A workload-dependent $M/G/1$ queue under a two-stage service policy

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Abstract

We consider an $M/G/1$ queueing system where the speed of the server depends on the amount of work present in the system. As a service policy, we adopt the $P_{M}^{λ,τ}$ release policy in a dam model. By using the level crossing theory and solving the corresponding integral equations, we obtain the stationary distribution of the workload in the system explicitly.

Keywords: $P_{M}^{λ,τ}$ policy; $M/G/1$ queue; stationary distribution; workload-dependent service speed

1 Introduction

The $P_{M}^{λ,τ}$ policy was introduced by Yeh [15] as a generalized release policy of the $P_{M}^{λ}$ policy of Faddy [11] for a dam with input formed by a Wiener process. Under the $P_{M}^{λ,τ}$ policy, the release rate is zero until the level of water reaches $λ(>0)$ and as soon as this occurs, water is released at rate $M(>0)$ until the level of water reaches $τ(0<τ<λ)$; once the water reaches level $τ$ the release rate remains zero until level $λ$ is reached again. Abdel-Hameed [1] considered the optimal control of a dam using $P_{M}^{λ,τ}$ policies when the input process is a

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compound Poisson process with positive drift. Bae et al. [4] determined the long-run average cost per unit time under the $P_{\lambda,\tau}^M$ policy in a finite dam with a compound Poisson input where they assumed the initial releasing rate to be a constant $a(\geq 0)$ instead of 0. Under the $P_{\lambda}^M$ policy, the stationary distribution of the workload in the $M/G/1$ queueing system with constant service speeds was derived in Bae et al. [3].

On the other hand, in queueing systems such as human-server systems the speed of the server often depends on the amount of work present. For example, Bertrand and Van Ooijen [7] described a production system where the speed of the server is relatively low when there is much work or when there is little work. See Bekker et al. [6] for more examples. In fact, models where the service speed is dependent on the workload originate from the studies of dams and storage processes. Dams with compound Poisson inputs and a general release rate function were studied by Asmussen [2], Cohen [10], Harrison and Resnick [12] and many others. For dams with more general input processes, see e.g. Brockwell et al. [8] and Kaspi et al. [13].

In this paper, we modify the $P_{\lambda,\tau}^M$ policy and introduce it as a two-stage service policy for the $M/G/1$ queueing system having workload-dependent service speeds; a server is initially idle and starts to serve, if a customer arrives, with service speed $r_1(\cdot)$ depending on the amount of work present in the system. The customers arrive according to a Poisson process of rate $\nu(> 0)$ and each customer brings a job consisting of an amount of work to be processed that is independently and identically distributed with a distribution function $G$ and a mean $m(> 0)$. If the workload exceeds threshold $\lambda(> 0)$, the server changes his service speed to another workload-dependent speed $r_2(\cdot)$ instantaneously and continues to follow that service speed until the workload level reaches $\tau(0 < \tau < \lambda)$. When the workload reaches level $\tau$, the service speed is changed again to $r_1(\cdot)$ instantaneously. The service speed $r_1(\cdot)$ is kept until the level up-crosses $\lambda$ again. We assume that $r_1(0) = 0$ and $r_1(\cdot)$ and $r_2(\cdot)$ are strictly positive, left continuous, and have a strictly positive right limit on $(0, \lambda)$ and on $(\tau, \infty)$, respectively. We also assume that $1/r_i(x)$, for $i = 1, 2$, are
integrable over any finite interval $[a, b] \subset (0, \infty)$. In particular, define

$$R_1(x) := \int_0^x \frac{1}{r_1(y)} \, dy, \quad 0 \leq x \leq \lambda,$$

representing the time required for the workload process with the service speed $r_1(\cdot)$ to become zero in the absence of any arrivals, starting with workload $x$. We assume that $R_1(x) < \infty$ for all $0 \leq x \leq \lambda$, which implies that the workload process has an atom at level 0 (see Asmussen [2, p. 288]). For the stability of the system, the speed functions must satisfy $\limsup_{x \to \infty} m \nu / r_i(x) < 1$, for $i = 1, 2$.

Bar-Lev and Perry [5] had discussed a similar model for a dam to ours and derived integral equations whose solutions determine the stationary distribution of the dam content. They also demonstrated such a determination explicitly for the case of exponential jumps. In this paper, using a similar method as in Bae et al. [3], we derive and solve the integro-differential equation for the distribution of the workload at the exit time from $(0, \lambda]$. Together with the level crossing theory, it enables us to determine the explicit stationary distribution of the workload not only for the case of exponential jumps but also for the case of generally distributed jumps.

In Section 2, we determine the distribution of the excess amount over $\lambda$ at an exit time from $[0, \lambda]$ along with the distribution of the workload at the exit time from $(0, \lambda]$. Adopting the level crossing analysis carried out in Bar-Lev and Perry [5], the stationary distribution of the workload is derived in Section 3. In Section 4, as an example we obtain an explicit distribution for the $M/M/1$ queueing system under the $P^M_{\lambda, \tau}$ policy with constant service speeds, which coincides with the result obtained in Bae et al. [3] in case $\tau = 0$. 

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2 The excess amount over $\lambda$ at the exit time from $[0, \lambda]$

Let $X(t)$ denote the workload of the system at time $t$ under the service policy described in the previous section. Let $X(0) = 0$ and define

\[ T_0^\lambda = \inf \{ t > 0 \mid X(t) > \lambda \}, \]
\[ T_0^\tau = \inf \{ t > T_0^\lambda \mid X(t) = \tau \}, \]

and for $n \geq 1$,

\[ T_n^\lambda = \inf \{ t > T_{n-1}^\tau \mid X(t) > \lambda \}, \]
\[ T_n^\tau = \inf \{ t > T_n^\lambda \mid X(t) = \tau \}, \]

then $\{X(t), t \geq 0\}$ is a delayed regenerative process having $T_0^\tau, T_1^\tau, T_2^\tau, \ldots$ as regeneration points. Figure 1 depicts a typical realization of the workload process $\{X(t), t \geq 0\}$.

Since $\{X(t), t \geq 0\}$ is non-Markovian, we decompose it into two Markov processes. Let $\{X_1(t), t \geq 0\}$ be a process obtained from $\{X(t), t \geq 0\}$ by deleting the time periods from $T_n^\lambda$ to $T_n^\tau$, for all $n \geq 0$, and by gluing together the remaining periods. Note that in the process $\{X_1(t), t \geq 0\}$ the system operates
with service speed function $r_1(\cdot)$. Let \{$X_2(t), t \geq 0$\} be formed similarly by separating and connecting the periods which start at $T^n_0$ and end at $T^n_\tau$, for all $n \geq 0$. Then, clearly the process \{$X_2(t), t \geq 0$\} has the service speed function $r_2(\cdot)$.

Now, we observe the excess amount over $\lambda$ at the first passage time through $\lambda$ of the process \{$X(t), t \geq 0$\}, that is, $X(T_\lambda) - \lambda$. Note that it is the same as the excess amount over $\lambda$ at the end of the cycle of the process \{$X_1(t), t \geq 0$\}.

Let us denote the exit time of the process \{$X_1(t), t \geq 0$\}, starting at $x$, by $T_x$, namely,

$$T_x = \inf\{t \geq 0 \mid X_1(t) \notin (0, \lambda], X_1(0) = x\}, \quad 0 \leq x \leq \lambda,$$

and define the distribution of the workload at the exit time $T_x$ by

$$Q(l, x) = P\{X_1(T_x) > \lambda + l\}, \quad l \geq 0, \quad 0 \leq x \leq \lambda.$$

Then, we obtain the following lemma:

**Lemma 1** For $l \geq 0$,

$$Q(l, x) = \begin{cases} 0, & x = 0, \\ \int_0^{\lambda-x} \tilde{q}(l, y) dy + \tilde{Q}(l, 0), & 0 < x \leq \lambda, \end{cases}$$

where $\tilde{q}(l, y)$ and $\tilde{Q}(l, 0)$ are determined in (4) and (5), respectively.

**Proof** Since $X_1(T_0) = 0$, it is clear that $Q(l, 0) = 0$ for all $l \geq 0$. For $0 < x \leq \lambda$, we employ the method of Kolmogorov’s backward differential equation to obtain $Q(l, x)$ as in Bae et al [3]. Conditioning on whether a customer arrives or not during the infinitesimal interval $(0, \Delta t]$ gives

$$X_1(T_x) = \begin{cases} X_1(T_{x-r_1(x)\Delta t}), & \text{if no customers arrive,} \\ X_1(T_{x-r_1(x)t+S-r_1(x)t+S}(\Delta t-t)), & \text{if a customer arrives at } t \in (0, \Delta t] \\
 & \text{and } S \leq \lambda - x + r_1(x)t, \\ x - r_1(x)t + S, & \text{if a customer arrives at } t \in (0, \Delta t] \\
 & \text{and } S > \lambda - x + r_1(x)t, \end{cases}$$

where $S$ denotes the amount of work that the arriving customer carries to the system. Noting that the conditional density of the arrival time given that an
arrival has occurred in the interval \((0, \Delta t]\) is \(1/\Delta t\), we have that for \(0 < x \leq \lambda\),

\[
Q(l, x) = (1 - \nu \Delta t)Q(l, x - r_1(x)\Delta t) + \nu \int_0^{\Delta t} \int_0^{\lambda - x + r_1(x)t} Q(l, x - r_1(x)t + s - r_1(x)t + s(\Delta t - t))dG(s)dt \\
+ \nu \int_0^{\Delta t} [1 - G(\lambda + l - x + r_1(x)t)]dt + o(\Delta t),
\]

where \(o(\Delta t)/\Delta t\) goes to zero as \(\Delta t \to 0\). Dividing each side of the above equation by \(\Delta t\) and then taking the limit as \(\Delta t \to 0\) yield the following integro-differential equation:

\[
-\tilde{r}_1(y) \frac{\partial}{\partial y} \tilde{Q}(l, y) = -\nu \tilde{Q}(l, y) + \nu \int_0^y \tilde{Q}(l, y - s)dG(s) + \nu [1 - G(y + 1)],
\]

where \(y = \lambda - x, \tilde{Q}(l, y) = Q(l, \lambda - y)\), and \(\tilde{r}_1(y) = r_1(\lambda - y)\) for the convenience of analysis. Let \(\tilde{q}(l, y) = \frac{\partial}{\partial y} \tilde{Q}(l, y)\). Then, from the fundamental theorem of calculus, it follows that

\[
\tilde{r}_1(y)\tilde{q}(l, y) = \nu \{\tilde{Q}(l, 0)[1 - G(y)] - [1 - G(y + 1)]\} \\
+ \nu \int_0^y \tilde{q}(l, z)dz - \nu \int_0^y \int_0^{y-s} \tilde{q}(l, z)dzdG(s). \tag{1}
\]

Observing that

\[
\int_0^y \int_0^{y-s} \tilde{q}(l, z)dzdG(s) = \int_0^y G(y - z)\tilde{q}(l, z)dz,
\]

(1) can be rewritten as

\[
\tilde{q}(l, y) = h(l, y) + \int_0^y \tilde{q}(l, z)K^1(y, z)dz, \tag{2}
\]

where

\[
h(l, y) = \frac{\nu \{\tilde{Q}(l, 0)[1 - G(y)] - [1 - G(y + 1)]\}}{\tilde{r}_1(y)}, \tag{3}
\]

and

\[
K^1(y, z) = \frac{\nu[1 - G(y - z)]}{\tilde{r}_1(y)}.
\]
In a manner analogous to that in Harrison and Resnick [12], define its iterates recursively by
\[
K^{n+1}(y, z) = \int_{z}^{y} K^n(y, w)K^1(w, z)dw \\
= \int_{z}^{y} K^1(y, w)K^n(w, z)dw, \quad 0 \leq z < y < \lambda,
\]
for \( n \geq 1 \). Using the bound \( K^1(y, z) \leq \nu/r_1(\lambda - y) \), it follows easily by induction that
\[
K^{n+1}(y, z) \leq \frac{\nu^{n+1} [R_1(\lambda - z) - R_1(\lambda - y)]^n}{r_1(\lambda - y)n!}, \quad 0 \leq z < y < \lambda,
\]
for all \( n \geq 1 \). Thus the kernel \( K^\ast(y, z) := \sum_{n=1}^{\infty} K^n(y, z) \) is well-defined because of the assumption that \( R_1(x) < \infty \) for all \( 0 \leq x \leq \lambda \). Iterating the relation (2) for \( N-1 \) times gives
\[
\tilde{q}(l, y) = h(l, y) + \int_{0}^{y} h(l, z) \sum_{n=1}^{N-1} K^n(y, z)dz + \int_{0}^{y} \tilde{q}(l, z)K^N(y, z)dz.
\]
Letting \( N \to \infty \) and using the dominated convergence theorem, we conclude that
\[
\tilde{q}(l, y) = h(l, y) + \int_{0}^{y} h(l, z)K^\ast(y, z)dz
\]
is the unique solution to (2).

From the boundary condition
\[
\lim_{y \to \lambda} \tilde{Q}(l, y) = \lim_{x \to 0} Q(l, x) = 0,
\]
we have
\[
\tilde{Q}(l, 0) = \left( 1 + \int_{0}^{\lambda} K^\ast(y, 0)dy \right)^{-1} \times \left\{ \int_{0}^{\lambda} \frac{\nu[1-G(y+l)]}{r_1(\lambda - y)}dy + \int_{0}^{\lambda} \int_{0}^{y} \frac{\nu[1-G(z+l)]}{r_1(\lambda - z)}K^\ast(y, z)dzdy \right\},
\]
which completes the function \( h(l, y) \) in (3). \( \blacksquare \)
Remark 1 $Q(0, x)$ is the probability that the process $\{X_1(t), t \geq 0\}$, starting from $0 < x \leq \lambda$, up-crosses level $\lambda$ without reaching level 0 and is given by

$$Q(0, x) = \frac{\int_{\lambda-x}^{\lambda} K^*(y, 0)dy}{1 + \int_{0}^{\lambda} K^*(y, 0)dy}$$

which coincides the first exit probability $U(x)$ for $a = 0$ and $b = \lambda$ in Harrison and Resnick [12].

In the next lemma, we express the distribution of the excess amount over $\lambda$ at the first passage time through $\lambda$ in terms of $Q(l, x)$ obtained in Lemma 1.

Lemma 2 The excess amount over $\lambda$ for the process $\{X_1(t), t \geq 0\}$, starting with $x(0 \leq x \leq \lambda)$, denoted by $L_x$, has the distribution function given by

$$P(l, x) := P\{L_x \leq l\} = 1 - Q(l, x) + [Q(0, x) - 1] \frac{1 - G(\lambda + l) + \int_{0}^{\lambda} Q(l, x)dG(x)}{1 - G(\lambda) + \int_{0}^{\lambda} Q(0, x)dG(x)}.$$

(6)

Proof From the Markovian property of the process $\{X_1(t), t \geq 0\}$, it follows that

$$P\{L_x > l\} = Q(l, x) + [1 - Q(0, x)]P\{L_0 > l\}, \quad l \geq 0, \; 0 \leq x \leq \lambda.$$

(7)

On the other hand, by conditioning on the amount of work which the first customer brings after an idle period, we have

$$P\{L_0 > l\} = 1 - G(\lambda + l) + \int_{0}^{\lambda} P\{L_x > l\}dG(x), \quad l \geq 0.$$

(8)

Substituting $P\{L_x > l\}$ of (7) into (8) yields that

$$P\{L_0 > l\} = \frac{1 - G(\lambda + l) + \int_{0}^{\lambda} Q(l, x)dG(x)}{1 - G(\lambda) + \int_{0}^{\lambda} Q(0, x)dG(x)}.$$

Combining the above equation and (7) results in (6).

3 The stationary distribution

As illustrated in the Figure 1, we denote by $C$, $C_1$, and $C_2$ the cycles of the processes $\{X(t), t \geq 0\}$, $\{X_1(t), t \geq 0\}$, and $\{X_2(t), t \geq 0\}$, respectively. Then, obviously $C = C_1 + C_2$.
Because \( \{X(t), t \geq 0\} \) and \( \{X_i(t), t \geq 0\} \) for \( i = 1, 2 \), are regenerative processes with finite mean cycles, each process has its stationary distribution function. Let \( F_i(x) \) be the stationary distribution function of \( \{X_i(t), t \geq 0\} \) for \( i = 1, 2 \), and let \( F(x) \) be that of \( \{X(t), t \geq 0\} \). Then it follows that

\[
F(x) = \beta F_1(x) + (1 - \beta) F_2(x),
\]

where \( \beta = E[C_1]/E[C] \). Note that \( F_2 \) is continuous and supported on \( [\tau, \infty) \), whereas \( F_1 \) is supported on \( [0, \lambda] \), has a jump at zero, and is continuous otherwise. We denote the jump size of \( F_1 \) at zero by \( \alpha \) and write

\[
F_1(x) = \alpha + (1 - \alpha) F_{1ac}(x),
\]

where \( F_{1ac} \) is the absolutely continuous part of \( F_1 \). Using (9), the distribution \( F \) can be written as

\[
F(x) = \alpha \beta + (1 - \alpha) \beta F_{1ac}(x) + (1 - \beta) F_2(x).
\]

For \( i = 1, 2 \), let \( D_i(x) \) and \( U_i(x) \) be the numbers of down- and up-crossings of level \( x \) by the process \( \{X_i(t), t \geq 0\} \) during the cycle \( C_i \), respectively, and \( N_i \) the number of arrivals during \( C_i \). By convention the arrival that causes \( \{X(t), t \geq 0\} \) to up-cross level \( \lambda \) for the first time during the cycle \( C \) is counted only in \( N_1 \).

By using the level crossing theory in Cohen [9], we have that for the number of down-crossings

\[
E[D_1(x)] = E[C_1] \cdot r_1(x)(1 - \alpha) f_{1ac}(x), \quad 0 < x < \lambda, \tag{10}
\]

\[
E[D_2(x)] = E[C_2] \cdot r_2(x) f_2(x), \quad \tau < x < \infty, \tag{11}
\]

where \( f_{1ac} \) and \( f_2 \) are densities corresponding to \( F_{1ac} \) and \( F_2 \), respectively. We also have that

\[
E[U_1(x)] = E[N_1] \cdot E[1_{\{X_1 \leq x\}} - 1_{\{X_1 + S \leq x\}}], \quad 0 < x < \lambda, \tag{12}
\]

\[
E[U_2(x)] = E[N_2] \cdot E[1_{\{X_2 \leq x\}} - 1_{\{X_2 + S \leq x\}}], \quad \tau < x < \infty, \tag{13}
\]
where $X_1$ and $X_2$ are the generic random variables with distributions $F_1$ and $F_2$, respectively, and $S$ is the generic random variable denoting the amount of work that the arriving customer carries to the system with distribution $G$.

Because the process $\{X(t), t \geq 0\}$ is the regenerative process having the same level $\tau$ at all regeneration points, the number of up-crossings of level $x$ equals the number of down-crossings of that level during the cycle. Therefore, it follows that

$$D_1(x) \equiv \begin{cases} U_1(x), & 0 < x < \tau, \\ U_1(x) - 1, & \tau \leq x < \lambda, \end{cases}$$

(14)

and

$$D_2(x) \equiv \begin{cases} U_2(x) + 1, & \tau < x < \lambda, \\ U_2(x) + U_1(x), & x \geq \lambda, \end{cases}$$

(15)

where $U_1(x)$ in (15) means the number of arrivals during the cycle $C_1$ that cause the process $\{X_1(t), t \geq 0\}$ to up-cross both level $\lambda$ and level $x (\geq \lambda)$ simultaneously, in other words, for $x \geq \lambda$,

$$U_1(x) = \begin{cases} 1, & X(T_{\alpha}^\lambda) > x, \\ 0, & \text{otherwise}. \end{cases}$$

Let $Q(\lambda)$ denote the probability that the process $\{X_1(t), t \geq 0\}$, leaving from 0, up-crosses level $\lambda$ before returning to 0. Then, conditioning on the amount of work which the arriving customer brings and using Remark 1, we get

$$Q(\lambda) = 1 - G(\lambda) + \int_0^\lambda Q(0, x) dG(x).$$

(16)

We are now ready for the following lemma:

**Lemma 3** The stationary densities $f_{1ac}$ and $f_2$ satisfy the following level crossing equations:

$$r_1(x)(1 - \alpha)f_{1ac}^x(x) = \begin{cases} \nu \int_0^x [1 - G(x - y)]dF_1(y), & 0 < x < \tau, \\ \nu \int_0^x [1 - G(x - y)]dF_1(y) - \frac{\alpha \nu Q(\lambda)}{1 - Q(0, \tau)}, & \tau \leq x < \lambda, \end{cases}$$

and

$$r_2(x)f_2(x) = \begin{cases} \nu \int_\tau^x [1 - G(x - y)]df_2(y) + \frac{\alpha \beta \nu Q(\lambda)}{(1 - \beta)[1 - Q(0, \tau)]^\alpha}, & \tau < x < \lambda, \\ \nu \int_\tau^x [1 - G(x - y)]df_2(y) + \frac{\alpha \beta \nu Q(\lambda)[1 - P(x - \lambda, \tau)]}{(1 - \beta)[1 - Q(0, \tau)]}, & x \geq \lambda, \end{cases}$$

where $x \geq \lambda$. 

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where $\alpha$ and $\beta$ are determined by two normalizing conditions given in (20) and (21), respectively.

**Proof** Note that, by Wald’s theorem (Ross [14, p. 105]), $E[N_i]/E[C_i] = \nu$, for $i = 1, 2$. By taking expectations in (14) and (15) and using the relations (10), (11), (12), and (13), we have

$$r_1(x)(1-\alpha)f_{1}^{ac}(x) = \begin{cases} \nu \int_{0}^{x}[1 - G(x-y)]dF_1(y), & 0 < x < \tau, \\ \nu \int_{0}^{\lambda}[1 - G(x-y)]dF_1(y) - \frac{1}{E[C_1]}, & \tau \leq x < \lambda, \end{cases}$$

and

$$r_2(x)f_2(x) = \begin{cases} \nu \int_{\tau}^{x}[1 - G(x-y)]f_2(y)dy + \frac{1}{E[C_2]}, & \tau < x < \lambda, \\ \nu \int_{\tau}^{\lambda}[1 - G(x-y)]f_2(y)dy + \frac{E[U_1(x)]}{E[C_2]}, & x \geq \lambda. \end{cases}$$

Letting $x \to 0^+$ in the first equation of (17) gives

$$r_1(0^+)(1-\alpha)f_{1}^{ac}(0^+) = \alpha\nu. \tag{18}$$

Since $D_1(x) \leq N_1$ with probability 1 for $0 < x < \lambda$, and $N_1$ is integrable, it follows, from the dominated convergence theorem, that $\lim_{x \to 0^+} E[D_1(x)] = E[D_1(0^+)]$. From the Markovian property of the process $\{X_1(t), t \geq 0\}$ and Remark 1, we can show that $D_1(0^+)$ has the following distribution:

$$P\{D_1(0^+) = n\} = \begin{cases} Q(0, \tau), & n = 0, \\ Q(\lambda)[1 - Q(0, \tau)][1 - Q(\lambda)]^{n-1}, & n \geq 1, \end{cases}$$

where $Q(\lambda)$ is given by (16). Therefore we have

$$E[D_1(0^+)] = \frac{1 - Q(0, \tau)}{Q(\lambda)}. \tag{19}$$

Substituting both (18) and (19) into (10) gives

$$E[C_1] = \frac{1 - Q(0, \tau)}{\alpha\nu Q(\lambda)}$$

and from the relation $\beta = E[C_1]/(E[C_1] + E[C_2])$, it follows that

$$E[C_2] = \frac{(1 - \beta)[1 - Q(0, \tau)]}{\alpha\beta\nu Q(\lambda)}.$$
On the other hand, note that the expectation $E[U_1(x)]$ for $x \geq \lambda$ is equal to the probability that the excess amount over $\lambda$ for the process $\{X_1(t), t \geq 0\}$ which starts at $\tau$, is greater than $x - \lambda$. That is exactly $1 - P(x - \lambda, \tau)$, where the probability $P(\cdot, \cdot)$ was derived in Lemma 2. Finally, $\alpha$ and $\beta$ can be obtained by the normalizing conditions

$$
\alpha + (1 - \alpha) \int_0^\lambda f^{ac}_1(x)dx = 1 \tag{20}
$$

and

$$
\int_\tau^\infty f_2(x)dx = 1. \tag{21}
$$

In Theorem 1 we provide expressions for the solutions $f^{ac}_1$ and $f_2$ to the level crossing equations of Lemma 3.

**Theorem 1** The stationary densities $f^{ac}_1$ and $f_2$ are given, respectively, by

$$
f_1^{ac}(x) = \begin{cases} \frac{\alpha}{1 - \alpha} K_1^*(x, 0), & \text{if } 0 < x < \tau, \\ \frac{\alpha}{1 - \alpha} \left\{ K_1^*(x, 0) - \frac{\nu Q(\lambda)}{1 - Q(0, \tau)} \left( \frac{1}{r_1(x)} + \int_x^\tau \frac{K_1^*(x, y)}{r_1(y)}dy \right) \right\}, & \text{if } \tau \leq x < \lambda, \\ 0, & \text{otherwise.} \end{cases} \tag{22}
$$

and

$$
f_2(x) = \begin{cases} \frac{\alpha \nu Q(\lambda)}{(1 - \beta)[1 - Q(0, \tau)]} \left\{ \frac{1}{r_2(x)} + \int_x^\tau \frac{K_2^*(x, y)}{r_2(y)}dy \right\}, & \text{if } \tau < x < \lambda, \\ \frac{\alpha \nu Q(\lambda)}{(1 - \beta)[1 - Q(0, \tau)]} \left\{ \frac{1 - P(x - \lambda, \tau)}{r_2(x)} + \int_\lambda^x \frac{1 - P(y - \lambda, \tau)}{r_2(y)} K_2^*(x, y)dy \right\}, & \text{if } x \geq \lambda, \\ 0, & \text{otherwise,} \end{cases} \tag{23}
$$

where $K_i^*(x, y) := \sum_{n=1}^\infty K_i^n(x, y)$ with $K_i^n(x, y)$ defined in (26), for $i = 1, 2$.

**Proof** Setting $f_1(x) = (1 - \alpha)f_1^{ac}(x)$ and $K_i(x, y) = \nu(1 - G(x - y))/r_i(x)$
for $i = 1, 2$ in the integral equations of Lemma 3 yields

\[
\begin{align*}
    f_1(x) &= \begin{cases} 
     \alpha K_1(x, 0) + \int_0^x K_1(x, y)f_1(y)dy, & 0 < x < \tau, \\
     \alpha K_1(x, 0) + \int_0^x K_1(x, y)f_1(y)dy - \frac{\alpha \nu Q(\lambda)}{[1 - Q(0, \tau)]r_1(x)} & \tau \leq x < \lambda, \\
     0, & \text{otherwise},
    \end{cases} \\
    f_2(x) &= \begin{cases} 
     \int_\tau^x K_2(x, y)f_2(y)dy + \frac{\alpha \beta \nu Q(\lambda)}{(1-\beta)[1-Q(0,\tau)]r_2(x)}, & \tau < x < \lambda, \\
     \int_\tau^x K_2(x, y)f_2(y)dy + \frac{\alpha \beta \nu Q(\lambda)[1-P(x-\lambda, \tau)]}{(1-\beta)[1-Q(0,\tau)]r_2(x)} & x \geq \lambda, \\
     0, & \text{otherwise}.
    \end{cases}
\end{align*}
\]  

(24)

and

\[
\begin{align*}
    f_2(x) &= \begin{cases} 
     \int_\tau^x K_2(x, y)f_2(y)dy + \frac{\alpha \beta \nu Q(\lambda)}{(1-\beta)[1-Q(0,\tau)]r_2(x)}, & \tau < x < \lambda, \\
     \int_\tau^x K_2(x, y)f_2(y)dy + \frac{\alpha \beta \nu Q(\lambda)[1-P(x-\lambda, \tau)]}{(1-\beta)[1-Q(0,\tau)]r_2(x)} & x \geq \lambda, \\
     0, & \text{otherwise}.
    \end{cases}
\end{align*}
\]  

(25)

Define recursively

\[
K_i^{n+1}(x, y) = \int_y^x K_i^n(x, z)K_i(z, y)dz = \int_y^x K_i(x, z)K_i^n(z, y)dz, \quad 0 \leq y < x,
\]

(26)

for $n \geq 1$ with $K_i^1 = K_i$ for $i = 1, 2$. Then, for all $n \geq 1$, it follows that

\[
K_i^{n+1}(x, y) \leq \nu^{n+1} \left( \frac{\int_y^x r_i(z)^{-1}dz}{r_i(x)n!} \right)^n, \quad 0 \leq y < x.
\]

Thus the kernels $K_i^n(x, y) = \sum_{n=1}^{\infty} K_i^n(x, y)$, for $i = 1, 2$, are well-defined. Therefore by using the method of successive substitutions of Harrison and Resnick [12] as described in the proof of Lemma 1, we can see that (22) and (23) are unique solutions of (24) and (25), respectively.

4 Example

In this section, we consider the special case of exponential jumps and positive constant service speeds, i.e., $G(x) = 1 - e^{-x/m}$, $r_1(x) = r_1$ and $r_2(x) = r_2$, for all $x > 0$. By the memoryless property of the exponential random variable, for all starting level $0 < x \leq \lambda$, the probability $P(l, x)$ in Lemma 2 is given by

\[
P(l, x) = 1 - e^{-l/m}, \quad l \geq 0.
\]
For $i = 1, 2$, let

$$\theta_i = \frac{1}{m} - \frac{\nu}{r_i}.$$ 

Since

$$K^*(x, y) = K_i^*(x, y) = \frac{\nu}{r_i} e^{-\theta_i (x-y)}, \quad 0 < y < x < \lambda,$$

and

$$K_2^*(x, y) = \frac{\nu}{r_2} e^{-\theta_2 (x-y)}, \quad \tau < y < x,$$

we can obtain that

$$Q(l, x) = \frac{\nu m e^{-\theta_1 \lambda} (e^{-\theta_1 (\lambda-x)} - e^{-\theta_1 \lambda})}{r_1 - \nu m e^{-\theta_1 \lambda}}$$

and

$$Q(\lambda) = \frac{(r_1 - \nu m) e^{-\theta_1 \lambda}}{r_1 - \nu m e^{-\theta_1 \lambda}}.$$ 

Therefore the densities can be derived by

$$f_1(x) = \begin{cases} \frac{\alpha \nu e^{-\theta_1 x}}{(1-\alpha)r_1}, & 0 < x < \tau, \\ \frac{\alpha \nu (e^{-\theta_1 x} - e^{-\theta_1 \lambda})}{(1-\alpha)(r_1 - \nu m e^{-\theta_1 \lambda})}, & \tau \leq x < \lambda, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$f_2(x) = \begin{cases} \frac{\alpha \beta \nu e^{-\theta_1 \lambda} (r_1 - \nu m)(r_2 - \nu m e^{-\theta_2 (x-\tau)})}{(1-\beta)r_2(r_2 - \nu m)(r_1 - \nu m e^{-\theta_1 (\lambda-\tau)})}, & \tau < x < \lambda, \\ \frac{\alpha \beta \nu e^{-\theta_1 \lambda} (r_1 - \nu m)(r_2 e^{-\theta_2 (x-\lambda)} - \nu m e^{-\theta_2 (x-\tau)})}{(1-\beta)r_2(r_2 - \nu m)(r_1 - \nu m e^{-\theta_1 (\lambda-\tau)})}, & x \geq \lambda, \\ 0, & \text{otherwise}. \end{cases}$$

By employing the normalizing conditions, the parameters $\alpha$ and $\beta$ are also given by

$$\alpha = \frac{(r_1 - \nu m)(r_1 - \nu m e^{-\theta_1 (\lambda-\tau)})}{r_1 (r_1 - \nu m e^{-\theta_1 (\lambda-\tau)}) - \nu(r_1 - \nu m)(\lambda + \tau) e^{-\theta_1 \lambda}},$$

and

$$\beta = \frac{r_2 - \nu m}{r_2 - \frac{r_1}{1-\alpha}}.$$ 

**Remark 2** When $r_1 = 1$ and $r_2 = M$, if $\tau$ decreases to 0, then our policy is reduced to the $P_x^M$ policy for the $M/M/1$ queue introduced by Bae et al. [3].
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References


