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Robust control for a class of uncertain nonlinear systems: adaptive fuzzy approach based on backstepping

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Abstract

A robust adaptive fuzzy control design approach is developed for a class of multivariable nonlinear systems with modeling uncertainties and external disturbances. The controller design for the overall systems has been carried out through a number of simpler controller designs for a series of the relevant auxiliary systems based on the backstepping design technique. For each auxiliary system, an adaptive fuzzy logic system is introduced to learn the behavior of unknown dynamics, and then a robust control algorithm is employed to efficiently compensate the approximation error and the external disturbances as well. It is shown that the resulting closed-loop systems guarantee a satisfactory transient and asymptotic performance. A simulation example is also presented to illustrate the design procedure and controller performance.

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1. Introduction

There exist many uncertain nonlinear systems in practice. For example, many of motion control systems such as robot manipulators and overhead crane mechanisms are nonlinear due to frictions and/or nonlinear actuators, and also of various uncertainties due to external disturbances and/or approximation of the modeling [1,2,12,14]. The study of such uncertain nonlinear systems has been one of active research

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topics during the last few years. A number of significant results have been obtained. For example, an adaptive control algorithm without overparameterization is given for a class of parametric strict-feedback nonlinear systems in [9]. Using perturbation method and backstepping technique, the problem of tracking and disturbance attenuation for parametric strict-feedback nonlinear systems is solved in [11]. The authors in [7] develop a robust adaptive backstepping scheme for this class of nonlinear systems with unmodeled dynamics. More recently, the authors in [10] develop a robust adaptive control method for this class of nonlinear systems by combining the adaptive backstepping technique with fuzzy control method, and the authors in [3] propose an adaptive fuzzy variable control via backstepping for a class of SISO nonlinear systems which can solve the model reference adaptive control problem in the presence of system uncertainties. Due to the nature of the backstepping technique, the systems considered are required to be of lower-triangular or upper-triangular structures, and affine in the control input. In fact, these two conditions have been commonly used in most of the existing robust and adaptive control schemes that are based on the backstepping design (see, [3,7,9–11]).

The system models studied by [7,9–11] are linearly parameterized and all the subsystems related are assumed to be scalar. However, these system models do not characterize many motion control systems with Lagrange form [1,12,14]. These motion control systems with Lagrange form are not linearly parametric and are often described by vector subsystems. This paper is concerned with the robust control for a class of multivariable nonlinear systems with uncertainties and external disturbances, which are described by vector subsystems. Using the backstepping technique and the fuzzy approximation method, a hybrid robust adaptive fuzzy controller is proposed. More specifically, the controller design for the overall system is carried out through a number of simpler controller designs for a series of the relevant auxiliary subsystems. For each auxiliary system, an adaptive fuzzy logic system is introduced to learn the behavior of unknown dynamics, and then a robust control algorithm is employed to efficiently compensate the approximation error and external disturbances as well. It is shown that all signals of the resulting closed-loop system are globally uniformly ultimately bounded and the norm of the system output exponentially converges to an arbitrarily specified small number.

The organization of this paper is as follows. In Section 2, a class of multivariable nonlinear systems with uncertainties and external disturbances, which are described by vector subsystems, and the model descriptions of fuzzy logic systems are presented. The controller design is given in Section 3. Performance analysis is presented in Section 4. In Section 5, a simulation example is illustrated. The paper ends with the concluding remarks in Section 6. The following standard notations will be used in this paper. The square norm of a vector $x \in R^m$ is denoted by $\|x\|^2$, G^T denotes transpose of matrix G , G^+ denotes Moore–Penrose inverse of G .

2. System description and problem statement

2.1. System model

Consider a class of nonlinear systems of the form

$$\begin{aligned} \dot{x}_i &= f_i(\bar{x}_i) + G_i(\bar{x}_i)x_{i+1} + N_i(\bar{x}_i) + d_i, \quad i = 1, \dots, n-1, \\ \dot{x}_n &= f_n(x) + G_n(x)u + N_n(x) + d_n, \\ y &= x_1, \end{aligned} \tag{1}$$

where for each $i \in \{1, 2, \dots, n\}$, $x_i \in R^{m_i}$ is a vector denoting the state of the i th subsystem, $\bar{x}_i = (x_1^\tau, x_2^\tau, \dots, x_i^\tau)^\tau$, $f_i(\bar{x}_i)$ and $N_i(\bar{x}_i)$ are vector-valued functions with $N_i(\bar{x}_i)$ denoting the unknown modeling uncertainties of the system, $G_i(\bar{x}_i)$ is a matrix-valued function, d_i denotes the external disturbance of the system, $x = \bar{x}_n$ denotes the state of the overall system, u denotes the control input and y denotes the output, and positive integer n denotes the number of the subsystems.

Remark 1. (1) If $n = 1$, Eq. (1) is an uncertain affine nonlinear system.

(2) If $m_i = 1$ for each $i \in \{1, 2, \dots, n\}$, every subsystem becomes scalar form and Eq. (1) becomes the system studied in [3,7,9–11].

(3) System (1) can characterize a large class of motion control systems such as systems studied in [1,12,14], which are described by vector form subsystems.

The problem we consider in this paper is to design a robust controller for the system such that all signals of the resulting closed-loop system are globally uniformly ultimately bounded and the norm of the system output exponentially converges to an arbitrarily specified small number. We make the following assumption for this class of systems.

Assumption 1. For all \bar{x}_i , $G_i(\bar{x}_i)$ is of full row rank, and $f_i, G_i^+, (i = 1, \dots, n)$ are sufficiently smooth.

2.2. Model description of fuzzy approximator

Fuzzy logic systems have been successfully employed to universally approximate the mathematical models of dynamical systems in recent years [1,10,15,16]. We consider the following fuzzy systems as a building block for the proposed adaptive control system:

$$R^{(l)} : \text{IF } x_1 \text{ is } F_1^l \text{ and } x_2 \text{ is } F_2^l \text{ and } \dots \text{ and } x_n \text{ is } F_n^l, \text{ THEN } y \text{ is } G^l,$$

where $x = (x_1, \dots, x_n)^\tau \in X = X_1 \times \dots \times X_n \subset R^n$ and $y \in Y \subset R$ are the input and output of the fuzzy logic system, respectively, F_i^l and G^l are labels of fuzzy sets in the universes of discourse X_i and Y , respectively, and $l = 1, 2, \dots, M$ with M being the number of rules in the rule base.

Lemma 2.1 (Lee and Tomizuka [10], Wang [15,16]). *The fuzzy logic systems with center average defuzzifier, product inference, and singleton fuzzifier are of the following form:*

$$y(x) = \frac{\sum_{l=1}^M \theta_l \left(\prod_{i=1}^n \mu_{F_i^l}(x_i) \right)}{\sum_{l=1}^M \left(\prod_{i=1}^n \mu_{F_i^l}(x_i) \right)}, \quad (2)$$

where $\mu_{F_i^l}(\cdot)$ is the membership function of the fuzzy set F_i^l and θ_l is the point at which μ_{G^l} achieves its maximum value (it is assumed here that $\mu_{G^l}(\theta_l) = 1$).

The above fuzzy logic system has been shown to be capable of uniformly approximating any well-defined nonlinear function over a compact set U to any degree of accuracy. The universal approximation theorem is quoted as follows.

Theorem 2.1 (Lee and Tomizuka [10], Wang [15,16]). *For any given real continuous function $f(x)$ on a compact subset $U \in \mathbb{R}^N$ and arbitrary $\varepsilon > 0$ there exists a fuzzy logic system $y(x)$ in the form of (2) such that $\max_{x \in U} \|f(x) - y(x)\| < \varepsilon$.*

Although the fuzzy logic system described above is of single-output, it is straightforward to show that a multi-output system can always be approximated by a group of single-output approximation systems.

2.3. Re-parameterization of the system

Using the fuzzy logic system introduced in the above section, the unknown term $N_i(\bar{x}_i)$ in Eq. (1) can be approximated by a fuzzy system $\hat{N}_i(\bar{x}_i, \vartheta_i)$ in any compact set $U_{\bar{x}_i}$, where ϑ_i is a regulating parameter vector. An adaptive algorithm will be constructed in the next section to learn the unknown parameters of the fuzzy logic system. Choose $\hat{N}_i(\bar{x}_i, \vartheta_i) = H_i(\bar{x}_i)\vartheta_i$, where $H_i(\bar{x}_i)$ denotes a basis function matrix. The following assumption will be used as a foundation for designing a fuzzy controller.

Assumption 2. Assume that for any $i \in \{1, \dots, n\}$, there is a finite parameter value $\theta_i \in \Omega_{\theta_i}$ known as the optimal approximation parameter, i.e., [1,10,15].

$$\theta_i = \arg \min_{\vartheta_i \in \Omega_{\theta_i}} \left(\max_{\bar{x}_i \in U_{\bar{x}_i}} \|\hat{N}_i(\bar{x}_i, \vartheta_i) - N(\bar{x}_i)\| \right). \quad (3)$$

The set Ω_{θ_i} contains parameter vector θ_i is defined as

$$\Omega_{\theta_i} \in \{\vartheta_i : \vartheta_i^T \vartheta_i \leq \eta_{i0}^2\},$$

where η_{i0} are some positive constants. Let

$$\Delta_i = N_i(\bar{x}_i) - H_i(\bar{x}_i)\theta_i + d_i \quad (4)$$

denote the approximation error plus the external disturbance. In many previous published works the approximation error is assumed to be bounded by a fixed constant. However, this may not be true in many cases since the number of fuzzy IF–THEN rules is finite in practice and there is no guarantee the compact convex set can be easily identified before the stability of the closed-loop system is established. Hence, we instead make the following assumption on the approximation error.

Assumption 3. For any $i \in \{1, \dots, n\}$, there exist a known positive function $\rho_i(\bar{x}_i)$ and two unknown positive constants such that

$$\|\Delta_i\| \leq \eta_{i1}\rho_i(\bar{x}_i) + \eta_{i2}. \quad (5)$$

It follows from (1) and (4) that

$$\begin{aligned} \dot{x}_i &= f_i(\bar{x}_i) + G_i(\bar{x}_i)x_{i+1} + H_i(\bar{x}_i)\theta_i + \Delta_i(t, \bar{x}_i), \quad i = 1, \dots, n-1, \\ \dot{x}_n &= f_n(x) + G_n(x)u + H_n(x)\theta_n + \Delta_n(t, x). \end{aligned} \quad (6)$$

Remark 2. If for some $i \in \{1, \dots, n\}$, the uncertain term $N_i(\bar{x}_i)$ has the linear parametric form $H_i(\bar{x}_i)\theta_i$ with known matrix-valued function $H_i(\bar{x}_i)$ and unknown parameter vector θ_i , we will not need any fuzzy logic system to approximate the term. In this case, $\Delta_i = d_i$.

Eq. (6) describes the system parameterized by fuzzy logic systems. In the next section, we will construct an adaptive control law for all the regulating parameters to learn the unknown dynamics of the system.

3. Robust adaptive fuzzy controller design

The control objective is to design a robust adaptive fuzzy control law such that under Assumptions 1–3, all the signals of the closed-loop system are globally uniformly ultimately bounded [8]. The design idea is motivated by the adaptive backstepping approach. For this purpose, we will consider a series of auxiliary systems. That is, for each $i \in \{1, \dots, n\}$, $\Sigma_i^{x_{i+1}}$ denotes the auxiliary system

$$\begin{aligned} \dot{x}_j &= f_j(\bar{x}_j) + G_j(\bar{x}_j)x_{j+1} + H_j(\bar{x}_j)\theta_j + \Delta_j(t, \bar{x}_j), \quad j = 1, \dots, i-1, \\ \dot{x}_i &= f_i(\bar{x}_i) + G_i(\bar{x}_i)x_{i+1} + H_i(\bar{x}_i)\theta_i + \Delta_i(t, \bar{x}_i), \end{aligned} \quad (7)$$

where $\bar{x}_i = (x_1^\tau, x_2^\tau, \dots, x_i^\tau)$ is its state vector, x_{i+1} is its virtual input vector. We will use a number of notations associated with each auxiliary system. V_i denotes a positive scalar function with respect to \bar{x}_i , $\alpha_i(\bar{x}_i)$ denotes a state feedback control law of $\Sigma_i^{x_{i+1}}$, and $\dot{V}_i(\Sigma_i^{x_{i+1}=\alpha_i})$ denotes the time derivative of V_i along the solutions of the system $\Sigma_i^{x_{i+1}}$ with $x_{i+1} = \alpha_i$, $i = 2, \dots, n-1$. $\partial V_i / \partial x_j^\tau$ denotes the row vector

$$\left[\frac{\partial V_i}{\partial x_{j1}}, \frac{\partial V_i}{\partial x_{j2}}, \dots, \frac{\partial V_i}{\partial x_{jr_j}} \right],$$

where x_{jk} , $k = 1, 2, \dots, r_j$ are the entries of x_j , and $\partial V_i / \partial x_j$ denotes the column vector

$$\left[\frac{\partial V_i}{\partial x_{j1}}, \frac{\partial V_i}{\partial x_{j2}}, \dots, \frac{\partial V_i}{\partial x_{jr_j}} \right]^\tau.$$

Throughout the paper, we will also simplify our notation of $f_i(\bar{x}_i)$, $\Delta_i(t, \bar{x}_i)$, $\rho_i(\bar{x}_i)$, $G_i(\bar{x}_i)$, $H_i(\bar{x}_i)$ and $\alpha_i(\bar{x}_i)$ to f_i , Δ_i , ρ_i , G_i , H_i and α_i , respectively, by dropping their arguments.

We will start by controlling the first equation of (7) considering x_2 to be its control, and then at each subsequent step, we will augment the designed subsystem by one equation. The detailed procedure is described below.

Step 1: Consider the auxiliary system $\Sigma_1^{x_2}$:

$$\dot{x}_1 = f_1(\bar{x}_1) + G_1(\bar{x}_1)x_2 + H_1(\bar{x}_1)\theta_1 + \Delta_1. \quad (8)$$

In this auxiliary system (8), \bar{x}_1 can be viewed as its state vector, x_2 as its input vector. The design target is to construct a triplet $\{V_1(\bar{x}_1), \alpha_1(\bar{x}_1), \zeta_1(x_1)\}$ such that the storage rate $\dot{V}_1(\Sigma_1^{x_2=\alpha_1})$ satisfies some requirements. Let

$$V_1(x_1, \vartheta_1) = \frac{1}{2}x_1^\tau x_1 + \frac{1}{2\gamma_1}(\theta_1 - \vartheta_1)^\tau(\theta_1 - \vartheta_1), \quad (9)$$

where γ_1 is a positive constant which will be chosen at later stage and ϑ_1 is the estimation of θ_1 . After some algebraic manipulation, the time derivative of V_1 along the solution of the system $\Sigma_1^{x_2}$ is given by

$$\begin{aligned} \dot{V}_1(\Sigma_1^{x_2}) &= x_1^\tau [f_1 + G_1 x_2 + H_1 \theta_1 + \Delta_1] + (\theta_1 - \vartheta_1)^\tau \left(-\frac{1}{\gamma_1} \dot{\vartheta}_1 \right) \\ &= x_1^\tau [f_1 + G_1 x_2 + H_1 \vartheta_1] + x_1^\tau \Delta_1 + (\theta_1 - \vartheta_1)^\tau \left[\left(-\frac{1}{\gamma_1} \dot{\vartheta}_1 \right) + H_1^\tau x_1 \right]. \end{aligned} \tag{10}$$

Using (5) and the following inequality:

$$ab \leq ka^2 + \frac{b^2}{4k} \quad \text{for any } k > 0,$$

it can be shown that

$$x_1^\tau \Delta_1 \leq k_{11}^{(1)} \rho_1^2 x_1^\tau x_1 + k_{11}^{(2)} x_1^\tau x_1 + \frac{\eta_{11}^2}{4k_{11}^{(1)}} + \frac{\eta_{12}^2}{4k_{11}^{(2)}} \tag{11}$$

for any positive constants $k_{11}^{(1)}$ and $k_{11}^{(2)}$.

It follows from (10) and (11) that

$$\begin{aligned} \dot{V}_1(\Sigma_1^{x_2}) &\leq x_1^\tau [f_1 + G_1 x_2 + H_1 \vartheta_1 + k_{11}^{(1)} \rho_1^2 x_1 + k_{11}^{(2)} x_1] \\ &\quad + (\theta_1 - \vartheta_1)^\tau \left[\left(-\frac{1}{\gamma_1} \dot{\vartheta}_1 \right) + H_1^\tau x_1 \right] + \frac{\eta_{11}^2}{4k_{11}^{(1)}} + \frac{\eta_{12}^2}{4k_{11}^{(2)}}. \end{aligned} \tag{12}$$

Let

$$\dot{\vartheta}_1 = \zeta_1 = \gamma_1 H_1^\tau x_1 - 2\gamma_1 l_1 \vartheta_1, \tag{13}$$

$$\alpha_1 = G_1^+ \left[-f_1 - H_1 \vartheta_1 - \frac{c_1}{2} x_1 - k_{11}^{(1)} \rho_1^2 x_1 - k_{11}^{(2)} x_1 \right], \tag{14}$$

where $c_1 > 0$ and $l_1 > 0$ are positive constants which will be chosen later on, G_1^+ denotes the Moore–Penrose inverse of G_1 . It is easy to see that $G_1^+ = G_1^\tau (G_1 G_1^\tau)^{-1}$ since G_1 is of full row rank.

Remark 3. It is noted that the adaptive law (13) is of fixed σ -modification form [4–6], which guarantees $\vartheta_1 \in L_\infty$ if $H_1^\tau x_1 \in L_\infty$.

From (12)–(14), one gets

$$\dot{V}_1(\Sigma_1^{x_2=\alpha_1}) \leq -\frac{c_1}{2} x_1^\tau x_1 + 2l_1 (\theta_1 - \vartheta_1)^\tau \vartheta_1 + \frac{\eta_{11}^2}{4k_{11}^{(1)}} + \frac{\eta_{12}^2}{4k_{11}^{(2)}}. \tag{15}$$

Recalling the inequality

$$2l_1 (\theta_1 - \vartheta_1)^\tau \vartheta_1 \leq l_1 \theta_1^\tau \theta_1 - l_1 (\theta_1 - \vartheta_1)^\tau (\theta_1 - \vartheta_1), \tag{16}$$

it then follows from (15) and (16) that

$$\dot{V}_1(\Sigma_1^{x_2=\alpha_1}) \leq -\frac{c_1}{2}x_1^\tau x_1 - l_1(\theta_1 - \vartheta_1)^\tau(\theta_1 - \vartheta_1) + \frac{\eta_{11}^2}{4k_{11}^{(1)}} + \frac{\eta_{12}^2}{4k_{11}^{(2)}} + l_1\theta_1^\tau\theta_1. \quad (17)$$

Considering $\theta_1^\tau\theta_1 \leq \eta_{10}^2$, and by rewriting $l_1 = 1/4k_{11}^{(0)}$, one has the following more compact expression:

$$\dot{V}_1(\Sigma_1^{x_2=\alpha_1}) \leq -\frac{c_1}{2}x_1^\tau x_1 - l_1(\theta_1 - \vartheta_1)^\tau(\theta_1 - \vartheta_1) + \sum_{j=0}^2 \frac{\eta_{1j}^2}{4k_{11}^{(j)}}. \quad (18)$$

It should be noted that the proposed stabilizing function α_1 in (14) for the auxiliary system $\Sigma_1^{x_2}$ consists of two parts. The first part $-G_1^+H_1\vartheta_1$ comes from the fuzzy logic system with an adaptation law $\dot{\vartheta}_1 = \zeta_1$ to learn the unknown dynamics N_1 of the auxiliary system $\Sigma_1^{x_2}$, and the second part $G_1^+[-(c_1/2)x_1 - k_{11}^{(1)}\rho_1^2x_1 - k_{11}^{(2)}x_1]$ is the robust controller to achieve uniformly ultimately bounded performance. Hence, this controller is in fact a hybrid robust adaptive controller.

Choosing $c_1 > 0$, $k_{11}^{(1)} > 0$, $k_{11}^{(2)} > 0$ and $l_1 = (1/4k_{11}^{(0)}) > 0$ properly and using (18), one can easily prove that all the signals of the auxiliary adaptive fuzzy closed-loop system $\Sigma_1^{x_2=\alpha_1}$ are uniformly ultimately bounded and that the norm of the signal x_1 exponentially converges to a arbitrarily specified small number. In this step, we obtain a triplet $\{V_1, \zeta_1, \alpha_1\}$ such that the storage rate satisfies the dissipative inequality (18).

Step 2–Step n – 1: We can follow the above procedure to design hybrid adaptive robust controllers for the subsequent auxiliary systems. Suppose that in the i th step, the triplets $\{V_r, \zeta_r, \alpha_r\}_{r=1}^i$ are constructed such that the storage rate of the resulting closed-loop system $\Sigma_i^{x_{i+1}=\alpha_i}$ satisfies

$$\begin{aligned} \dot{V}_i(\Sigma_i^{x_{i+1}=\alpha_i}) \leq & -\sum_{j=1}^i \frac{c_j}{2}(x_j - \alpha_{j-1})^\tau(x_j - \alpha_{j-1}) - \sum_{s=1}^i \sum_{r=1}^s (-\beta_{rs})(\theta_r - \vartheta_r)^\tau(\theta_r - \vartheta_r) \\ & + \sum_{j=0}^2 \sum_{s=1}^i \sum_{r=1}^s \frac{\eta_{rj}^2}{4k_{rs}^{(j)}}, \end{aligned} \quad (19)$$

where $c_j > 0$, $k_{rs}^{(j)} > 0$, β_{rs} are constants ($j = 1, 2; r = 1, \dots, i; s \geq r$), $\beta_{jj} = -l_j$, $l_j = (1/4k_{jj}^{(0)}) > 0$ ($j = 1, \dots, i$) and for any $r < s$, $s = 1, \dots, i$, define $\eta_{r0}^2/4k_{rs}^{(0)} = 0$. Then at the $(i + 1)$ th step, we consider the auxiliary system $\Sigma_{i+1}^{x_{i+2}}$:

$$\dot{x}_j = f_j + G_j x_{j+1} + H_j \theta_j + \Delta_j, \quad j = 1, \dots, i + 1. \quad (20)$$

Let

$$\begin{aligned} V_{i+1} &= \frac{1}{2} \sum_{j=1}^{i+1} (x_j - \alpha_{j-1})^\tau(x_j - \alpha_{j-1}) + \sum_{j=1}^{i+1} \frac{1}{2\gamma_j} (\theta_j - \vartheta_j)^\tau(\theta_j - \vartheta_j) \\ &= V_i + \frac{1}{2}(x_{i+1} - \alpha_i)^\tau(x_{i+1} - \alpha_i) + \frac{1}{2\gamma_{i+1}}(\theta_{i+1} - \vartheta_{i+1})^\tau(\theta_{i+1} - \vartheta_{i+1}). \end{aligned} \quad (21)$$

The time derivative of V_{i+1} along the solution of the system $\Sigma_{i+1}^{x_{i+2}}$ is given by

$$\begin{aligned}
 \dot{V}_{i+1}(\Sigma_{i+1}^{x_{i+2}}) &= \dot{V}_i(\Sigma_i^{x_{i+1}}) + (x_{i+1} - \alpha_i)^\tau (\dot{x}_{i+1} - \dot{\alpha}_i) + (\theta_{i+1} - \vartheta_{i+1})^\tau \left(-\frac{1}{\gamma_{i+1}} \dot{\vartheta}_{i+1} \right) \\
 &= \dot{V}_i(\Sigma_i^{x_{i+1}=\alpha_i}) + (x_{i+1} - \alpha_i)^\tau \left[f_{i+1} + G_{i+1}x_{i+2} + H_{i+1}\theta_{i+1} + \Delta_{i+1} + G_i^\tau \frac{\partial V_i}{\partial x_i} \right. \\
 &\quad \left. - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial x_j^\tau} (f_j + G_j x_{j+1} + H_j \vartheta_j + \Delta_j) - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial \vartheta_j^\tau} \dot{\vartheta}_j \right] \\
 &\quad + (\theta_{i+1} - \vartheta_{i+1})^\tau \left(-\frac{1}{\gamma_{i+1}} \dot{\vartheta}_{i+1} \right) \\
 &= \dot{V}_i(\Sigma_i^{x_{i+1}=\alpha_i}) + (x_{i+1} - \alpha_i)^\tau \left[f_{i+1} + G_{i+1}x_{i+2} + H_{i+1}\vartheta_{i+1} \right. \\
 &\quad \left. + G_i^\tau \frac{\partial V_i}{\partial x_i} - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial x_j^\tau} (f_j + G_j x_{j+1} + H_j \vartheta_j) - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial \vartheta_j^\tau} \dot{\vartheta}_j \right] \\
 &\quad - \sum_{r=1}^i (x_{i+1} - \alpha_i)^\tau \frac{\partial \alpha_i}{\partial x_r^\tau} H_r (\theta_r - \vartheta_r) \\
 &\quad + (\theta_{i+1} - \vartheta_{i+1})^\tau \left[-\frac{1}{\gamma_{i+1}} \dot{\vartheta}_{i+1} + H_{i+1}^\tau (x_{i+1} - \alpha_i) \right] \\
 &\quad - (x_{i+1} - \alpha_i)^\tau \left[\sum_{r=1}^i \frac{\partial \alpha_i}{\partial x_r^\tau} \Delta_r - \Delta_{r+1} \right]. \tag{22}
 \end{aligned}$$

For purpose of compact notation, define $(\partial \alpha_i / \partial x_{i+1}^\tau) = -I$ ($i = 1, \dots, n-1$). Hence (22) can be rewritten as

$$\begin{aligned}
 \dot{V}_{i+1}(\Sigma_{i+1}^{x_{i+2}}) &= \dot{V}_i(\Sigma_i^{x_{i+1}=\alpha_i}) + (x_{i+1} - \alpha_i)^\tau \left[f_{i+1} + G_{i+1}x_{i+2} + H_{i+1}\vartheta_{i+1} + G_i^\tau \frac{\partial V_i}{\partial x_i} \right. \\
 &\quad \left. - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial x_j^\tau} (f_j + G_j x_{j+1} + H_j \vartheta_j) - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial \vartheta_j^\tau} \dot{\vartheta}_j \right] \\
 &\quad - \sum_{r=1}^i (x_{i+1} - \alpha_i)^\tau \frac{\partial \alpha_i}{\partial x_r^\tau} H_r (\theta_r - \vartheta_r) - \sum_{r=1}^{i+1} (x_{i+1} - \alpha_i)^\tau \frac{\partial \alpha_i}{\partial x_r^\tau} \Delta_r \\
 &\quad + (\theta_{i+1} - \vartheta_{i+1})^\tau \left[-\frac{1}{\gamma_{i+1}} \dot{\vartheta}_{i+1} + H_{i+1}^\tau (x_{i+1} - \alpha_i) \right]. \tag{23}
 \end{aligned}$$

Consider the term $(x_{i+1} - \alpha_i)^\tau (\partial \alpha_i / \partial x_r^\tau) \Delta_r$, ($r = 1, \dots, i + 1$). It follows from (5) that

$$-(x_{i+1} - \alpha_i)^\tau \frac{\partial \alpha_i}{\partial x_r^\tau} \Delta_r \leq \sum_{j=1}^2 \left[\delta_{ri+1}^{(j)} k_{ri+1}^{(j)} (x_{i+1} - \alpha_i)^\tau \frac{\partial \alpha_i}{\partial x_r^\tau} \frac{\partial \alpha_i^\tau}{\partial x_r} (x_{i+1} - \alpha_i) \right] + \sum_{j=1}^2 \frac{\eta_{rj}^2}{4k_{ri+1}^{(j)}}. \quad (24)$$

where

$$\delta_{ri+1}^{(j)} = \begin{cases} \rho_r^2, & j = 1 \\ 1, & j = 2 \end{cases} \quad \text{and} \quad k_{ri+1}^{(j)} > 0 \quad (j = 1, 2, \quad r = 1, \dots, i + 1)$$

are constants.

Note that

$$\begin{aligned} -(x_{i+1} - \alpha_i)^\tau \frac{\partial \alpha_{i+1}}{\partial x_r^\tau} H_r (\theta_r - \vartheta_r) &\leq \beta_{ri+1} (\theta_r - \vartheta_r)^\tau (\theta_r - \vartheta_r) \\ &+ \frac{1}{4\beta_{ri+1}} (x_{i+1} - \alpha_i)^\tau \frac{\partial \alpha_{i+1}}{\partial x_r^\tau} H_r H_1^\tau \frac{\partial \alpha_{i+1}^\tau}{\partial x_r} (x_{i+1} - \alpha_i). \end{aligned} \quad (25)$$

We have

$$\begin{aligned} \dot{V}_{i+1}(\Sigma_{i+1}^{x_{i+2}}) &\leq \dot{V}_i(\Sigma_i^{x_{i+1}=\alpha_i}) + (x_{i+1} - \alpha_i)^\tau \left[f_{i+1} + G_{i+1}x_{i+2} + H_{i+1}\vartheta_{i+1} \right. \\ &+ G_i^\tau \frac{\partial V_i}{\partial x_i} - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial \vartheta_j^\tau} \dot{\vartheta}_j + \sum_{r=1}^i \frac{1}{4\beta_{ri+1}} \frac{\partial \alpha_i}{\partial x_r^\tau} H_r H_r^\tau \frac{\partial \alpha_i^\tau}{\partial x_r} (x_{i+1} - \alpha_i) \\ &\left. - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial x_j^\tau} (f_j + G_j x_{j+1} + H_j \vartheta_j) + \sum_{r=1}^{i+1} \sum_{j=1}^2 \delta_{ri+1}^{(j)} k_{ri+1}^{(j)} \frac{\partial \alpha_i}{\partial x_r^\tau} \frac{\partial \alpha_i^\tau}{\partial x_r} (x_{i+1} - \alpha_i) \right] \\ &+ \sum_{r=1}^{i+1} \sum_{j=1}^2 \frac{\eta_{rj}^2}{4k_{ri+1}^{(j)}} + \sum_{r=1}^i \beta_{ri+1} (\theta_r - \vartheta_r)^\tau (\theta_r - \vartheta_r) \\ &+ (\theta_{i+1} - \vartheta_{i+1})^\tau \left[-\frac{1}{\gamma_{i+1}} \dot{\vartheta}_{i+1} + H_{i+1}^\tau (x_{i+1} - \alpha_i) \right]. \end{aligned} \quad (26)$$

Let

$$\dot{\vartheta}_{i+1} = \zeta_{i+1} = \gamma_{i+1} H_{i+1}^\tau (x_{i+1} - \alpha_i) - 2l_{i+1} \gamma_{i+1} \vartheta_{i+1} \quad (27)$$

for some constant $l_{i+1} > 0$.

Remark 4. It is noted that the adaptive law (27) is also of fixed σ -modification form [4–6], which guarantees $\vartheta_{i+1} \in L_\infty$ if $H_{i+1}^\tau x_{i+1} \in L_\infty$.

It follows from (26) and (27) that

$$\begin{aligned} \dot{V}_{i+1}(\Sigma_{i+1}^{x_{i+2}}) &\leq \dot{V}_i(\Sigma_i^{x_{i+1}=\alpha_i}) + (x_{i+1} - \alpha_i)^\tau \left[f_{i+1} + G_{i+1}x_{i+2} + H_{i+1}\vartheta_{i+1} \right. \\ &\quad + G_i^\tau \frac{\partial V_i}{\partial x_i} - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial \vartheta_j^\tau} \dot{\vartheta}_j + \sum_{r=1}^i \frac{1}{4\beta_{ri+1}} \frac{\partial \alpha_i}{\partial x_r^\tau} H_r H_r^\tau \frac{\partial \alpha_i^\tau}{\partial x_r} (x_{i+1} - \alpha_i) \\ &\quad \left. - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial x_j^\tau} (f_j + G_j x_{j+1} + H_j \vartheta_j) + \sum_{r=1}^{i+1} \sum_{j=1}^2 \delta_{ri+1}^{(j)} k_{ri+1}^{(j)} \frac{\partial \alpha_i}{\partial x_r^\tau} \frac{\partial \alpha_i^\tau}{\partial x_r} (x_{i+1} - \alpha_i) \right] \\ &\quad + \sum_{r=1}^{i+1} \sum_{j=1}^2 \frac{\eta_{rj}^2}{4k_{ri+1}^{(j)}} + \sum_{r=1}^i \beta_{ri+1} (\theta_r - \vartheta_r)^\tau (\theta_r - \vartheta_r) \\ &\quad + 2l_{i+1} (\theta_{i+1} - \vartheta_{i+1})^\tau \vartheta_{i+1}. \end{aligned} \tag{28}$$

Recalling the following inequality:

$$2l_{i+1} (\theta_{i+1} - \vartheta_{i+1})^\tau \vartheta_{i+1} \leq l_{i+1} \theta_{i+1}^\tau \theta_{i+1} - l_{i+1} (\theta_{i+1} - \vartheta_{i+1})^\tau (\theta_{i+1} - \vartheta_{i+1}) \tag{29}$$

and for notational simplification, let

$$l_{i+1} = \frac{1}{4k_{i+1i+1}^{(0)}} > \sum_{r=1}^i \beta_{ri+1}, \quad \frac{\eta_{i+10}^2}{4k_{i+1i+1}^{(0)}} = 0, \quad 1 \leq r < i + 1$$

and $\beta_{i+1i+1} = -l_{i+1}$, then one has

$$\begin{aligned} \dot{V}_{i+1}(\Sigma_{i+1}^{x_{i+2}}) &\leq \dot{V}_i(\Sigma_i^{x_{i+1}=\alpha_i}) + (x_{i+1} - \alpha_i)^\tau \left[f_{i+1} + G_{i+1}x_{i+2} + H_{i+1}\vartheta_{i+1} \right. \\ &\quad + G_i^\tau \frac{\partial V_i}{\partial x_i} - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial \vartheta_j^\tau} \dot{\vartheta}_j + \sum_{r=1}^i \frac{1}{4\beta_{ri+1}} \frac{\partial \alpha_i}{\partial x_r^\tau} H_r H_r^\tau \frac{\partial \alpha_i^\tau}{\partial x_r} (x_{i+1} - \alpha_i) \\ &\quad \left. - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial x_j^\tau} (f_j + G_j x_{j+1} + H_j \vartheta_j) + \sum_{r=1}^{i+1} \sum_{j=1}^2 \delta_{ri+1}^{(j)} k_{ri+1}^{(j)} \frac{\partial \alpha_i}{\partial x_r^\tau} \frac{\partial \alpha_i^\tau}{\partial x_r} (x_{i+1} - \alpha_i) \right] \\ &\quad + \sum_{r=1}^{i+1} \sum_{j=0}^2 \frac{\eta_{rj}^2}{4k_{ri+1}^{(j)}} - \sum_{r=1}^{i+1} (-\beta_{ri+1}) (\theta_r - \vartheta_r)^\tau (\theta_r - \vartheta_r). \end{aligned} \tag{30}$$

Let

$$\begin{aligned} \alpha_{i+1} = & -G_{i+1}^+ \left[\frac{c_{i+1}}{2} + f_{i+1} + H_{i+1}\vartheta_{i+1} - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial x_j^\tau} (f_j + G_j x_{j+1} + H_j \vartheta_j) \right. \\ & + G_i^\tau \frac{\partial V_i}{\partial x_i} - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial \vartheta_j^\tau} \dot{\vartheta}_j + \sum_{r=1}^i \frac{1}{4\beta_{ri+1}} \frac{\partial \alpha_i}{\partial x_r^\tau} H_r H_r^\tau \frac{\partial \alpha_i^\tau}{\partial x_r} (x_{i+1} - \alpha_i) \\ & \left. - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial x_j^\tau} (f_j + G_j x_{j+1} + H_j \vartheta_j) + \sum_{r=1}^{i+1} \sum_{j=1}^2 \delta_{ri+1}^{(j)} k_{ri+1}^{(j)} \frac{\partial \alpha_i}{\partial x_r^\tau} \frac{\partial \alpha_i^\tau}{\partial x_r} (x_{i+1} - \alpha_i) \right], \end{aligned} \quad (31)$$

where $c_{i+1} > 0$ is constant.

It follows from (19) and (31) that

$$\begin{aligned} \dot{V}_{i+1}(\Sigma_{i+1}^{x_{i+2}=\alpha_{i+1}}) \leq & - \sum_{j=1}^{i+1} \frac{c_j}{2} (x_j - \alpha_{j-1})^\tau (x_j - \alpha_{j-1}) \\ & - \sum_{s=1}^{i+1} \sum_{r=1}^s (-\beta_{rs}) (\theta_r - \vartheta_r)^\tau (\theta_r - \vartheta_r) + \sum_{j=0}^2 \sum_{s=1}^{i+1} \sum_{r=1}^s \frac{\eta_{rj}^2}{4k_{rs}^{(j)}}. \end{aligned} \quad (32)$$

Similar to the previous steps, the proposed stabilizing function α_{i+1} in (31) for the auxiliary system $\Sigma_{i+1}^{x_{i+2}}$ consists of two parts. The first part is a fuzzy logic system with adaptation law $\dot{\vartheta}_r = \zeta_r$, $r = 1, \dots, i + 1$ to learn the unknown dynamics N_r , $r = 1, \dots, i + 1$ of the system $\Sigma_{i+1}^{x_{i+2}}$, and the second part is the robust controller to achieve uniformly ultimately bounded performance.

Choosing $c_j > 0$, $k_{rs}^{(j)} > 0$ ($j = 1, 2$; $r = 1, \dots, i + 1$, $s \geq r$), and $\beta_{rs} > 0$ ($r < s$, $r = 1, \dots, i + 1$) and $l_j = (1/4k_{jj}^{(0)}) > \sum_{r=1}^{j-1} \beta_{rj}$ ($j = 2, \dots, i + 1$) properly and using (32), one can prove that all the signals of the auxiliary adaptive fuzzy closed-loop system $\Sigma_{i+1}^{x_{i+2}=\alpha_{i+1}}$ are uniformly ultimately bounded and that the norm of the signal x_1 exponentially converges to an arbitrarily specified small number. In these $i + 1$ steps, $i + 1$ triplets $\{V_r, \zeta_r, \alpha_r\}_{r=1}^{i+1}$ are obtained. Note that the triplet $\{V_{i+1}, \zeta_{i+1}, \alpha_{i+1}\}$ obtained in the $(i + 1)$ th step has recursive form. The dissipative inequality (32) is the same as (19), if one replaces the index i by $i + 1$. The backstepping process can be continued from $i = 2$ to $i = n - 1$ using the same methodology.

Step n: At the step n , the actual control appears explicitly in the overall system Σ_n^u , which allows us to obtain the explicit control law for the overall system. In this step, we obtain n triplets $\{V_r, \zeta_r, \alpha_r\}_{r=1}^n$ as follows:

$$\begin{aligned} V_{i+1} &= \frac{1}{2} \sum_{j=1}^{i+1} (x_j - \alpha_{j-1})^\tau (x_j - \alpha_{j-1}) + \sum_{j=1}^{i+1} \frac{1}{2\gamma_j} (\theta - \vartheta_j)^\tau (\theta - \vartheta_j) \\ &= V_i + \frac{1}{2} (x_{i+1} - \alpha_i)^\tau (x_{i+1} - \alpha_i) + \frac{1}{2\gamma_{i+1}} (\theta_{i+1} - \vartheta_{i+1})^\tau (\theta_{i+1} - \vartheta_{i+1}) \end{aligned} \quad (33)$$

$$\dot{\vartheta}_{i+1} = \zeta_{i+1} = \gamma_{i+1} H_{i+1}^\tau (x_{i+1} - \alpha_i) - 2l_{i+1}\gamma_{i+1}\vartheta_{i+1} \quad (34)$$

$$\begin{aligned} \alpha_{i+1} = & -G_{i+1}^+ \left[\frac{c_{i+1}}{2} + f_{i+1} + H_{i+1}\vartheta_{i+1} - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial x_j^\tau} (f_j + G_j x_{j+1} + H_j \vartheta_j) \right. \\ & + G_i^\tau \frac{\partial V_i}{\partial x_i} - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial \vartheta_j^\tau} \dot{\vartheta}_j + \sum_{r=1}^i \frac{1}{4\beta_{ri+1}} \frac{\partial \alpha_i}{\partial x_r^\tau} H_r H_r^\tau \frac{\partial \alpha_i^\tau}{\partial x_r} (x_{i+1} - \alpha_i) \\ & \left. - \sum_{j=1}^i \frac{\partial \alpha_i}{\partial x_j^\tau} (f_j + G_j x_{j+1} + H_j \vartheta_j) + \sum_{r=1}^{i+1} \sum_{j=1}^2 \delta_{ri+1}^{(j)} k_{ri+1}^{(j)} \frac{\partial \alpha_i}{\partial x_r^\tau} \frac{\partial \alpha_i^\tau}{\partial x_r} (x_{i+1} - \alpha_i) \right], \end{aligned} \quad (35)$$

where $i = 0, \dots, n - 1, \alpha_0 = 0,$

$$\delta_{rs}^{(j)} = \begin{cases} \rho_r^2, & j = 1 \\ 1, & j = 2 \end{cases} \text{ for all } r \leq s, s = 1, \dots, n \text{ and } \frac{\partial \alpha_i}{\partial x_{i+1}^\tau} = -I \ (i = 1, \dots, n - 1).$$

The robust adaptive control law for the overall system is then given by

$$\begin{aligned} \dot{\vartheta}_r &= \zeta_r, \quad r = 1, \dots, n, \\ u &= \alpha_n \end{aligned} \quad (36)$$

and the design process is thus completed.

In addition, the selection of the design parameters in the above process can be summarized as follows:

- Constants $k_{rs}^{(j)} > 0$ ($j = 1, 2; r = 1, \dots, n, s \geq r$) can be chosen arbitrarily, and for $r < s, s = 1, \dots, n,$ we define $\eta_{r0}^2/4k_{rs}^{(0)} = 0.$
- Positive constants γ_r ($r < s, s = 1, \dots, n$) can be chosen arbitrarily.
- For $r = 1, \dots, n,$ choose $c_r = 2\delta,$ where δ is a positive constant that can be chosen arbitrarily.
- Constants $\beta_{rs} > 0$ ($r < s, s = 1, \dots, n$) can be chosen arbitrarily.
- For $r = 1, \dots, n - 1,$ choose $l_r = -\beta_{rr} = (1/4k_{rr}^{(0)}) = (\delta/\gamma_r) + \sum_{j=r+1}^n \beta_{rj}$ and $l_n = -\beta_{nn} = (1/4k_{nn}^{(0)}) = (\delta/\gamma_n).$

4. Performance analysis of closed loop system

In this section, we will give performance analysis for the resulting closed-loop system discussed in the last section.

Theorem 4.1. Consider system (1) under Assumptions 1–3 and the robust adaptive control law (33)–(36). All the signals of the resulting closed-loop system are globally uniformly ultimately bounded. Moreover, $\forall \varepsilon > 0,$ the design parameters can be chosen properly such that the output y satisfies

$$\|y\| \leq e^{-\delta(t-t_0)} \sqrt{2V_n|_{t=t_0}} + \varepsilon, \quad (37)$$

where $V_n|_{t=t_0}$ stands for the value of the storage function of the closed-loop system at $t_0.$ The following lemma will be used to prove the theorem.

Lemma 4.1. Let $V : [0, \infty) \rightarrow R$ satisfy the inequality

$$\dot{V} \leq -2\lambda V + M \quad \forall t \geq 0, \tag{38}$$

where λ and M are positive constants. Then

$$V(t) \leq V(t_0) \exp[-2\lambda(t - t_0)] + \frac{M}{2\lambda} \quad \forall t \geq t_0. \tag{39}$$

Proof. It follows from $\dot{V}(\tau) + 2\lambda V(\tau) \leq M$ that

$$e^{2\lambda\tau} \dot{V}(\tau) + 2\lambda e^{2\lambda\tau} V(\tau) \leq e^{2\lambda\tau} M.$$

Therefore,

$$\frac{d}{d\tau} [e^{2\lambda\tau} V(\tau)] \leq e^{2\lambda\tau} M.$$

Integrating this inequality from t_0 to t yields

$$e^{2\lambda t} V(t) - e^{2\lambda t_0} V(t_0) \leq \frac{M}{2\lambda} (e^{2\lambda t} - e^{2\lambda t_0}) \leq \frac{M}{2\lambda} e^{2\lambda t}.$$

Hence

$$V(t) \leq V(t_0) \exp[-2\lambda(t - t_0)] + \frac{M}{2\lambda}.$$

This concludes the proof of the lemma. \square

Proof of Theorem 4.1. The storage function candidate of the closed-loop system $\Sigma_n^{u=\alpha_n}$ is chosen as

$$V_n = \frac{1}{2} \sum_{j=1}^n (x_j - \alpha_{j-1})^\tau (x_j - \alpha_{j-1}) + \sum_{j=1}^n \frac{1}{2\gamma_j} (\theta_j - \vartheta_j)^\tau (\theta_j - \vartheta_j). \tag{40}$$

It is clear from the design process that

$$\begin{aligned} \dot{V}_n(\Sigma_n^{u=\alpha_n}) &\leq - \sum_{j=1}^n \frac{c_j}{2} (x_j - \alpha_{j-1})^\tau (x_j - \alpha_{j-1}) \\ &\quad - \sum_{s=1}^n \sum_{r=1}^s (-\beta_{rs}) (\theta_r - \vartheta_r)^\tau (\theta_r - \vartheta_r) + \sum_{j=0}^2 \sum_{s=1}^n \sum_{r=1}^s \frac{\eta_{rj}^2}{4k_{rs}^{(j)}}. \end{aligned} \tag{41}$$

Note that

$$\sum_{s=1}^n \sum_{r=1}^s (-\beta_{rs}) (\theta_r - \vartheta_r)^\tau (\theta_r - \vartheta_r) = \sum_{r=1}^n \sum_{s=r}^n (-\beta_{rs}) (\theta_r - \vartheta_r)^\tau (\theta_r - \vartheta_r). \tag{42}$$

With the chosen parameters, one gets

$$\begin{aligned} \sum_{s=1}^n \sum_{r=1}^s (-\beta_{rs})(\theta_r - \vartheta_r)^\tau (\theta_r - \vartheta_r) &= \sum_{r=1}^n \left[l_r - \sum_{s=r+1}^n (\beta_{rs}) \right] (\theta_r - \vartheta_r)^\tau (\theta_r - \vartheta_r) \\ &= 2\delta \sum_{r=1}^n \frac{1}{2\gamma_r} (\theta_r - \vartheta_r)^\tau (\theta_r - \vartheta_r). \end{aligned} \tag{43}$$

It follows from (41) and (43) that

$$\dot{V}_n \leq -2\delta V_n + M, \tag{44}$$

where

$$M = \sum_{j=0}^2 \sum_{s=1}^n \sum_{r=1}^s \frac{\eta_{rj}^2}{4k_{rs}^{(j)}}. \tag{45}$$

Thus

$$\dot{V}_n \leq -2\delta V_n + M. \tag{46}$$

It follows from Lemma 4.1 that

$$V_n(t) \leq V_n(t)|_{t=t_0} e^{-2\delta(t-t_0)} + \frac{M}{2\delta}. \tag{47}$$

This implies that all the signals of the resulting closed-loop system are globally uniformly ultimately bounded, and that $\|x_1(t)\|^2 \leq 2V_n(t) \leq 2V_n|_{t=t_0} e^{-2\delta(t-t_0)} + M/\delta$, that is

$$\|y\| = \|x_1(t)\| \leq \sqrt{2V_n|_{t=t_0}} \exp(-\delta(t-t_0)) + \sqrt{\frac{M}{\delta}}. \tag{48}$$

We now show how to choose the parameters such that $\sqrt{(M/\delta)} < \varepsilon$ for any arbitrarily small positive constant $\varepsilon > 0$. For $j = 1, 2; r = 1, \dots, n, s \geq r$, choose $k_{rs}^{(j)} > 0$ such that

$$k_{rs}^{(j)} > \frac{\eta_{rj}^2 n^2}{\delta \varepsilon^2}. \tag{49}$$

For $r = 1, \dots, n$, choose $k_{rr}^{(0)}$ such that

$$k_{rr}^{(0)} > \frac{\eta_{r0}^2 n}{2\delta \varepsilon^2}. \tag{50}$$

For $r < s, s = 1, \dots, n$, recalling $l_r = -\beta_{rr} = (1/4k_{rr}^{(0)}) = (\delta/\gamma_r) + \sum_{j=r+1}^n \beta_{rj}$ leads to

$$0 < \beta_{rs} < \frac{\delta \varepsilon^2}{2n^2 \eta_{r0}^2}, \quad \gamma_r > \frac{2\eta_{r0}^2 n}{\varepsilon^2}. \tag{51}$$

It follows from (49) and (50) that $\sqrt{(M/\delta)} < \varepsilon$. This ends the proof. \square

Remark 5. It can also be shown that $x = \bar{x}_n$ and $\vartheta = \bar{\vartheta}_n$ of the resulting closed-loop system converge to the set along the trajectories:

$$\mathfrak{R} = \left\{ (x, \vartheta) : \sum_{j=1}^n c_j (x_j - \alpha_{j-1})^\tau (x_j - \alpha_{j-1}) + \sum_{j=1}^n \gamma_j (\theta_j - \vartheta_j)^\tau (\theta_j - \vartheta_j) \leq 2M \right\},$$

where $\vartheta^\tau = [\vartheta_1^\tau, \dots, \vartheta_n^\tau]$.

Remark 6. Note that the adaptive law $\dot{\vartheta}_n = \gamma_n H_n^\tau (x_n - \alpha_{n-1}) - 2l_n \gamma_n \vartheta_n$ is also of fixed σ -modification form. This form can be replaced by switching σ -modification form [6] for robustness improvement and the above stability and performance result still holds.

Remark 7. It can be observed from Eqs. (35), (36) and (51) that the controllers are of high gain form. For example, α_n is of terms with the factors $1/4\beta_{rn}$, $r = 1, \dots, n - 1$. The high gain terms in the controllers are used to compensate the approximation errors $\tilde{\theta}_r = \theta_r - \vartheta_r$, $r = 1, \dots, n$ and other uncertainties.

Remark 8. It is interesting to note that the systems studied in [3] are fully feedback linearizable. Its feedback linearized form is a very special case of Eq. (1). The controller given in [3] can be easily obtained without recursive design, and the resulting closed-loop system can achieve the H_∞ tracking performance for a prescribed attenuation level. However, controller (36) for the much more general nonlinear system (1) is obtained via recursive design, and the resulting closed-loop control system can only achieve the uniform ultimate bound performance.

5. A simulation example

We consider two inverted pendulums connected by a moving spring mounted on two carts [13]. We assume that the pivot position of the moving spring is a function of time that can change along the length l of the pendulums. The motion of the carts is specified. For this example we specify this motion as sinusoidal trajectories. The input to each pendulum is the torque u_i applied at the pivot point. See Fig. 1 for illustration.

The dynamic equations of the inverted double pendulums on carts can be described as

$$\ddot{\varphi}_1 = \frac{g}{cl} \varphi_1 + \frac{1}{cm l^2} u_1 + N_{21}(\varphi_1, \dot{\varphi}_1) + \left[\frac{k(a(t) - cl)}{cm l^2} (-a(t)\varphi_1 + a(t)\varphi_2 - y_1 + y_2) \right], \quad (52)$$

$$\ddot{\varphi}_2 = \frac{g}{cl} \varphi_2 + \frac{1}{cm l^2} u_2 + N_{22}(\varphi_2, \dot{\varphi}_2) + \left[\frac{k(a(t) - cl)}{cm l^2} (-a(t)\varphi_2 + a(t)\varphi_1 - y_1 + y_2) \right], \quad (53)$$

where φ_i and $\dot{\varphi}_i$ are the angles and angular velocities of the pendulums, respectively, with respect to vertical axes, u_1 and u_2 are the control torques applied to the pendulums, $c = m/(M + m)$, k and g are spring and gravity constants, respectively, $N_{21}(\varphi_1, \dot{\varphi}_1)$ and $N_{22}(\varphi_2, \dot{\varphi}_2)$ are uncertain terms.

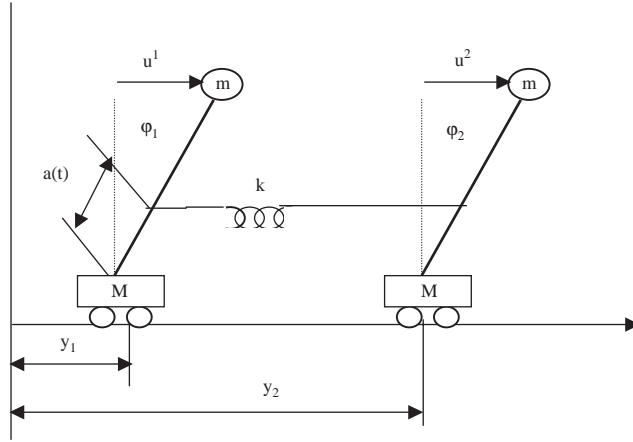


Fig. 1. Two inverted pendulums on carts.

Defining the state vector $x_1 = [x_{11}, x_{12}]^\tau = [\varphi_1, \varphi_2]^\tau$ and $x_2 = [x_{21}, x_{22}]^\tau = [\dot{\varphi}_1, \dot{\varphi}_2]^\tau$, the dynamic equations of the inverted double pendulums on carts can be rewritten as

$$\dot{x}_{11} = x_{21},$$

$$\dot{x}_{12} = x_{22},$$

$$\dot{x}_{21} = \frac{g}{cl}x_{11} + \frac{1}{cm}u_1 + N_{21}(x_{11}, x_{21}) + \left[\frac{k(a(t) - cl)}{cm}(-a(t)x_{11} + a(t)x_{12} - y_1 + y_2) \right],$$

$$\dot{x}_{22} = \frac{g}{cl}x_{12} + \frac{1}{cm}u_2 + N_{22}(x_{12}, x_{22}) + \left[\frac{k(a(t) - cl)}{cm}(-a(t)x_{12} + a(t)x_{11} - y_1 + y_2) \right], \quad (54)$$

or in vector form as

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = \begin{bmatrix} \frac{g}{cl}x_{11} \\ \frac{g}{cl}x_{12} \end{bmatrix} + \frac{1}{cm} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u + \begin{bmatrix} N_{21} \\ N_{22} \end{bmatrix} + \begin{bmatrix} \left[\frac{k(a(t)-cl)}{cm}(-a(t)x_{11} + a(t)x_{12} - y_1 + y_2) \right] \\ \left[\frac{k(a(t)-cl)}{cm}(-a(t)x_{12} + a(t)x_{11} - y_1 + y_2) \right] \end{bmatrix}, \quad (55)$$

where $u = [u_1 \ u_2]^\tau$. It can be seen that the system has the form of (1) with

$$f_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad d_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad m_1 = 2, \quad f_2 = \begin{bmatrix} \frac{g}{cl}x_{11} \\ \frac{g}{cl}x_{21} \end{bmatrix},$$

$$G_2 = \frac{1}{cm_l^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N_2 = \begin{bmatrix} N_{21} \\ N_{22} \end{bmatrix}, \quad d_2 = \begin{bmatrix} \left[\frac{k(a(t)-cl)}{cm_l^2} (-a(t)x_{11} + a(t)x_{12} - y_1 + y_2) \right] \\ \left[\frac{k(a(t)-cl)}{cm_l^2} (-a(t)x_{12} + a(t)x_{11} - y_1 + y_2) \right] \end{bmatrix},$$

$$m_2 = 2.$$

In order to approximate the uncertain vector $N_2 = \begin{bmatrix} N_{21} \\ N_{22} \end{bmatrix}$, we define three fuzzy sets for each x_{ij} , $i = 1, 2$, $j = 1, 2$ with labels negative (N_{ij}), zero (Z_{ij}) and positive (P_{ij}). Fuzzy membership functions for these labels are defined as

$$\begin{aligned} \mu_{N_{11}} &= e^{-0.03(x_{11}+20)^2}, & \mu_{Z_{11}} &= e^{-0.03x_{11}^2}, & \mu_{P_{11}} &= e^{-0.03(x_{11}-20)^2}, \\ \mu_{N_{12}} &= e^{-0.03(x_{12}+20)^2}, & \mu_{Z_{12}} &= e^{-0.03x_{12}^2}, & \mu_{P_{12}} &= e^{-0.03(x_{12}-20)^2}, \\ \mu_{N_{21}} &= e^{-10(x_{21}+1)^2}, & \mu_{Z_{21}} &= e^{-10x_{21}^2}, & \mu_{P_{21}} &= e^{-10(x_{21}-1)^2}, \\ \mu_{N_{22}} &= e^{-10(x_{22}+1)^2}, & \mu_{Z_{22}} &= e^{-10x_{22}^2}, & \mu_{P_{22}} &= e^{-10(x_{22}-1)^2}. \end{aligned}$$

The approximation of the unknown nonlinearity N_{21} can be calculated by $H_{21}^\tau \theta_{12}$ with $\theta_{21} \in R^9$, and N_{22} by $H_{22}^\tau \theta_{22}$ with $\theta_{22} \in R^9$, where the entries of $H_{21}^\tau = [h_{211}, h_{212}, \dots, h_{219}]$ and $H_{22}^\tau = [h_{221}, h_{222}, \dots, h_{229}]$ can be calculated by (2) in terms of the above membership functions.

The robust adaptive control law is given as

$$\dot{\vartheta}_2 = \zeta_2 = \gamma_2 H_2^\tau (x_2 - \alpha_1) - 2\sigma l_2 \gamma_2 \vartheta_2, \tag{56}$$

$$\begin{aligned} u = -G_2^+ & \left[\frac{c_2}{2} (x_2 - \alpha_1) + f_2 + H_2 \vartheta_2 + x_1 - \frac{\partial \alpha_1}{\partial x_1^\tau} x_2 \right. \\ & \left. + \frac{c_1}{4} k_{12}^{(2)} (x_2 - \alpha_1) + \rho_2^2 k_{22}^{(2)} (x_2 - \alpha_1) \right], \end{aligned} \tag{57}$$

where

$$\alpha_1 = -\frac{c_1}{2} x_1, \quad \rho_2^2 = x_{11}^2 + x_{12}^2, \quad H_2 = \begin{bmatrix} H_{21}^\tau & 0 \\ 0 & H_{22}^\tau \end{bmatrix},$$

and

$$\sigma = \begin{cases} 1 & \text{if } \|\vartheta_2\| \geq \mu, \\ 0 & \text{if } \|\vartheta_2\| < \mu, \end{cases} \tag{58}$$

with μ being a positive design constant.

In simulation, we choose $g = l = 1, k = 1, M = m = 50, y_1 = \sin(\omega_1 t), y_2 = \sin(\omega_2 t) + L, L = 2, a(t) = \sin(\omega t), N_{21} = -(m/M)x_{21}^2 \sin(x_{11}),$ and $N_{22} = -(m/M)x_{22}^2 \sin(x_{12}).$

A number of simulations have been carried out with different initial conditions. These simulation results demonstrate consistent transient and steady performance. One set of typical results is recorded in Fig. 2

with $c_1 = c_2 = k_{12}^{(2)} = 10, k_{22}^{(2)} = \gamma_2 = 1, l_2 = 5,$ where the states x_{11} and x_{12} are shown in (a), x_{21} and x_{22} in (b), a few selected parameters of estimation in (c), and the control inputs in (d). Fig. 3(a)

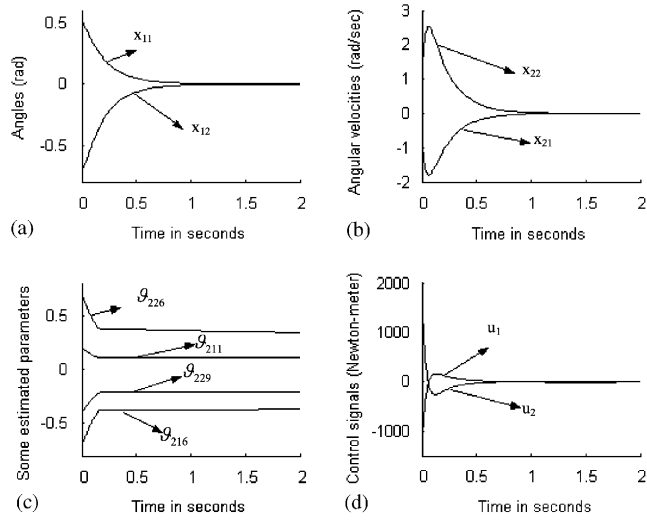


Fig. 2. Time responses of the closed-loop system.

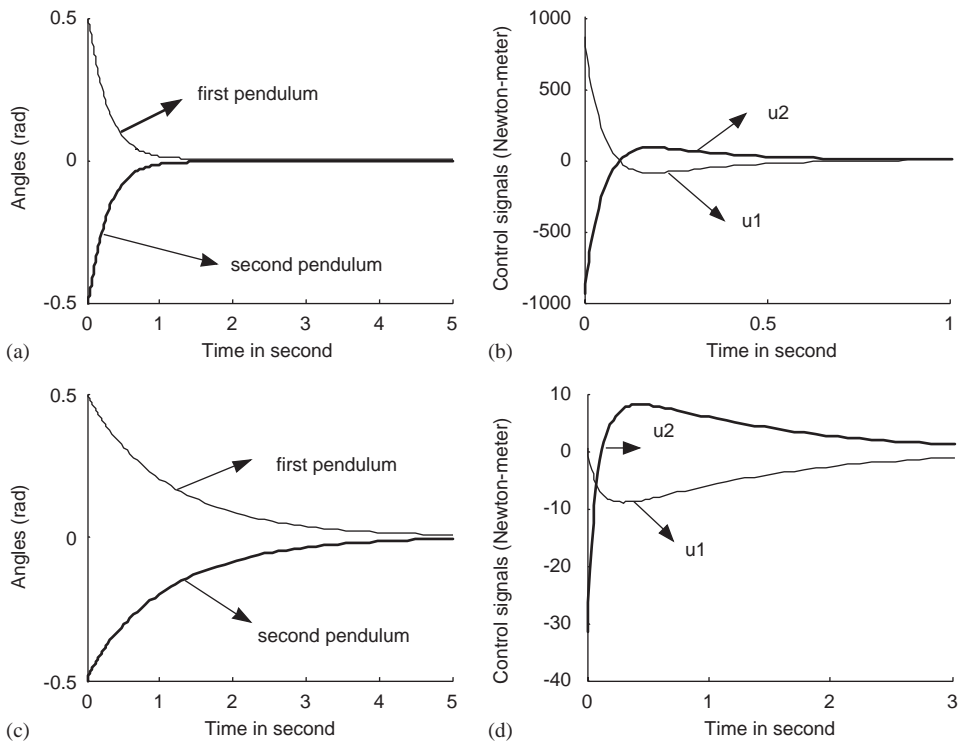


Fig. 3. Comparison of time responses of the closed-loop system.

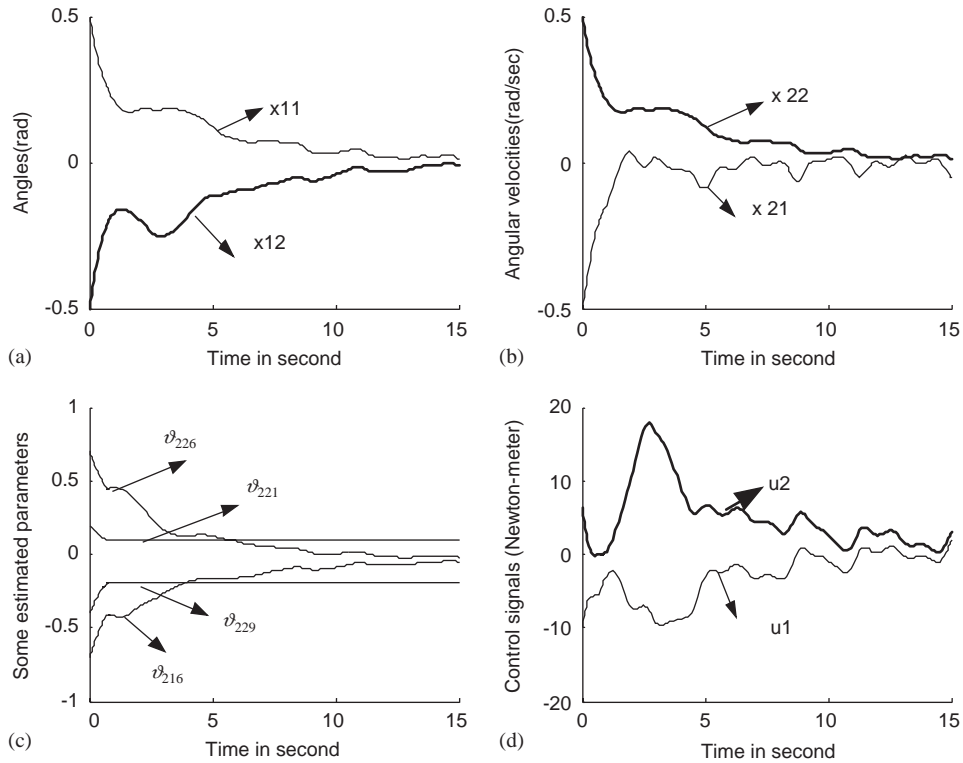


Fig. 4. Time responses of the closed-loop system with different design parameters.

and (b) provide the responses for our controller with $c_1 = c_2 = 9$, $k_{12}^{(2)} = 8$, $k_{22}^{(2)} = 1$, $\gamma_2 = 1$, $l_2 = 4.5$ and Fig. 3(c) and (d) provide the responses for the semi-adaptive controller outlined in Theorem 2 in [13]. It can be observed that the proposed robust adaptive controller with appropriate design parameters can achieve better transient and steady performance compared with the simulation results reported in [13]. In addition, another set of simulation results is recorded in Fig. 4 with parameters set to $c_1 = c_2 = 1$, $k_{12}^{(2)} = 1$, $k_{22}^{(2)} = 0.5$, $\gamma_2 = 1$, $l_2 = 0.5$, where the states x_{11} and x_{12} are shown in (a), x_{21} and x_{22} in (b), a few selected parameters of estimation in (c), and the control inputs are shown in (d). It is observed from Figs. 2 and 4 that the larger design parameters ($c_1 = c_2 = 2\delta > 0$, $k_{12}^{(2)} > 0$, $k_{22}^{(2)} > 0$, $\gamma_2 > 0$, $l_2 = \delta/\gamma_2$) lead to better transient and steady performance, while requiring larger control signals.

6. Conclusions

Using the backstepping technique and fuzzy approximation method, a hybrid robust adaptive fuzzy controller is proposed for a class of multivariable nonlinear systems with uncertain dynamics and external disturbances. This controller has an adaptation mechanism to approximate the unknown modeling uncertainties and high gain terms to compensate the approximation errors. All signals of the resulting closed-loop system are globally uniformly ultimately bounded and the norm of the output exponentially

converges to an arbitrarily specified small number. The systems we studied can characterize a large class of motion control systems with Lagrange form. Our study is an extension of the results in [1,7,9–11]. A simulation example is utilized to illustrate the controller design procedure and control performance.

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