A five-point binary subdivision scheme with a parameter $u$ is presented. The generating polynomial method and the Hölder exponent are used to investigate the uniform convergence and $C^k$-continuity of this subdivision scheme, where $k$ depends on the choice of the parameter $u$. Moreover, the conditions on the initial points are discussed for the given limit curve to be convexity preserving and an example is given to illustrate our conclusion.

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1. Introduction

Subdivision techniques are being widely used to create smooth curves and surfaces in many fields such as in CAGD, CG and geometric modeling. Besides convergence and smoothness, convexity preserving is also a hot topic in curves design. A lot of papers have been published on this topic during the past decades. Cai [1] gave a convexity preserving algorithm for classic four-point interpolating subdivision scheme [2], and claimed that a convex curve can be constructed by means of the four-point interpolating subdivision scheme for arbitrary convex set of discrete points with no three points collinear. Literature [3] discussed the convergence of four-point subdivision scheme in nonuniform control points and its error estimation. Dyn et al. [4] derived the conditions on the parameter of the classic four-point interpolatory subdivision scheme presented in [2] to preserve convexity. Hassan et al. [5] introduced a four-point ternary interpolatory subdivision scheme, that can generate $C^2$ continuous limit curves. The convexity preserving condition was given in [6] for four-point ternary interpolatory subdivision scheme in [5]. Based on quintic B-spline basis functions, Siddiai et al. [7] constructed a binary six-point approximating subdivision scheme, which generates $C^3$ continuous limit curves. The sufficient conditions on the parameters of the scheme to preserve convexity and generate $C^r$ ($r = 7, 8, 9$) limit curves in [7] were introduced by Hao et al. [8]. Under the nonuniform initial control vertices, Kuijit et al. [9] proposed a class of shape preserving binary subdivision schemes, and a sufficient condition for preservation of convexity was deduced. Amat et al. [10] introduced a new approach to prove the convexity preserving properties for interpolatory subdivision schemes through reconstruction operators. A new $C^5$ continuous binary five-point relaxation subdivision scheme with tension parameter was presented by Cao et al. [11].

Based on the above research, we present a five-point binary subdivision scheme with parameter $u$ in this paper and discuss the continuity of the limit curve as well as its convexity preservation. The rest of the paper is organized as follows. In Section 2, a five-point binary subdivision scheme with parameter $u$ is introduced, the continuity of the limit curve is discussed and the Hölder exponent is calculated. In Sections 3 and 4, the convexity preserving properties of limit curves with different choices of parameter are analyzed. Numerical examples and the conclusion is drawn in Section 5.
2. A five point binary subdivision scheme with a parameter

Based on the literature [11], first of all, we give the definition of a new 5-point binary subdivision scheme.

**Definition 1.** Given a set of initial control points \( p^0 = \{ p^0_i \in \mathbb{R}^d \}_{i=-2}^{n+2} \), let \( p^k = \{ p^k_i \in \mathbb{R}^d \}_{i=-2}^{n+2} \) be the set of control points at \( k \)th level of refinement. Then the refined polygon \( \{ p^k \}_{i=-2}^{n+2} \) is obtained by applying the following subdivision rules:

\[
\begin{align*}
\{ p^k_{3i+1} \} &= \frac{u}{16} p_{3i}^k + (\frac{15}{16} + \frac{3}{16}) p_{3i-1}^k + (\frac{3}{16} - \frac{1}{16}) p_{3i-2}^k, \\
\{ p^k_{3i+2} \} &= \frac{1}{16} p_{3i+1}^k + (\frac{15}{16} - \frac{3}{16}) p_{3i}^k + (\frac{1}{16} - \frac{1}{16}) p_{3i-1}^k + (\frac{1}{16} + \frac{1}{16}) p_{3i-2}^k.
\end{align*}
\]

(1)

In practice, the new five-point binary subdivision scheme is the special case of literature [11] when \( u = 128w \).

The convergence and up to \( C^5 \) continuity of the five-point binary subdivision scheme (1) are analyzed and discussed as follows.

**Theorem 1.** The new five-point binary subdivision scheme (1) generates a limiting curve of continuity up to \( C^5 \).

**Proof.** From subdivision rules it follows:

\[
a_{-4} = a_4 = \frac{u}{128}, \quad a_{-3} = a_3 = 0, \quad a_{-2} = a_2 = \frac{u}{32} + \frac{3}{16}, \quad a_{-1} = a_1 = \frac{15}{32} - \frac{u}{32}, \quad a_0 = \frac{5}{8} - \frac{5u}{64},
\]

Thus, the generating polynomial \( a(z) \) for the mask of the subdivision scheme can be written as

\[
a(z) = \left( \frac{1 + z}{2\sqrt{2}} \right)^6 \left( \frac{u}{2} + \frac{(4 - 2u)z + uz^2}{2} \right).
\]

Then according to literature [12], we have the following generating polynomials for \( S_j(j = 1, 2, 3, 4, 5, 6) \):

\[
\begin{align*}
a^{(1)}(z) &= \frac{2z}{1 + z} a(z) = \left( \frac{1 + z}{2\sqrt{2}} \right)^5 \left( \frac{u}{2} + \frac{(4 - 2u)z + uz^2}{2} \right), \\
a^{(2)}(z) &= \left( \frac{2z}{1 + z} \right)^2 a(z) = \left( \frac{1 + z}{2\sqrt{2}} \right)^4 \left( \frac{u}{2} + \frac{(4 - 2u)z + uz^2}{2} \right), \\
a^{(3)}(z) &= \left( \frac{2z}{1 + z} \right)^3 a(z) = \left( \frac{1 + z}{2\sqrt{2}} \right)^3 \left( \frac{u}{2} + \frac{(4 - 2u)z^2 + uz^2}{2} \right), \\
a^{(4)}(z) &= \left( \frac{2z}{1 + z} \right)^4 a(z) = \left( \frac{1 + z}{2\sqrt{2}} \right)^2 \left( \frac{u}{2} + \frac{(4 - 2u)z^2 + uz^2}{2} \right), \\
a^{(5)}(z) &= \left( \frac{2z}{1 + z} \right)^5 a(z) = \frac{1 + z}{2} \left( \frac{u}{2} + \frac{(4 - 2u)z^2 + uz^2}{2} \right), \\
a^{(6)}(z) &= \left( \frac{2z}{1 + z} \right)^6 a(z) = \frac{1}{2} \left( \frac{u}{2} + \frac{(4 - 2u)z^2 + uz^2}{2} \right).
\end{align*}
\]

Therefore,

\[
\sum_{k=2i}^{2i+1} a_{2i+1}^{(j)} = \sum_{k=2i}^{2i+1} a_{2i+1}^{(j)} = 1, \quad j = 0, 1, 2, 3, 4, 5,
\]

and when \(-9 < u < 14.4\), we have

\[
\left\| \frac{1}{2} a^{(j)}_{2i} \right\|_\infty = \max \left\{ \left\| \frac{3u}{128} + \frac{1}{32} \right\|_\infty + \left\| \frac{5u}{128} \right\|_\infty + \left\| \frac{5u}{128} + \frac{1}{32} \right\|_\infty + \left\| \frac{u}{128} \right\|_\infty \right\} < 1;
\]

when \(-6 < u < 9\), we have

\[
\left\| \frac{1}{2} a^{(j)}_{2i} \right\|_\infty = 4 \max \left\{ \left\| \frac{u}{64} + \frac{1}{32} \right\|_\infty + \left\| \frac{u}{64} + \frac{1}{32} \right\|_\infty + \left\| \frac{u}{64} + \frac{3}{32} \right\|_\infty \right\} < 1;
\]

when \(-2 < u < 6\), we have

\[
\left\| \frac{1}{2} a^{(j)}_{2i} \right\|_\infty = 4 \max \left\{ \left\| \frac{u}{128} + \frac{1}{32} \right\|_\infty + \left\| \frac{u}{128} + \frac{3}{32} \right\|_\infty \right\} < 1;
\]
\[ \left\| \frac{1}{2} S_4 \right\|_\infty = 8 \max \left\{ 2 \left| \frac{u}{128} \right| + \left| \frac{u}{64} + \frac{1}{16} \right|, \frac{1}{16} \right\} < 1, \]
\[ \left\| \frac{1}{2} S_5 \right\|_\infty = 16 \max \left\{ -\frac{u}{128} + \frac{1}{32} \right\} < 1; \]
when \( u = 1 \), we have
\[ \sum_{i,j} a_{2i}^{(6)} = \sum_{i,j} a_{2i+1}^{(6)} = 1 \]
and
\[ \left\| \frac{1}{2} S_6 \right\|_\infty = 32 \max \left\{ \frac{1}{128} + \frac{1}{128} \right\} < 1. \]

Thus, according to literature \([13]\), the new five-point binary subdivision (1) generates a limiting curve of continuity up to \( C^5 \). Furthermore, according to Rioul’s method \([17]\), we have \( \| S_n \|_\infty = \frac{|4u^3 + w|}{8} \) when \(-2 < u < 6\). Therefore from \( 2^{-v} = \| S_n \|_\infty = \frac{|4u^3 + w|}{8} \), it follows \( v = 3 - \log_2 |4u^3 + w| \) when \(-2 < u < 6\), which takes the maximum 1 when \( 0 \leq u \leq 4 \). Hence we find that the Hölder exponent for our scheme (1) is \( \tilde{C}^5 \) when \( 0 \leq u \leq 4 \) and \( \tilde{C}^\mu \) where \( 4 < \mu < 5 \) when \(-2 < u < 0 \) or \( 4 < u < 6 \). In the particular case when \( u = 1 \), the limiting curve generated by the scheme (1) is \( C^5 \) continuous and from \( 2^{-v} = \| S_n \|_\infty = \frac{1}{2} \), it follows \( v = 1 \), which means that the Hölder exponent for scheme (1) when \( u = 1 \) is \( \tilde{C}^6 \).

As known to all, the quality of subdivided curves can be assessed quantitatively by measuring curvature, and curvature measures the rate at which any non-straight curve tends to depart from its tangent. With the same initial control points, using different subdivision schemes will generate the limit curves with different curvatures. Shown in Fig. 1 are the graphs of the curvatures corresponding to the limit curves generated by the subdivision schemes in \([14–16]\) and ours \((u = -1)\) with the same initial control points \( (0,0), (1,0), (2,0), (3,1), (4,0), (5,0), (6,0) \). As we can see from Fig. 1, the largest curvature of our scheme is not more than 0.8.

Next, we will discuss what conditions should be satisfied so that the subdivision scheme (1) is of convexity preservation.

3. Convexity preservation of five-point binary subdivision scheme

As pointed out in Section 2, when \(-2 < u < 6\), our subdivision scheme (1) is at least \( C^4 \) continuous. As shape preserving of curve is an important subject in the geometric design. It is interesting to know what choices of \( u \) in \((-2,6)\) can make the scheme (1) not only own high order continuity, but also preserve convexity with the given initial points.

![Fig. 1. The curvatures of the limit curves corresponding to 4 different subdivision schemes.](image-url)
Given a set of initial control points \( \{ p_i^0 \}_{i \in Z} \), \( p_i^0 = (x_i^0, f_i^0) \), which are strictly convex, where \( \{ x_i^0 \}_{i \in Z} \) are equidistant. For convenience, we assume that \( \Delta x^0 = x^0_{i+1} - x^0_i = 1 \). By subdivision scheme (1), we know \( \Delta x^{k+1} = x^{k+1}_{i+1} - x^{k+1}_i = \frac{1}{2} \Delta x^k = \frac{1}{2^k} \).

Denote the second order divided difference by \( d_i^k = f[x^0_{i+1}, x^0_i, x^0_{i-1}] = 2^{2k-1}(f^k_{i+1} - 2f^k_i + f^k_{i-1}) \).

Then we have the following.

**Theorem 2.** Suppose the initial control points \( \{ p_i^0 \}_{i \in Z} \), \( p_i^0 = (x_i^0, f_i^0) \) are strictly convex, i.e., \( d_i^0 > 0, \forall i \in Z \). Then the subdivision scheme (1) preserves convexity when \( 0 < u < 6 \).

**Proof.** It is not difficult to obtain by using (1)

\[
d_{2i+1}^{k+1} = 2^{2k+1}((2^{k+1}_i + 2^{k+1}_{i+1}) + f_{2i+1}^{k+1}) = \frac{1}{16} \left( (2 + u)d_{i-1}^{k} + (12 - 2u)d_{i}^{k} + (u + 2)d_{i+1}^{k} \right),
\]

(2)

\[
d_{2i}^{k+1} = 2^{2k+1}(f_{2i}^{k+1} - 2f_{2i-1}^{k+1} + f_{2i+2}^{k+1}) = \frac{1}{16} \left( ud_{i-1}^{k} + (16 - u)d_{i}^{k} + (u - 2)d_{i+1}^{k} \right).
\]

(3)

Assume \( d_i^k > 0 \) for some \( k \in Z \), \( k \geq 0 \), then when \( 0 < u < 6 \), from (2) and (3) it follows \( d_{2i+1}^{k+1} > 0 \) and \( d_{2i+1}^{k+1} > 0 \). Therefore \( d_i^k > 0, \forall k \in Z, k \geq 0 \).

The proof of Theorem 2 is completed. \( \square \)

Combining Theorem 2 and the arguments in the foregoing section, we draw the conclusion that when \( 0 < u < 6 \), the subdivision scheme (1) generates a convexity preserving \( C^4 \) continuous limit curve for the given initial convex data.

4. Convexity preservation of the subdivision scheme (1) when \(-2 < u < 0\)

As proved in the previous section, when \( 0 < u < 6 \), the subdivision scheme (1) generates a convexity preserving limit curve, which is at least \( C^4 \) continuous, for the given initial convex data. Then a question arises that under what conditions the subdivision scheme (1) can generate a convexity preserving \( C^4 \) continuous limit curve for the given initial convex data if \(-2 < u < 0\). To this end, we consider what conditions should be satisfied so that the subdivision scheme (1) is of convexity preservation when \(-2 < u < 0\).

**Theorem 3.** Denote \( r_i^k = \frac{d_i^k}{d_i^{k+1}}, R_k = \max \{ r_i^k, \frac{1}{r_i^k} \}, \forall k \geq 0, k \in Z \). Suppose the initial control points \( \{ p_i^0 \}_{i \in Z} \), \( p_i^0 = (x_i^0, f_i^0) \) are strictly convex, i.e., \( d_i^0 > 0, \forall i \in Z \). Further suppose \( \frac{3u-8 - \sqrt{9u^2 - 8u + 32}}{4u+8} < \lambda < \min \{ \frac{16 - u + \sqrt{9u^2 - 16u + 256}}{4u+8}, \frac{12 - 3u + \sqrt{7u^2 + 24u + 144}}{2u} \} \), then when \( R_k < \lambda \), we have

\[
d_i^k > 0, \quad R_k < \lambda, \quad \forall k \geq 0, k \in Z, \quad \forall i \in Z
\]

(4)

i.e., the subdivision scheme (1) preserves convexity when \(-2 < u < 0\).

**Proof.** We use mathematical induction to prove \( d_i^k > 0, R_k < \lambda, \forall k \geq 0, k \in Z, \forall i \in Z \).

(1) When \( k = 0, d_i^0 > 0, \forall i \in Z, R_k < \lambda, \) then (4) clearly holds.

(2) Assume \( d_i^k > 0 \) and \( R_k < \lambda \) for \( \forall i \in Z \) and some \( k \geq 1 \), then it follows \( \frac{1}{\lambda} < r_i^k < \lambda \), \forall k \in Z, \forall i \in Z \).

By (2) and (3) we have

\[
d_{2i+1}^{k+1} = \frac{1}{16} d_i^k \left( (2 + u) \frac{1}{r_i^k} + (12 - 2u) + (u + 2)r_i^k \right),
\]

(5)

\[
> \frac{1}{16\lambda} d_i^k \left[ (12 - 2u) + 2(u + 2) \right],
\]

(5)

\[
> 0.
\]

\[
d_{2i}^{k+1} = \frac{1}{16} \left( \frac{1}{r_i^{k+1}} + (u + 1)r_i^k \right) + d_{i+1}^{k+1} \left[ (16 - u + ur_i^k) \right],
\]

(6)

\[
> \frac{1}{16\lambda} \left( d_i^k [16 + (\lambda - 1)u] + d_{i+1}^k [16 + (\lambda - 1)u] \right),
\]

(6)

\[
> 0.
\]

Combining (5) with (6) gives \( d_i^{k+1} > 0, \forall i \in Z \). Therefore \( d_i^k > 0, \forall k \geq 0, k \in Z, \forall i \in Z \).
(3) To prove \( R^k < \lambda \quad \forall k \geq 0, \quad k \in \mathbb{Z} \), it suffices to prove
\[
\frac{1}{\lambda} < r^k < \lambda, \quad \forall k \geq 0, \quad k \in \mathbb{Z}, \quad \forall i \in \mathbb{Z}.
\]
Notice that we have assumed \( R^k < \lambda \), for some \( k \geq 0 \), namely, \( \max_i(r^k_{t_i} \frac{1}{r^k_{t_i}}) < \lambda \), which implies \( \frac{1}{\lambda} < r^k < \lambda \), \( \forall i \in \mathbb{Z} \) for some \( k \geq 0 \). Therefore by induction we only need to show \( \frac{1}{\lambda} < r^{k+1} < \lambda \), \( \forall i \in \mathbb{Z} \). Since
\[
r^{k+1}_{2i} = \frac{d^{k+1}_{2i-1}}{d^{k+1}_{2i}} = \frac{1}{\lambda} d^k_{2i} \left[ \frac{1}{r^k_{t_i-1}} + (16 - u) + (16 - u) r^k_{t_i} + u r^k_{t_i} r^k_{t_i-1} \right]
\]
\[
= \frac{1}{\lambda} d^k_{2i} \left[ (u + 2) \frac{1}{r^k_{t_i-1}} + (12 - 2u) + (u + 2) r^k_{t_i} \right]
\]
\[
u + (16 - u) r^k_{t_i} + (16 - u) r^k_{t_i} r^k_{t_i-1} + u r^k_{t_i} r^k_{t_i-1} \right]
\]
\[
= \frac{1}{\lambda} d^k_{2i} \left[ (u + 2) \frac{1}{r^k_{t_i-1}} + (12 - 2u) + (u + 2) r^k_{t_i} \right]
\]
we have
\[
r^{k+1}_{2i} - \frac{1}{\lambda} u - (16 - u - 24 \lambda + 4u \lambda) r^k_{t_i} + (16 - u - 2u \lambda - 4 \lambda) r^k_{t_i} r^k_{t_i-1} + u r^k_{t_i} r^k_{t_i-1} - (2u + 4) \lambda
\]
\[
\Delta \frac{A}{B}.
\]
By (5), the denominator of the above expression is \( \frac{32a^{k+1}}{\phi} r^{k+1} r^{k+1}_{t_i-1} > 0 \). And the numerator satisfies
\[
A = u - (2u + 4) \lambda + (16 - u - 24 \lambda + 4u \lambda) r^k_{t_i} + (16 - u - 2u \lambda - 4 \lambda) r^k_{t_i} r^k_{t_i-1} + u r^k_{t_i} r^k_{t_i-1} = u - (2u + 4) \lambda + (16 - u - 24 \lambda + 4u \lambda) r^k_{t_i} + (16 - u - 2u \lambda - 4 \lambda + \frac{u}{\lambda}) r^k_{t_i} r^k_{t_i-1}
\]
\[
< u - (2u + 4) \lambda + (16 - u - 24 \lambda + 4u \lambda) r^k_{t_i} + \left( 16 - u - 2u \lambda - 4 \lambda + \frac{u}{\lambda} \right) r^k_{t_i} r^k_{t_i-1}
\]
\[
< u - (2u + 4) \lambda + (16 - 8 \lambda + 3u \lambda - 2u \lambda^2 - 4 \lambda^2) r^k_{t_i} r^k_{t_i-1}
\]
\[
< u - (2u + 4) \lambda + (16 - 8 \lambda + 3u \lambda - 2u \lambda^2 - 4 \lambda^2) \frac{1}{\lambda}
\]
\[
\frac{4}{\lambda} \lambda^2 + (u - 2) \lambda + 4
\]
\[
\lambda(\lambda - 1)(u + 2) + \lambda + 4
\]
\[
< 0.
\]
where we have used the fact that \( \frac{3u - 8 + \sqrt{9u^2 + 80u + 320}}{4u + 8} < \lambda < \min \left\{ \frac{16 - u + \sqrt{9u^2 - 16u + 256}}{4u + 8}, \frac{12 - 3u - \sqrt{-7u^2 + 24u + 144}}{2u} \right\} \), which results in
\[
1 \frac{1}{\lambda} (2u \lambda^2 - 4 \lambda^2 - u \lambda + 16 \lambda + u) = \frac{2u + 4}{\lambda} \left( \lambda + 16 - u + \frac{\sqrt{9u^2 - 16u + 256}}{4u + 8} \right) \left( \lambda + 16 - u - \frac{\sqrt{9u^2 - 16u + 256}}{4u + 8} \right) > 0
\]
\[
16 - 8 \lambda + 3u \lambda - 2u \lambda^2 - 4 \lambda^2 = (2u - 4) \left( \lambda - \frac{3u - 8 + \sqrt{9u^2 + 80u + 320}}{4u + 8} \right) \left( \lambda - \frac{3u - 8 - \sqrt{9u^2 + 80u + 320}}{4u + 8} \right) < 0
\]
Therefore
\[
r^{k+1}_{2i} < \lambda, \quad \forall i \in \mathbb{Z}.
\]
Since
\[
r^{k+1}_{2i+1} = \frac{d^{k+1}_{2i+1}}{d^{k+1}_{2i}} = \frac{1}{\lambda} d^k_{2i} \left[ (u + 2) \frac{1}{r^k_{t_i-1}} + (12 - 2u) + (u + 2) r^k_{t_i} \right]
\]
\[
= \frac{1}{\lambda} d^k_{2i} \left[ u \frac{1}{r^k_{t_i-1}} + (16 - u) + (16 - u) r^k_{t_i} + u r^k_{t_i} r^k_{t_i-1} \right],
\]
we have
\[
 r_{2i+1}^k - \lambda = \frac{2[u + 2 + (12 - 2u)r_i^k + (u + 2)r_i^k r_{i+1}^k] - \lambda \left[ u \frac{1}{r_i^k} + (16 - u) + (16 - u)r_i^k + ur_i^k r_{i+1}^k \right]}{u \frac{1}{r_i^k} + (16 - u) + (16 - u)r_i^k + ur_i^k r_{i+1}^k}.
\]

where
\[
 A = 2[u + 2 + (12 - 2u)r_i^k + (u + 2)r_i^k r_{i+1}^k] - \lambda \left[ u \frac{1}{r_i^k} + (16 - u) + (16 - u)r_i^k + ur_i^k r_{i+1}^k \right],
\]
\[
 B = u \frac{1}{r_i^k} + (16 - u) + (16 - u)r_i^k + ur_i^k r_{i+1}^k.
\]

It follows from (6),
\[
 B = \frac{32d_{2i+1}^k}{d_i^k} > 0,
\]
\[
 A = 2(2 + u) + (24 - 4u - 16\lambda + u\lambda)r_i^k + (2u + 4 - u\lambda)r_i^k r_{i+1}^k - u\lambda \frac{1}{r_i^k} - (16 - u)\lambda
\]
\[
 < 2(2 + u) + (24 - 4u + 3u\lambda - 12\lambda - u\lambda^2)r_i^k - u\lambda^2 - (16 - u)\lambda.
\]

When
\[
 \frac{3u - \sqrt{9u^2 - 16u + 120}}{4u^3 + 8u + 120} < \lambda < \min \left\{ \frac{16 - u + \sqrt{9u^2 - 16u + 256}}{4u^3 + 8u + 120}, \frac{12 - 3u - \sqrt{7u^2 + 24u + 144}}{4u^3 + 8u + 120} \right\},
\]
we have
\[
 24 - 4u + 3u\lambda - 12\lambda - u\lambda^2 = -u \left( \frac{\lambda}{2} - \frac{12 - 3u + \sqrt{-7u^2 + 24u + 144}}{2u} \right) \left( \frac{\lambda}{2} - \frac{12 - 3u - \sqrt{-7u^2 + 24u + 144}}{2u} \right) \geq 0
\]
and
\[
 A < 2(2 + u) + (24 - 4u + 3u\lambda - 12\lambda - u\lambda^2)\lambda - u\lambda^2 - (16 - u)\lambda
\]
\[
 = -u\lambda^3 + (2u - 12)\lambda^2 + (8 - 3u)\lambda + 2(2 + u)
\]
\[
 = (\lambda - 1)[-u\lambda^2 + (u - 12)\lambda - 2u - 4] < 0.
\]

When
\[
 \frac{12 - 3u - \sqrt{7u^2 + 24u + 144}}{2u} < \lambda < \min \left\{ \frac{16 - u + \sqrt{9u^2 - 16u + 256}}{4u^3 + 8u + 120}, \frac{12 - 3u + \sqrt{7u^2 + 24u + 144}}{4u^3 + 8u + 120} \right\},
\]
we have
\[
 24 - 4u + 3u\lambda - 12\lambda - u\lambda^2 = -u \left( \frac{\lambda}{2} - \frac{12 - 3u + \sqrt{-7u^2 + 24u + 144}}{2u} \right) \left( \frac{\lambda}{2} - \frac{12 - 3u - \sqrt{-7u^2 + 24u + 144}}{2u} \right) < 0
\]
and
\[
 A < 2(2 + u) + \frac{24 - 4u}{\lambda} + 3u - 12 - u\lambda - u\lambda^2 - (16 - u)\lambda
\]
\[
 = \frac{1}{\lambda} [-u\lambda^3 + 16\lambda^2 + (5u - 8)\lambda + (24 - 4u)]
\]
\[
 = \frac{1}{\lambda} (\lambda - 1)[-u\lambda^2 + (u + 16)\lambda + 4u - 24] < 0.
\]

Therefore
\[
 r_{2i+1}^k < \lambda, \quad \forall i \in \mathbb{Z}.
\]

Since
\[
 \frac{1}{r_{2i+1}^k} = \frac{d_{2i+1}^k}{d_i^k} = \frac{1}{12} \frac{d_i^k \left[ 2 + u \right] + (12 - 2u) + (u + 2)r_i^k}{d_i^k \left[ u \frac{1}{r_i^k} + (16 - u) + (16 - u)r_i^k + ur_i^k r_{i+1}^k \right]} \frac{1}{r_i^k} + (12 - 2u) + (u + 2)r_i^k \right] + \frac{2}{u \frac{1}{r_i^k} + (16 - u) + (16 - u)r_i^k + ur_i^k r_{i+1}^k}.
\]
we have

\[
\frac{1}{r_{21}^{\lambda+1}} - \lambda = \frac{2(2 + u) + 2(12 - 2u)r_{41}^{\lambda} + 2(2 + u)r_{41}^{\lambda}r_{41}^{\lambda}}{u + (16 - u)r_{41}^{\lambda} + (16 - u)r_{41}^{\lambda}r_{21}^{\lambda} + \text{ur}_{41}^{\lambda}r_{21}^{\lambda}r_{41}^{\lambda}}.
\]

By (6), the denominator of the above expression is $\frac{32A^{\lambda+1}}{\alpha} > 0$. And the numerator satisfies

\[
A = 2(2 + u) + 2(12 - 2u)r_{41}^{\lambda} + 2(2 + u)r_{41}^{\lambda}r_{41}^{\lambda} - u\lambda - (16 - u)\lambda r_{41}^{\lambda} - (16 - u)\lambda r_{41}^{\lambda}r_{41}^{\lambda} - u\lambda r_{41}^{\lambda}r_{21}^{\lambda}r_{41}^{\lambda} = 2(2 + u) + (24 - 4u - 16\lambda + u\lambda)r_{41}^{\lambda} + (4 + 2u - 16\lambda + u\lambda)r_{41}^{\lambda}r_{41}^{\lambda} - u\lambda - u\lambda r_{41}^{\lambda}r_{21}^{\lambda}r_{41}^{\lambda} < 2(2 + u) - u\lambda + (24 - 4u - 16\lambda + u\lambda)r_{41}^{\lambda} + (4 + 2u - 16\lambda + u\lambda - u\lambda^2)r_{41}^{\lambda}r_{41}^{\lambda} < 2(2 + u) - u\lambda + (24 - 4u - 16\lambda + u\lambda + \frac{4 + 2u}{\lambda} - 16 + u - u\lambda)r_{41}^{\lambda}
\]

\[
= 2(2 + u) - u\lambda + \frac{1}{\lambda}[-16\lambda^2 + (8 - 3u)\lambda + 4 + 2u]r_{41}^{\lambda} < 2(2 + u) - u\lambda + \frac{1}{\lambda}[-16\lambda^2 + (8 - 3u)\lambda + 4 + 2u]
\]

\[
= \frac{1}{\lambda}(-u\lambda^3 + (2u - 12)\lambda^2 + (8 - 3u)\lambda + 4 + 2u)
\]

\[
= \frac{1}{\lambda}(\lambda - 1)[-u\lambda^2 + (u - 12)\lambda - 4 - 2u] < 0,
\]

where we have used the fact that $1 < \frac{3u - 8 + \sqrt{9u^2 - 80u + 320}}{40 + 8} < \lambda < \min\left\{\frac{16 - u + \sqrt{9u^2 - 16u + 256}}{40 + 8}, \frac{12 - 3u + \sqrt{7u^2 + 24u + 144}}{2u} \right\}$, which leads to

\[
4 + 2u - 16\lambda + u\lambda - u\lambda^2 = -u\left(\lambda - \frac{16 - u + \sqrt{9u^2 - 16u + 256}}{2u}\right) < 0,
\]

\[
-16\lambda^2 + (8 - 3u)\lambda + 4 + 2u = -16\left(\lambda - \frac{3u - 8 + \sqrt{9u^2 + 80u + 320}}{32}\right) < 0.
\]

So

\[
\frac{1}{r_{21}^{\lambda+1}} < \lambda, \quad \forall \lambda \in \mathbb{Z}. \tag{9}
\]

Finally from

\[
\frac{1}{r_{21}^{\lambda+1}} = \frac{d_{21}^{\lambda+1}}{d_{21+2}^{\lambda+1}} = \frac{\frac{32}{\lambda} d_{21}^{\lambda}}{}[u \frac{1}{\tau_{21}} + (16 - u) + (16 - u)r_{41}^{\lambda} + ur_{41}^{\lambda}r_{41}^{\lambda}]
\]

\[
= \frac{32}{\lambda} d_{21}^{\lambda}[2 + u + (12 - 2u)r_{41}^{\lambda} + (u + 2)r_{41}^{\lambda}r_{41}^{\lambda}]
\]

\[
u \frac{1}{\tau_{21}} + (16 - u) + (16 - u)r_{41}^{\lambda} + ur_{41}^{\lambda}r_{41}^{\lambda} = \frac{2}{2 + u + (12 - 2u)r_{41}^{\lambda} + (u + 2)r_{41}^{\lambda}r_{41}^{\lambda}}
\]

it follows

\[
\frac{1}{r_{21}^{\lambda+1}} - \lambda = \frac{u \frac{1}{\tau_{21}} + (16 - u) + (16 - u)r_{41}^{\lambda} + ur_{41}^{\lambda}r_{41}^{\lambda} - 2\lambda[2 + u + (12 - 2u)r_{41}^{\lambda} + (u + 2)r_{41}^{\lambda}r_{41}^{\lambda}]}{2[2 + u + (12 - 2u)r_{41}^{\lambda} + (u + 2)r_{41}^{\lambda}r_{41}^{\lambda}]}
\]

\[
\Delta = \frac{A}{B}.
\]

By (5) and (6), the denominator of the above expression is $\frac{32A^{\lambda+1}}{\alpha} > 0$. And the numerator satisfies
\[ A = u \frac{1}{r_{i-1}^k} + (16 - u) - 2(2 + u)\lambda + (16 - u - 24\xi + 4u\lambda)\frac{r_{i}^k}{\lambda} + (u - 2u\lambda - 4\lambda)\frac{r_{i+1}^k}{\lambda^2} \]

\[ < u \frac{1}{\lambda} + (16 - u) - 2(2 + u)\lambda + \frac{16 - u - 24\xi + 4u\lambda}{\lambda} + \frac{u - 2u\lambda - 4\lambda}{\lambda^2} \]

\[ < u \frac{1}{\lambda} + (16 - u) - 2(2 + u)\lambda + \frac{16 - u - 24\xi + 4u\lambda}{\lambda} + \frac{u - 2u\lambda - 4\lambda}{\lambda^2} \]

\[ = -\frac{1}{\lambda^2} [2(2 + u)\lambda^2 - (3u - 8)\lambda^2 - (12 - 2u)\lambda - u] \]

\[ = -\frac{1}{\lambda^2} (\lambda - 1)(4 + 2u)\lambda^2 + (12 - u)\lambda + u] < 0. \]

Thus
\[ \frac{1}{r_{2i+1}^k} < \lambda, \quad \forall i \in \mathbb{Z}. \quad (10) \]

Combining (7)–(10) gives
\[ \frac{1}{\lambda} < r_{i+1}^k < \lambda, \quad \forall i \in \mathbb{Z}. \]

By induction we have
\[ \frac{1}{\lambda} < r_{i}^k < \lambda, \quad \forall k \geq 0, \quad i \in \mathbb{Z}. \]

Considering \( R^k = \max \left\{ r_1^k, \frac{1}{r_1^k} \right\} \), we have \( R^k < \lambda, \quad \forall k \geq 0, \quad i \in \mathbb{Z} \).

This completes the proof. \( \square \)

5. Numerical example and conclusion

In this section, we give two examples to illustrate our result. In Fig. 2, given the initial control points, \{(-4, 8.5), (-3.5), (-2.5), (-1.1), (0, 0.5), (1, 1), (2.5, 2.5), (3.5, 3.5), (4.8.5)\}, it is easy to get \( R^0 = 2 \). Clearly the initial control points are strictly convex. When \( u = -\frac{1}{4}, -\frac{1}{2}, -\frac{3}{4}, -\frac{5}{2} \), the limit curves generated by the scheme (1) are also convex, as shown in Fig. 2, where \( R^0 \in \left( \frac{3u - 8 \sqrt{9u^2 - 81u + 256}}{4u - 8}, \min \left\{ \frac{16 - u \sqrt{9u^2 - 288u + 1296}}{12 - 3u \sqrt{7u^2 - 24u + 144}} \right\} \right) \). Shown in Fig. 3 are the graphs of the

\[ \text{(a) when } u = -\frac{1}{4} \quad \text{(b) when } u = -\frac{1}{2} \]

\[ \text{(c) when } u = -\frac{3}{4} \quad \text{(d) when } u = -\frac{5}{2} \]

**Fig. 2.** The limit curves generated by the subdivision scheme (1) when \(-2 < u < 0\).
convexity preserving corresponding to the limit curves generated by the subdivision schemes in [1, 6, 8] and our scheme (when $u = C_0 1$) with the same initial control points {(-4,8.5), (-3,5), (-2,2.5), (-1,1), (0,0.5), (1,1), (2,2.5), (3,5), (4,8.5)}. As we can see from Fig. 3, our scheme not only own high order continuity, but also preserve convexity with the given initial points. As shown in Table 1, we compare our scheme with the schemes in [1, 6, 8, 14–16]. By comparing the indicators, obviously our method is effective.

In summary, if the initial control points are strictly convex and satisfy
\[
\begin{align*}
R^0 &\in \left( \frac{3u-8+\sqrt{9u^2+80u+320}}{40u}, \min \left\{ \frac{16-u+\sqrt{9u^2-16u+256}}{16-8}, \frac{12-3u+\sqrt{-7u^2+24u+144}}{-24} \right\} \right),
\end{align*}
\]
then we can choose the parameter $-2 < u < 0$ such that the limit curve generated by the subdivision scheme (1) is also convex.

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References


