

$$0 < \alpha \leq 0.3164 \Rightarrow \{1, 2, 3, 4, 5, 7\}, \{6\},$$

$$0.3164 < \alpha \leq 0.4218 \Rightarrow \{2, 3, 4, 5, 7\}, \{1, 6\} \text{ etc.,}$$

$$0.4218 < \alpha \leq 0.5625 \Rightarrow \{3, 4, 5, 7\}, \{1, 2\}, \{6\} \text{ etc.,}$$

$$0.5625 < \alpha \leq 0.75 \Rightarrow \{1, 2\}, \{3, 4\}, \{5, 7\}, \{6\} \text{ etc.,}$$
0.75 < \alpha \leq 1 \Rightarrow \{1\}, \{2\}, \{3\}, \ldots, \{7\}.

(b) the $t_2$-transitive lower approximation or opening $R_{t_2}$ derived from Algorithm 2
0 < \alpha \leq 0.25 \Rightarrow \{1, 2, 5, 7\}, \{3, 4, 6\},
0.25 < \alpha \leq 0.5 \Rightarrow \{1, 2, 7\}, \{3, 4\}, \{5\}, \{6\},
0.5 < \alpha \leq 0.75 \Rightarrow \{1, 2\}, \{3, 4\}, \{5\}, \{6\}, \{7\},
0.75 < \alpha \leq 1 \Rightarrow \{1\}, \{2\}, \{3\}, \ldots, \{7\}.

(c) the weighted quasiarithmetic mean of $R^{t_2}$ and $R_{t_2}$ with weight $p = 0.5$
0 < \alpha \leq 0.3061 \Rightarrow \{1, 2, 5, 7\}, \{3, 4, 6\},
0.3061 < \alpha \leq 0.3247 \Rightarrow \{1, 2, 5, 7\}, \{3, 4\}, \{6\},
0.3247 < \alpha \leq 0.5303 \Rightarrow \{1, 2, 7\}, \{3, 4\}, \{5\}, \{6\},
0.5303 < \alpha \leq 0.6123 \Rightarrow \{1, 2\}, \{3, 4\}, \{5, 7\}, \{6\},
0.6123 < \alpha \leq 0.75 \Rightarrow \{1, 2\}, \{3, 4\}, \{5\}, \{6\}, \{7\},
0.75 < \alpha \leq 1 \Rightarrow \{1\}, \{2\}, \{3\}, \ldots, \{7\}.

(3) The clustering results were obtained when the proposed clustering algorithm was used
on (a), (b), and (c), as follows:

(a) the $t_3$-transitive closure $R^{t_3}$ derived from Algorithm 3
0 < \alpha \leq 0.5 \Rightarrow \{1, 2, 3, 4, 5, 6, 7\},
0.5 < \alpha \leq 0.75 \Rightarrow \{1, 2, 3, 4, 5, 7\}, \{6\},
0.75 < \alpha \leq 1 \Rightarrow \{1\}, \{2\}, \{3\}, \ldots, \{7\}.

(b) the $t_3$-transitive lower approximation or opening $R_{t_3}$ derived from Algorithm 3
0 < \alpha \leq 0.25 \Rightarrow \{1, 2, 5, 7\}, \{3, 4\}, \{6\},
0.25 < \alpha \leq 0.75 \Rightarrow \{1, 2\}, \{3, 4\}, \{5, 7\}, \{6\},
0.75 < \alpha \leq 1 \Rightarrow \{1\}, \{2\}, \{3\}, \ldots, \{7\}.

(c) the ‘closer’ $t_3$-transitive approximation of $R$ derived from Algorithm 3
0 < \alpha \leq 0.25 \Rightarrow \{1, 2, 3, 4, 5, 6, 7\},
0.25 < α ≤ 0.4375 ⇒ \{1, 2, 3, 4, 5, 7\}, \{6\},
0.4375 < α ≤ 0.5 ⇒ \{1, 2, 5, 7\}, \{3, 4\}, \{6\},
0.5 < α ≤ 0.75 ⇒ \{1, 2\}, \{3, 4\}, \{5, 7\}, \{6\},
0.75 < α ≤ 1 ⇒ \{1\}, \{2\}, \{3\}, \ldots, \{7\}.

The clustering results of \(t_1\) and \(t_2\)-indistinguishabilities appeared to be comparably more reasonable and flexible than those of \(t_3\)-indistinguishabilities. It was also observed that a weighted quasiarithmetic mean with weight \(p = 0.5\) or a “closer” \(t_3\)-transitive approximation could be used to combine clustering results based on \(R_{t_i}\) and \(R_{t_i}\). The use of a weighted quasiarithmetic mean with weight \(p = 0.5\) or a “closer” \(t_3\)-transitive approximation should be more appropriate than \(R_{t_i}\) and \(R_{t_i}\).

**Example 2.** Portraits of 15 members from three families \(A, B\) and \(C\) are collected. Numbers 1–15 are marked corresponding to the portraits, as follows:

<table>
<thead>
<tr>
<th></th>
<th>Dad</th>
<th>Mom</th>
<th>Children</th>
</tr>
</thead>
<tbody>
<tr>
<td>family (A):</td>
<td>4</td>
<td>12</td>
<td>2,7</td>
</tr>
<tr>
<td>family (B):</td>
<td>8</td>
<td>11</td>
<td>1,5,9,14</td>
</tr>
<tr>
<td>family (C):</td>
<td>3</td>
<td>15</td>
<td>6,10,13</td>
</tr>
</tbody>
</table>

\[
R = \begin{bmatrix}
1 \\
0.2 & 1 \\
0 & 0.2 & 1 \\
0.4 & 0.6 & 0 & 1 \\
0.8 & 0.2 & 0 & 0 \\
0.4 & 0.2 & 0.6 & 0.4 & 0.2 & 1 \\
0.2 & 0.8 & 0.2 & 0.8 & 0.4 & 0.2 & 1 \\
0.8 & 0.8 & 0 & 0 & 0.6 & 0.4 & 0.4 & 1 \\
0.8 & 0.4 & 0.2 & 0 & 0.6 & 0.2 & 0.2 & 0.4 & 1 \\
0.2 & 0 & 0.6 & 0.2 & 0.2 & 0.8 & 0 & 0 & 0.4 & 1 \\
0.8 & 0 & 0 & 0 & 0.6 & 0.2 & 0.4 & 0 & 0.8 & 0.2 & 1 \\
0.2 & 0.8 & 0 & 0.2 & 0.2 & 0.2 & 0.8 & 0 & 0 & 0 & 0 & 1 \\
0.4 & 0.2 & 0.6 & 0 & 0.2 & 0.8 & 0.2 & 0 & 0.2 & 0.6 & 0 & 0 & 1 \\
0.8 & 0.2 & 0 & 0 & 0.8 & 0.4 & 0.2 & 0.6 & 0.8 & 0.4 & 0 & 0.4 & 1 \\
0 & 0 & 0 & 0 & 0.2 & 0.4 & 0.4 & 0 & 0.2 & 0.8 & 0 & 0 & 0.8 & 0.4 & 1
\end{bmatrix}
\]
Subjective similarity was assigned using the following values: \textit{same}=1, \textit{very similar}=0.8, \textit{similar}=0.6, \textit{not so similar}=0.4, \textit{different}=0.2, and \textit{quite different}=0. According to this assignment of subjective similarities, we obtained the proximity relation \( R \) of these 15 portraits.

The clustering results were obtained when the proposed clustering algorithm was used on (a), (b), and (c), as follows:

(a) the weighted quasiarithmetic mean of \( R^{t_1} \) and \( R_{t_1} \) with weight \( p = 0.5 \)

\[
0 < \alpha \leq 0.1 \Rightarrow \{1, 2, 4, 5, 6, 7, 8, 9, 11, 12, 14\}, \{3, 10, 13, 15\} \text{ etc.,}
\]
\[
0.1 < \alpha \leq 0.2 \Rightarrow \{1, 2, 5, 6, 7, 8, 9, 11, 14\}, \{3, 10, 13, 15\}, \{4, 12\} \text{ etc.,}
\]
\[
0.2 < \alpha \leq 0.3 \Rightarrow \{1, 5, 8, 9, 11, 14\}, \{6, 10, 13, 15\}, \{2, 4, 7, 12\}, \{3\} \text{ etc.,}
\]
\[
0.3 < \alpha \leq 0.4 \Rightarrow \{1, 5, 8, 9, 14\}, \{6, 10, 13, 15\}, \{2, 4, 7, 12\}, \{3\}, \{11\} \text{ etc.,}
\]
\[
0.4 < \alpha \leq 0.5 \Rightarrow \{1, 5, 9, 14\}, \{6, 10, 13, 15\}, \{2, 7, 12\}, \{3\}, \{4\}, \{11\} \text{ etc.,}
\]
\[
0.5 < \alpha \leq 0.6 \Rightarrow \{1, 5, 9, 14\}, \{6, 10, 13\}, \{2, 7, 12\}, \{3\}, \{4\}, \{8\}, \{11\}, \{15\} \text{ etc.,}
\]
\[
0.6 < \alpha \leq 0.7 \Rightarrow \{1, 9, 14\}, \{2, 7, 12\}, \{6, 13\}, \{10, 15\}, \{3\}, \{4\}, \{5\}, \{8\}, \{11\} \text{ etc.,}
\]
\[
0.7 < \alpha \leq 0.8 \Rightarrow \{1, 5, 14\}, \{2, 7\}, \{6, 10\}, \{3\}, \{4\}, \{8\}, \{9\}, \{11\}, \{12\},
\{13\}, \{15\} \text{ etc.,}
\]
\[
0.8 < \alpha \leq 1 \Rightarrow \{1\}, \{2\}, \{3\}, \ldots, \{15\}.
\]

(b) the weighted quasiarithmetic mean of \( R^{t_2} \) and \( R_{t_2} \) with weight \( p = 0.5 \)

\[
0 < \alpha \leq 0.2862 \Rightarrow \{1, 2, 5, 7, 8, 9, 14\}, \{6, 10, 13, 15\}, \{4, 12\}, \{3\}, \{11\} \text{ etc.,}
\]
\[
0.2862 < \alpha \leq 0.32 \Rightarrow \{1, 2, 5, 7, 8\}, \{6, 10, 13, 15\}, \{4, 12\}, \{9, 14\}, \{3\}, \{11\} \text{ etc.,}
\]
\[
0.32 < \alpha \leq 0.3577 \Rightarrow \{1, 5, 8, 9, 14\}, \{6, 10, 13, 15\}, \{2, 7\}, \{4, 12\}, \{3\}, \{11\} \text{ etc.,}
\]
0.3577 < \alpha \leq 0.5059 \Rightarrow \{1, 5, 8, 9, 14\}, \{6, 10, 13, 15\}, \{2, 7\}, \{3\}, \{4\}, \{11\}, \{12\} etc.,

0.5059 < \alpha \leq 0.6196 \Rightarrow \{1, 5, 8, 14\}, \{6, 10, 13\}, \{2, 7\}, \{3\}, \{4\}, \{9\}, \{11\},

\{12\}, \{15\} etc.,

0.6196 < \alpha \leq 0.6531 \Rightarrow \{1, 5, 14\}, \{2, 7\}, \{6, 10\}, \{13, 15\}, \{3\}, \{4\}, \{8\}, \{9\}, \{11\}, \{12\},

\{13\}, \{15\} etc.,

0.6531 < \alpha \leq 0.8 \Rightarrow \{1, 5, 14\}, \{2, 7\}, \{6, 10\}, \{3\}, \{4\}, \{8\}, \{9\}, \{11\}, \{12\},

\{13\}, \{15\} etc.,

0.8 < \alpha \leq 1 \Rightarrow \{1\}, \{2\}, \{3\}, ..., \{15\}.

(c) the “closer” $t_3$-transitive approximation of $R$

$0 < \alpha \leq 0.156 \Rightarrow \{1, 2, ..., 15\},$

$0.156 < \alpha \leq 0.1833 \Rightarrow \{1, 2, 4, 5, 7, 8, 9, 11, 12, 14\}, \{3, 6, 10, 13, 15\},$

$0.1833 < \alpha \leq 0.45 \Rightarrow \{1, 5, 8, 9, 11, 14\}, \{3, 6, 10, 13, 15\}, \{2, 4, 7, 12\},$

$0.45 < \alpha \leq 0.6 \Rightarrow \{1, 5, 8, 9, 11, 14\}, \{6, 10, 13, 15\}, \{2, 4, 7, 12\}, \{3\},$

$0.6 < \alpha \leq 0.65 \Rightarrow \{1, 5, 8, 14\}, \{6, 10, 13, 15\}, \{2, 4, 7\}, \{9, 11\}, \{3\}, \{12\},$

$0.65 < \alpha \leq 0.7 \Rightarrow \{1, 5, 8, 14\}, \{2, 4, 7\}, \{6, 10\}, \{9, 11\}, \{13, 15\}, \{3\}, \{12\},$

$0.7 < \alpha \leq 0.8 \Rightarrow \{1, 5\}, \{2, 7\}, \{6, 10\}, \{9, 11\}, \{13, 15\}, \{3\}, \{4\}, \{8\}, \{12\}, \{14\},$

$0.8 < \alpha \leq 1 \Rightarrow \{1\}, \{2\}, \{3\}, ..., \{15\}.$

Clearly, choosing different $t_r$-indistinguishabilities yields different clustering results.

Using the proposed clustering algorithm, it is possible to distinguish between families $A, B,$ and $C$ based on (a) with $0.2 < \alpha \leq 0.3$, (b) with $0.32 < \alpha \leq 0.3577$ or (c) with $0.1833 < \alpha \leq 0.45$. We also observed that it is possible to distinguish the father,
mother, and children from the family based on (a) using $0.7 < \alpha \leq 0.8$ and (b) using $0.6531 < \alpha \leq 0.8$, but (c) does not yield this result. Contrasting the clustering results from (a), (b), and (c), shows that (a) demonstrates the superiority and usefulness of the proposed clustering algorithm. Thus, a weighted quasi-arithmetic mean of $R_{t_1}$ and $R_{t_1}$ with $p = 0.5$ recommended as a $T$-indistinguishability for our proposed clustering algorithm.

5. Application to evaluating the performance of higher education in Taiwan

The proposed clustering algorithm and weighted quasi-arithmetic mean of $R_{t_1}$ and $R_{t_1}$ with $p = 0.5$ were applied to an actual example obtained from the Taiwan Assessment and Evaluation Association (TWAEA) concerning the evaluation of the performance of academic departments of higher education in Taiwan (see also Guh¹⁰).

Higher education is vital to country’s global competitiveness. Higher education in Taiwan developed rapidly over the past decade, expanding to approximately 160 university-level institutions. To maintain competitiveness, most higher education institutions in Taiwan strive to offer quality teaching, research, and services. To evaluate the performance of these institutions, a nonprofit organization, the TWAEA was established in 2000 to provide a third-party evaluation of the performance of these universities.

Based on the SWOT (strength, weakness, opportunity, threat) concept, the TWAEA established performance evaluation models for various academic departments, as well as the following 10 evaluation criteria: Teaching Innovations (TI:1), Teaching Quality (TQ:2), Teaching Material (TM:3), Journal Paper (JP:4), Research Grant (RG:5), Academic Award (AA:6), Patent Acquisition (PA:7), Student Consultation (SC:8), Professional Service (PS:9), and University Service (US:10). After determining the proximity matrix for these 10 criteria, it is possible to organize these criteria into a hierarchical evaluation structure. After discussion by the committee to avoid bias in the structure selection, the following 6 degrees of similarities for each pair of criteria were determined:
same = 1, very similar = 0.8, similar = 0.6, not so similar = 0.4, different = 0.2, and very different = 0. According to these subjectively assigned similarities, the following proximity relation $R$ was obtained.

$$
R = \begin{bmatrix}
1 & 0 & 0.8 & 0.8 & 0.8 & 0.3 & 0.2 & 0.2 & 0.2 & 0.2 \\
0.8 & 1 & 0.8 & 0.8 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\
0.8 & 0.8 & 1 & 0.5 & 0.4 & 0.9 & 0.6 & 0.9 & 0.9 & 0.9 \\
0.3 & 0.4 & 0.5 & 1 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\
0.2 & 0.3 & 0.2 & 0.8 & 1 & 0.2 & 0.3 & 0.3 & 0.3 & 0.3 \\
0.2 & 0.4 & 0.3 & 0.9 & 0.6 & 1 & 0.2 & 0.4 & 0.4 & 0.4 \\
0.2 & 0.3 & 0.3 & 0.8 & 0.8 & 0.5 & 1 & 0.2 & 0.5 & 0.5 \\
0.2 & 0.1 & 0.1 & 0.8 & 0.9 & 0.4 & 0.2 & 1 & 0.2 & 0.1 \\
0.2 & 0.3 & 0.2 & 0.2 & 0.2 & 0.6 & 0.1 & 0.5 & 0.6 & 1
\end{bmatrix}
$$

We applied our proposed clustering algorithm to this proximity relation $R$ to derive a hierarchical evaluation structure. The clustering results were obtained by using the proposed clustering algorithm on the weighted quasiarithmetic mean of $R^o$ and $R_t$ derived from Algorithms 1 and 2 with weight $p = 0.5$.

$0 < \alpha \leq 0.1 \Rightarrow \{1, 2, \ldots, 10\}$,

$0.1 < \alpha \leq 0.2 \Rightarrow \{1, 2, 3, 4, 5, 6, 7, 9, 10\}, \{8\}$,

$0.2 < \alpha \leq 0.25 \Rightarrow \{4, 5, 6, 7, 9\}, \{1, 2, 3, 8\}, \{10\}$,

$0.25 < \alpha \leq 0.5 \Rightarrow \{4, 5, 6, 7, 9\}, \{1, 2, 3\}, \{8, 10\}$,

$0.5 < \alpha \leq 0.55 \Rightarrow \{4, 5, 6, 9\}, \{1, 2, 3\}, \{7\}, \{8\}, \{10\}$,

$0.55 < \alpha \leq 0.8 \Rightarrow \{4, 6, 9\}, \{1, 2, 3\}, \{5, 7\}, \{8\}, \{10\}$,

$0.8 < \alpha \leq 0.9 \Rightarrow \{4, 6\}, \{1\}, \{2\}, \{3\}, \{5\}, \{7\}, \{8\}, \{9\}, \{10\}$,

$0.9 < \alpha \leq 1 \Rightarrow \{1\}, \{2\}, \{3\}, \ldots, \{10\}$.
To avoid bias in structure selection, we compared the clustering results. The suggested rough hierarchical evaluation structure was \{4,5,6,7,9\}, \{1,2,3\}, \{8,10\}, i.e. \{JP, RG, AA, PA, PS\}, \{TI, TQ, TM\}, \{SC, US\}, and the suggested detailed hierarchical evaluation structure was \{4,6,9\}, \{1,2,3\}, \{5,7\}, \{8\}, \{10\}, i.e. \{JP, AA, PS\}, \{TI, TQ, TM\}, \{RG, PA\}, \{SC\}, \{US\}. Committee members considered this structure more flexible and appropriate than the outcomes originating from Guh\(^{10}\). The proposed clustering method has potential as a tool for constructing hierarchical evaluation structures to analyze management problems.

6. Conclusion

Yang and Shih\(^ {23}\) proposed a max-\(t\) \(n\)-step procedure to produce a max-\(t\) similarity relation so that a clustering algorithm could be constructed for clustering on a proximity relation \(R\). In this paper, we claim that a finite step exists to produce a max-\(t\) similarity relation through a max-\(t\) \(n\)-step procedure, and the obtained max-\(t\) similarity relation should be the \(t\)-transitive closure \(R^t\) of \(R\). By contrast, Garmendia et al.\(^ {9}\) provided an algorithm to gain a \(t\)-indistinguishability \(R_t\) from a proximity relation \(R\) so that \(R_t\) is expected to be the largest \(t\)-indistinguishability satisfying \(R \geq R_t\). Hence, we claim that the weighted quasiarithmetic mean of \(R^t\) and \(R_t\) is still a \(t\)-indistinguishability, but the weighted quasiarithmetic mean method cannot be applied for the minimum \(t\)-norm. We used an algorithm from Garmendia et al.\(^ {7}\) to obtain a min-transitive opening and a “closer” min-transitive approximation of a proximity relation so that the outputs of the algorithm are min-indistinguishabilities. This paper presents a proposed clustering method based on a weighted quasiarithmetic mean of \(T\)-transitive fuzzy relations as an improved version of the clustering algorithm used by Yang and Shih\(^ {23}\). Two examples were used to compare the fitness of clustering results based on three critical \(t_i\)-indistinguishabilities: minimum\((t_3)\), product\((t_2)\), and Lukasiewicz\((t_1)\). A weighted quasiarithmetic mean of \(R_i^t\) and \(R_t^i\) with weight \(p = 0.5\) demonstrated the superiority and usefulness of the proposed algorithm when clustering on an associated proximity relation \(R\). The algorithm was
then applied to evaluating the performance of higher education in Taiwan. The proposed clustering algorithm is an effective method for clustering on a proximity relation matrix.

References


