Frequency domain methods for optimal sensor placement and scheduling of spatially distributed systems arising in environmental and meteorological applications

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Abstract—We consider the problem of sensor placement (both spatial location and number) for a class of parameter-dependent diffusion-advection processes that model environmental processes. To distinguish between advection and diffusion dominated environmental processes, the Péclet number, Pe, which takes values over a set of a priori defined values, is utilized to optimize the number and spatial location of the sensors required. The minimum number of sensors is defined using system theoretic measures and essentially considers the smallest number of sensors that would render the system observable, thereby facilitating the design of a state observer. The optimization metric is defined with respect to the spatial $H_2$ norm of the dominant system modes, which may differ for different values of Pe. For each value of Pe, a set of optimal sensor locations and number is found and the associated state estimator is designed. The supervisory scheme then schedules the sensors corresponding to the Péclet number that describes the process at a given time by putting in sleep mode all sensors associated with a different value of Pe and activating the sensors that are optimal for the current value of Pe. At the same time, the state estimator also switches by using the filter gain corresponding to the current value of the Péclet number and the active sensors. Extensive simulation studies are included to provide further inside on parameter-dependent sensor and observer scheduling for environmental processes.

Index Terms—Distributed parameter systems; spatial $H_2$ norms; parameter-dependent sensor scheduling.

I. INTRODUCTION

This work is concerned with the sensor placement in systems representing environmental applications. The added feature in the established and mature area of sensor optimization is the hybrid nature of the models that mathematically representing such process. For the specific example under consideration, the underlying partial differential equation (PDE) that models the physical process depends on a parameter that changes the nature of the process from diffusion-dominated to advection-dominated process and back.

The contribution is twofold: for a prescribed value of the physical parameter that dictates the behavior of the process (diffusion-dominated vs advection-dominated process) find the optimal sensor locations that optimize an appropriate measure of sensor and estimator performance and then provide a scheduling policy that switches from one set of optimal sensor locations to the other sensor set in a seamless fashion without compromising the performance of the monitoring scheme.

The mathematical framework utilizing the concept of the spatial $H_2$ norm is taken from the earlier work [1]. Specifically, spatial norms are employed to quantify the sensor network efficiency. The sensor placement question is then addressed by formulating an optimization problem with constraints that allow us to preferentially consider the important system dynamics and confer robustness to changing environmental conditions, high frequency system dynamics and nonlinear behavior. In the next session we present the system under consideration and introduce basic concepts needed for the optimization formulation. In Section III we propose a method to synthesize the sensor network and present the problem formulation and the solution procedure. The sensor scheduling scheme and the associated state estimation scheme are summarized in Section IV. An extensive numerical study is presented in Section V and conclusions follow in Section VI.

II. MATHEMATICAL FORMULATION AND PROBLEM DESCRIPTION

When considering weather prediction and monitoring, many environmental processes are captured by PDEs whose state may represent velocity field or chemical species’ concentration [2]. The selection of sensing devices is of significant importance to successfully monitor and predict weather patterns.

While many approaches exist that address the optimal placement of sensing devices within the spatial domains of interest (see the review articles [3], [4], a rather meager set of results address the issue of such a location optimization for systems with time varying parameters. Representative of such processes is the advection-diffusion equation whose diurnal variation of the Péclet number, Pe, renders the PDE time varying. In its simplest approximation, this can be modeled as a switched dynamical system with a parameter that takes distinct values over small set; for the ensuing example, the set consists of two values, a high Péclet number one representing advective-dominated process and a low Péclet number one representing diffusive-dominated process.

The simplest form of the diffusion-advection equation is the one restricted on the real line and given by the 1D PDE
with source/sink term
\[
\frac{\partial}{\partial t} \bar{u}(t,z) = D \frac{\partial^2}{\partial z^2} \bar{u}(t,z) - \nu \frac{\partial}{\partial z} \bar{u}(t,z) + f(u) + \bar{d}(z)w(t). \tag{1}
\]

Typical choices for the boundary conditions are Dirichlet BCs where the concentration is specified on a boundary and Fourier BCs at an inlet boundary when considering a problem defined on a semi-infinite domain \( \xi > 0 \) with inlet boundary at \( \xi = 0 \). In the latter case, the flux is required to be continuous across the boundary.

After non-dimensionalizing (1), one arrives at [5]
\[
\frac{\partial}{\partial t} \bar{u}(t,\xi) = \frac{1}{Pe^2} \frac{\partial^2}{\partial \xi^2} \bar{u}(t,\xi) - \frac{\partial}{\partial \xi} \bar{u}(t,\xi) + f(u) + \bar{d}(\xi)w(t). \tag{2}
\]

The above can be written as an evolution equation in the Hilbert subspace \( H \subset L_2(\Omega) \) of functions defined in \( \Omega \) that satisfy the boundary conditions
\[
\bar{u} = \mathcal{A}(Pe)u + f(u) + dw(t) \tag{3}
\]
where the parameter-dependent spatial operator \( \mathcal{A}(Pe) \) is defined as
\[
\mathcal{A}(Pe)\phi = \frac{1}{Pe} \frac{\partial^2}{\partial \xi^2} \phi(\xi) - \frac{\partial}{\partial \xi} \phi(\xi), \quad \phi \in H^2(\Omega).
\]

The state space \( H \) is equipped with the standard inner product of a weighted integral
\[
\langle \phi_1, \phi_2 \rangle = \int_{\Omega} r(\xi)\phi_1^*(\xi)\phi_2(\xi) d\xi,
\]
and norm \( ||\phi||_1 = \sqrt{\langle \phi, \phi \rangle} \), where \( r(\xi) \) is the weight function. Associated with (1) are the measurements obtained by the sensing devices
\[
y(t) = \int_{\Omega} c(\xi;\xi_s)u(t,\xi)d\xi \tag{4}
\]
where the vector \( y(t) \) denotes the vector of measured variables and \( c(\xi;\xi_s) = [c_1(\xi;\xi_{s1}) \ldots c_N(\xi;\xi_{sN})]^T \) is an \( N \) dimensional vector that denotes the sensor shape in \( \Omega \) and essentially describes the sensors. The vector \( \xi_s \in (\Omega)^N \) denotes the vector of sensor locations or sensor centroids. The sensor shapes may be different to account for a heterogeneous network of sensors. Typical choices for sensors include

- the Dirac function representing pointwise in space measurements of the state with \( c_i(\xi;\xi_{si}) = \delta(\xi - \xi_{si}); \) then\]
\[
c_i = \delta(\xi - \xi_{si}) \quad \text{and} \quad y_i(t) = \int_{\Omega} \delta(\xi - \xi_{si})u(t,\xi)d\xi = u(t,\xi_{si})
\]

- the boxcar function representing local-in-space averaged measurements with
\[
c_i(\xi;\xi_{si}) = \begin{cases} \frac{1}{\xi} & \text{for } \xi_{si} - \frac{\xi}{2} \leq \xi \leq \xi_{si} + \frac{\xi}{2} \\ 0 & \text{otherwise} \end{cases}
\]

and
\[
y_i(t) = \frac{1}{\xi} \int_{\xi_{si} - \frac{\xi}{2}}^{\xi_{si} + \frac{\xi}{2}} u(t,\xi)d\xi \approx \frac{u(t,\xi_{si} + \frac{\xi}{2}) + u(t,\xi_{si} - \frac{\xi}{2})}{2}
\]

For emphasis, the dependence of the measured output on sensor location (centroid) \( \xi_s \) is presented explicitly as
\[
y(t;\xi_s) = [y_1(t;\xi_{s1}) \ldots y_N(t;\xi_{sN})]^T \tag{5}
\]

The solution to the evolution equation (3) can be computed using the eigenfunctions of the spatial operator, \( \phi(\xi) \), as a basis of \( H \) to express the system state, \( u \), as a function of system eigenmodes \( x_i \). Note that the eigenfunctions depend on \( Pe \). To simplify the exposition we only present this dependency when necessary: \( u(t,\xi) = \sum_i \phi_i(\xi)x_i(t) \).

To analyze the evolution equation, initially, the linear component of \( \bar{f}(u) \approx ku \) in (1) is retained, leading to the following infinite dimensional ODE expression within \( H \),
\[
x(t) = \mathcal{A}(Pe)x(t) + kx(t) + Dw(t)
\]
\[
y(t;\xi_s) = C(\xi_s)x(t). \tag{6}
\]

Using Laplace transforms with \( X_i(s) = \mathcal{L}[x_i(t)] \) one obtains the following transfer function model for the \( i \)th eigenmode
\[
X_i(s) = \frac{1}{s - (\lambda_i + k)}x_i(0) + \frac{\langle \phi_i, d \rangle}{s - (\lambda_i + k)}W(s)
\]
where \( \lambda_i \) denotes the \( i \)th eigenvalue of \( \mathcal{A} \). Similarly, one can obtain the Laplace transform of the measured output
\[
Y(s;\xi_s) = \sum_{i=1}^{\infty} \left( \int_{\Omega} c(\xi;\xi_s)\phi_i(\xi)d\xi \right) X_i(s).
\]
To express \( Y(s;\xi_s) \) in terms of the state initial condition and the disturbance, we define
\[
C_i(\xi_s,Pe) = \int_{\Omega} c(\xi;\xi_s)\phi_i(\xi)d\xi \quad \text{and} \quad D_i(Pe) = \langle \phi_i, d \rangle
\]
to obtain
\[
Y(s;\xi_s) = \sum_{i=1}^{\infty} \frac{C_i(\xi_s)}{s - (\lambda_i + k)}X_i(0) + \sum_{i=1}^{\infty} \frac{C_i(\xi_s)D_i}{s - (\lambda_i + k)}W(s)
\]
\[
= \sum_{i=1}^{\infty} G_i(s;\xi_s)X_i(0) + \sum_{i=1}^{\infty} H_i(s;\xi_s)W(s) \tag{7}
\]
\[
= \mathcal{G}(s;\xi_s,Pe)X(0) + \mathcal{H}(s;\xi_s,Pe)W(s).
\]
Here, \( G_i \) denotes the transfer function between the measurement vector and the \( i \)th eigenmode.

At this stage, one can proceed with the sensor location optimization using "open-loop" techniques. In essence, one chooses the sensor locations, through the vector \( \xi_s \), so that the useful information is accentuated and the effects of the disturbance are minimized; i.e. it is desired to choose \( \xi_s \) to maximize a norm of \( \mathcal{G}(s;\xi_s) \) and minimize a norm of \( \mathcal{H}(s;\xi_s) \). To proceed further, one can add another level of optimization whereby the sensor locations are found that maximize the useful information against the worst case of the disturbance [6]; this is represented by \( D_i \).

As the ultimate goal is the reconstruction of the state of the process, one must consider the optimal sensor placement in the context of filter design. Using the sensor location-parametrized output \( y(t;\xi_s) \), one can obtain the equation of the state estimate \( \hat{x}(t;\xi_s) \)
\[
\hat{x} = (\mathcal{A}(Pe) + kl - L(\xi_s,Pe)C(\xi_s,Pe))\hat{x} + L(\xi_s,Pe)y(t;\xi_s) \tag{8}
\]
where \( L(\xi_s,Pe) \) denotes the location-parametrized filter gain; such a gain can be obtained by Kalman filter design or
Luenberger observer design. The associated state error \( e(t) = x(t) - \hat{x}(t) \) is governed by

\[
\dot{e}(t) = (A(Pe) + kI - L(\xi_s, Pe)C(\xi_s, Pe))e(t) + D(\xi_s)w(t)
\]  

(9)

Using the above equation, one can describe the requirements for optimal sensor placement: find the optimal locations \( \xi_s \) that minimize the effects of \( w(t) \) on the state error \( e(t) \) and also minimize an appropriately chosen energy cost of the state error. An alternate requirement would be find \( \xi_s \) that enhance the convergence of the state error to zero.

Similar to the process state, we expand the state estimation error as \( e(t, \xi) = \sum_{i=1}^{M} \phi_i(\xi)e_i(t) \), and its Laplace transform as

\[
E(s) = \sum_{i=1}^{\infty} \frac{e_i(0)}{s-\mu_i(\xi_s)} + \sum_{i=1}^{\infty} D_i(\xi_s)W(s)
\]

\[
= T(s; \xi_s, Pe)e(0) + U(s; \xi_s, Pe)W(s)
\]

where \( \mu_i(\xi_s) \) denote the eigenvalues of the parameter-dependent closed loop operator \((A(Pe) - L(\xi_s, Pe)C(\xi_s, Pe))\). The term \( B_i \) represents the projection of the spatial distribution of the disturbances with respect to the basis function set \( \{\phi_i(\xi)\}_{i=1}^{\infty} \), where \( \phi_i(\xi) \) is the \( i \)th eigenfunction of the parameter-dependent closed loop operator \((A(Pe) - L(\xi_s, Pe)C(\xi_s, Pe))\).

Using the above, we can now state the design objective.

**Problem statement:** Find the optimal sensor locations for each value of parameter \( Pe \) in the set \( \Theta = \{Pe_1, Pe_2, \ldots \} \). For each \( Pe \in \Theta \) the \( N \) optimal values of the sensor locations \( \xi_s \) should maximize the spatial \( H_2 \) norm of \( G(s; \xi_s, Pe) \) and minimize the \( H_2 \) norm of \( T(s; \xi_s, Pe) \). Then find a sensor scheduling policy that would activate the sensors that correspond to the current value of parameter \( Pe \).

**Remark 1:** Please note that for a given value of \( Pe \in \Theta \), different sensor locations \( \xi_s \) yield different eigenfunctions \( \phi_i(\xi) \) of the closed-loop operator \((A(Pe) + kI - L(\xi_s, Pe)C(\xi_s, Pe))\).

### III. Sensor placement scheme

The question of optimal sensor placement for processes with significant disturbances is posed as a constrained nonlinear optimization problem. We initially introduce the concepts used for the definition of the objective and constraint functions. We then present the optimization formulation. At this stage we consider the sensor placement problem for each value of \( Pe \in \Theta \) independently. Furthermore, the dimensionality of the observer and the number of sensors used within the sensor network will vary, which is expected due to the profound effect that \( Pe \) has on (1).

**A. Spatial norms and observability measures**

In order to place sensors at locations of high sensitivity to the distributed process state \( \tilde{u} \) we first need define a quantitative metric based on the mathematical description as expressed with respect to Hilbert space, \( H \). The spatial \( H_2 \) norm [7] of transfer function \( G(s; \xi_s; \xi_s) \) in (7) is defined as

\[
\|G\|_{H_2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\xi_s} r(\xi)tr\{G^*(j\omega, \xi_s, \xi_s)G(j\omega, \xi_s, \xi_s)\} d\xi d\omega
\]

The spatial \( H_2 \) norm at location \( \xi_s \) is a measure of sensor sensitivity placed at location \( \xi_s \) over the entire spatial domain in an average sense.

With the aid of the spatial \( H_2 \) norm, numerous metrics of sensor placement effectiveness can be proposed. Based on the modal expansion of the state the orthogonality property of the eigenfunctions, \( \|G\|_{\xi_s}^2 \) can be expressed as the sum of the modal norms.

\[
\|G\|_{\xi_s}^2 = \sum_{i=1}^{M} \int_{\xi_s} r(\xi)tr\{G_i^*(j\omega, \xi_s, \xi_s)G_i(j\omega, \xi_s, \xi_s)\} d\xi d\omega
\]

(10)

We define, \( f_i(\xi_s) \triangleq \|G_i(s; \xi_s; \xi_s)\|_2 \), the \( i \)th modal norm at location \( \xi_s \) which is a measure of sensor network sensitivity placed at location \( \xi_s \) to the \( i \)th eigenmode of the system.

We further define the \( j \)th component \( i \)th modal norm

\[
f_{i,j}(\xi_s) \triangleq \frac{1}{(s - \lambda_i)^{100\%}} \int_{\xi_s} c_j(\xi_s, \xi_s, \phi_i(\xi)) d\xi d\omega
\]

which describes the sensitivity of the \( j \)th sensor over the \( i \)th mode. If the modal norm at a specific sensor location of a given mode is zero, it means that the specific mode is unobservable by that sensor.

Note that from the definitions we have \( f_i^2(\xi_s) = \sum_{j=1}^{k} f_{i,j}^2(\xi_s) \).

Capitalizing on the property of strong elliptic operators that the higher modes become progressively more stable \( (\lambda_{i+1} < \lambda_i \) and \( \lambda_i \to -\infty \) as \( i \to \infty \)\), we need only consider a finite number of modes to accurately approximate \( \|G(s; \xi_s; \xi_s)\|_{\xi_s}^2 \) [8]:

\[
\|G(s; \xi_s; \xi_s)\|_{\xi_s}^2 = \sum_{i=1}^{M} f_i^2(\xi_s) = \sum_{i=1}^{M} f_i^2(\xi_s) + \sum_{i=M+1}^{\infty} f_i^2(\xi_s)
\]

(11)

\[
= H^2(\xi_s) + S^2(\xi_s)
\]

where \( H^2(\xi_s) \triangleq \sum_{i=1}^{M} \|G_i(s; \xi_s)\|_2^2 = \sum_{i=1}^{M} f_i^2(\xi_s) \) denotes the truncation of the \( H_2 \) spatial norm to the first \( M \) modes and \( S^2(\xi_s) \triangleq \sum_{i=M+1}^{\infty} \|G_i(s; \xi_s)\|_2^2 = \sum_{i=M+1}^{\infty} f_i^2(\xi_s) \) denotes the “spillover” of the higher modes’ dynamics on sensor accuracy when estimating the dominant modes.

**Note:** \( H^2(\xi_s) \) is a measure of sensor sensitivity to the dominant dynamic behavior of the system when placed at location \( \xi_s \) and averaged over the entire spatial domain.

Following [9], the \( i \)th modal observability of the \( j \)th sensor at location \( \xi_{s,j} \) can now be defined as

\[
M_{i,j}(\xi_{s,j}) = \frac{f_{i,j}(\xi_{s,j})}{\max_{\xi_{s,j}} f_{i,j}(\xi_{s,j})} \times 100\%.
\]

(12)

\( M_{i,j}(\xi_{s,j}) \) describes the \( j \)th sensor sensitivity to the \( i \)th mode. If the modal observability at a specific location \( \xi_{s,j} \) of a given mode is zero, it means that the specific mode is unobservable by the specific sensor.

To measure the observability of a mode by the sensor network, we also define the \( i \)th modal observability at sensor
locations $\xi_s = \{\xi_{s,1}, \xi_{s,2}, \ldots, \xi_{s,K}\}$ as

$$M_i(\xi_s) = \frac{f_i(\xi_s)}{\max_{\xi \in \Omega_s} f_i(\xi)} \times 100\%, \quad (13)$$

which describes the total sensitivity of the sensor network to the $i^{th}$ mode. If the modal observability of a given mode is zero at specific locations $\xi_s$, it means that none of the sensors can observe the mode, rendering the specific mode unobservable. This coincides with the notion of approximate observability for the class of PDEs with Riesz-spectral operators [10]. The requirement for that in this case is $\langle c, \phi_i \rangle \neq 0$, $\forall i$. The $i^{th}$ mode has a zero modal observability at location $\xi_s$, when $f_i(\xi_s) \equiv 0$ or $C_i(\xi_s) = 0$, or $\langle c, \phi_i \rangle = 0$.

B. Sensitivity to fast transients

While the concept of spatial $H_2$ norm and its truncation to the dominant system behavior, $H$, allows us to identify sensor locations that confer high sensitivity to the sensor network, care must be exercised in order to avoid choosing locations that might also confer high sensitivity to the fast dynamics of the system (the higher modes of the Hilbert system representation). Such locations would lead to performance deterioration as initially there may be large error between observer predictions and measurements. The effect of the higher modes to the sensor network is mathematically described by the term $S(\xi_s)$ in (11), which, in general, cannot be computed since it is an infinite sum. Capitalizing again on the property of strongly elliptic spatial operators, it can be similarly assumed that this performance deterioration is due to medium range modes only; medium range modes are modes that are at immediate proximity of the first $M$ modes. Thus, when computing $S(\xi_s)$ these medium range modes $i = M + 1, \ldots, P$ need only be considered, i.e. $S_2(\xi_s) \simeq \sum_{i=M+1}^{P} f_i^2(\xi_s)$. Thus, $S_2(\xi_s)$ is an open-loop measure of the deteriorating effect that fast transients over the entire spatial domain have on average on the observer predictions when the sensor network is placed at locations $\xi_s$.

C. Measures of sensor clusterrization and redundancy

A problem commonly encountered during the synthesis of sensor networks for spatially distributed systems that consist of multiple sensors is the issue of clusterrization, i.e. placing sensors at or near the same location. An excellent exposure of the optimization and computational issues associated with multiple sensors, including that of sensor clusterrization, can be found in [11]. However, in a distributed system there are also locations that provide certain spatial symmetry with respect to measurement even though the sensors are physically placed at different locations.

Another related issue that is particular to spatially distributed systems is the issue of sensor redundancy, i.e. less sensors can provide the same amount of information as the current sensor network. This issue can appear in the absence of clustering and it is significantly impacted by the sensor location. For completeness we present the metrics employed to quantify these concepts; the interested reader may refer to [6] for a detailed analysis. We define the spatial observability matrix $\mathcal{M} \in \mathbb{R}^{K \times N}$ as:

$$\mathcal{M}(\xi_s) = \begin{bmatrix} M_1(\xi_s) & \ldots & M_N(\xi_s) \\ \vdots & \ddots & \vdots \\ M_K(\xi_s) & \ldots & M_K(\xi_s) \end{bmatrix}. \quad (14)$$

Assuming that more dominant modes are considered than the number of used sensors $K \geq N$, the proximity in an $L_2$ sense of the sensor locations and thus the sensor network redundancy is identified through linear dependence of the columns of $\mathcal{M}$. If two sensors are placed at the “same” position, two columns of $\mathcal{M}$ are the same and thus linearly dependent. The proximity to linear dependence of the columns is captured by the condition number of $\mathcal{M}$, denoted by $\kappa(\mathcal{M})$. This implies that there is a direct relationship between the value of the condition number and the redundancy in the actuator network. An in depth presentation of the physical interpretation of the specific measure can be found in [6].

D. Measure of disturbance effect

Finally, we need to consider sensor locations that provide an adequate level of modal observability while at the same time can minimize the effects of disturbance on the estimation error. In the proposed approach we assume that the spatial distribution of the disturbance is known. Based on this we can similarly define metrics of the disturbance effects. Specifically, we define the $i^{th}$ disturbance modal norm, a measure of the effect of all the disturbances over the specific $i^{th}$ eigenmode of the system,

$$g_i(\xi_s) = ||H_i(s; \xi_s)||_2 = \left| \frac{C_i(\xi_s, Pe)D_i(\Pi)}{s-(\lambda_i(\Pi)+k)} \right|_2. \quad (15)$$

Capitalizing again on the property of strong elliptic operators that the higher modes become progressively more stable ($\lambda_{i+1} \leq \lambda_i$ and $\lambda_i \to \infty$ as $i \to \infty$), and assuming that the frequency of the disturbance term is bounded, we only need consider a finite number of modes to compute an accurate approximation of the effect of disturbances. Cases may arise when the disturbance distribution function $D$ and the frequency bound on $w(t)$ is such that the disturbance excites specific higher modes to the system. In such a case, a sensor network that is insensitive to the system fast transients will also be insensitive to the noise.

We may now define the spatial robustness index as

$$R(\xi_s, Pe) = \sqrt{\sum_{i=1}^{K} S_i^2(\xi_s, Pe)}. \quad (16)$$

The robustness index is a measure of the sensor network sensitivity to the effects of noise on the dominant dynamic behavior of the system when placed at location $\xi_s$ and averaged over the entire spatial domain. In general the lower the index is, the smaller the effect of noise on the measurement.
E. Synthesis of sensor network

Once the various measures have been defined, the synthesis of the sensor network problem can now be recast as an optimization problem. The objective is then to identify \( N \) sensor locations \( \xi_{sj} \in \Omega_s, \ j = 1, \ldots, N \) that

1. maximize the network sensitivity to dominant system dynamics,
2. minimize redundancy,
3. minimize the effects of fast transients on estimation error,
4. maintain a reasonable level of sensor sensitivity over each slow mode,
5. reject at the sensor level disturbance effects on the state estimates.

Note that the optimization problem needs to be solved independently for each value of the Péclet number.

The constrained optimization problem for the sensor network is then formulated as:

\[
\begin{aligned}
\xi^{*}_s &= \arg \max_{\xi_s \in D_s} \left\{ \mathbf{H}^2(\xi_s) - \omega_s \mathbf{S}^2(\xi_s) - \omega_d \mathbf{R}(\xi_s) \right\} \\
\text{s.t.} \\
\kappa(\mathcal{M}(\xi_s)) &\leq \beta_s, \\
\mathcal{M}_i(\xi_s) &\geq \beta_{m_i}, \quad \forall i = 1, 2, \ldots, K.
\end{aligned}
\]  

(Pc-I)

The condition number constraint imposes a minimum distance of \( D_s \) between all sensors in the network in an \( L_2 \) sense. Thus the clustering and redundancy issues are avoided. The use of modal observability in (Pc-I) enforces a reasonable level of sensor sensitivity over each slow mode. Measurement sensitivity to fast transients is averted through the inclusion of term \( \mathbf{S}^2(\xi_s) \). The relative importance of minimizing these “spillover” effects compared to obtaining a sensor network that maximizes the spatial observability can be tuned through the selection of the weight \( \omega_d \).

Due to the nature of the problem, we prefer to express any constraints associated with the disturbance as a soft constraint (resembling a formulation for penalty methods). The severity of the sensor sensitivity to the disturbance can be tuned by the parameter \( \omega_d \) in the objective function. Note that the penalty itself is automatically adjusting to modes that have high sensitivity to the disturbance, since the modal norm is in effect an observability and a controllability grammian; as a result, modes that are not affected by the disturbance will have \( D_s \) values close to zero.

The formulated optimization problems can be solved using standard search algorithms such as Newton-based, interior-point or direct-search methods [13]. Due to the nonlinear nature of the objective function and the inequality constraints, and the “rugged” topology of the objective function preferable methods are derivative free and direct search algorithms [13], particle swarm optimization [14], simulated annealing, and funneling [15]. Genetic algorithms [16] are commonly used for such problems due to their robustness to all the above issues (it is also used in the next section).

A significant question that arises during the synthesis of the sensor network is how many sensors should be employed.

This question is addressed indirectly, by identifying for each Péclet number the value \( N \) that renders the optimization problem feasible. Following the solution of the optimization problem using open-loop criteria, a second “outer” optimization problem can then be formulated to identify the closed-loop observer gain \( L \) that now minimizes the closed-loop spatial observability gramian with respect to the disturbance. This leads to a two-level optimization formulation. Even though this procedure does not necessarily converge to a globally optimal sensor placement/observer gain pair, it is preferable since the full problem \( s \) has a very complex topology that can lead to convergence problems.

Remark 2: Assigning values to constraint parameters \( \beta_m \) and \( \beta_s \) in (Pc-I) may become an issue. It can be relieved by initially solving (Pc-I) for the process in the absence of disturbances (i.e., \( \mathcal{M}' = 0 \)) to obtain the best case scenario values of \( \mathbf{H}^*(\xi_s), \mathbf{S}^*(\xi_s) \). The constraints can then be evaluated based on the optimal non-disturbance values.

Remark 3: An issue which often arises in formulations using modal methods is the objective function weights that should be assigned to each mode. In (Pc-I) the weight that is assigned to each eigenmode depends on the associated eigenvalue. Thus, modes that are less stable have a greater contribution to \( \mathbf{H}^*(\xi_s) \) and as a result the sensors will gravitate towards locations with greater sensitivity to these modes.

IV. SENSOR SCHEDULING AND STATE ESTIMATION SCHEME

Once the optimal set of sensor location \( \xi_s \) for each value of \( Pe \in \Theta \) has been found, one can proceed with a scheduling policy of both the sensors and the estimator gains. We denote the output matrix associated with the optimal locations for a given value of the parameter \( Pe_k \in \Theta \), \( \xi^{*}_s \), by

\[
C(\xi^{*}_s; Pe_k), \quad Pe_k \in \Theta
\]

and denote the optimal filter gain associated with the pair \((\mathcal{A}(Pe_k), C(\xi^{*}_s; Pe_k))\) by

\[
L(\xi^{*}_s; Pe_k), \quad Pe_k \in \Theta.
\]

Then, for each value of \( Pe_k \in \Theta \) we implement the following estimator that utilizes the current set of sensors defined by \( C(\xi^{*}_s; Pe_k) \) via

\[
\begin{aligned}
\hat{x}(t; \xi^{*}_s) &= (\mathcal{A}(Pe_k) - L(\xi^{*}_s; Pe_k)C(\xi^{*}_s; Pe_k))\hat{x}(t; \xi^{*}_s) \\
&+ L(\xi^{*}_s; Pe_k)C(\xi^{*}_s; Pe_k)y(t; x(t)).
\end{aligned}
\]  

Algorithm 1 Sensor scheduling for process monitoring

1: \textbf{measure} physical parameter \( Pe_k \in \Theta \)
2: \textbf{switch} to optimal sensors \( \xi^{*}_s \) associated with current \( Pe_k \in \Theta \)
3: \textbf{output} process measurements \( y(t; \xi^{*}_s) \)
4: \textbf{repeat} for next value of \( Pe_k \in \Theta \)
Algorithm 2 Sensor and estimation gain scheduling for state estimation

1: read physical parameter \( P_{ek} \in \Theta \)
2: obtain process measurements with current set of sensors \( \xi^k \)
3: compute filter gain \( L(\xi^k, P_{ek}) \) corresponding to current optimal sensor locations
4: implement state estimator (17)
5: repeat for next value of \( P_{el} \in \Theta \)

V. EXAMPLE

To illustrate the theoretical concepts of the previous section, let us consider a representative environmental system within a bounded region of interest. Specifically, we consider the dispersion of a chemical species, denoted by \( A \), in one dimension in the presence of periodically varying air flow. We assume that the species slowly degrades in the environment following first order reaction kinetics.

The system can be mathematically described by the following one dimensional PDE in dimensionless form

\[
\frac{\partial}{\partial t} c(t, \xi) = \frac{1}{P_e} \frac{\partial^2}{\partial \xi^2} c(t, \xi) - \frac{\partial}{\partial \xi} c(t, \xi) - kc(t, \xi)
\]  

(18)

where \( c \) denotes the concentration of species \( A \), in the dimensionless spatial domain of interest \( \Omega = [0,1] \). The degradation rate used is \( k = 0.01 \). At the left boundary of \( \Omega \) we assume that there is a flux of species \( A \) entering the domain providing us with a Robin boundary condition. We also assume that \( A \) fully disperses within \( \Omega \) which is captured by a Neumann boundary condition

\[
\frac{\partial}{\partial \xi} c \bigg|_{\xi=0} = P_{e} c(t, 0), \quad \frac{\partial}{\partial \xi} c \bigg|_{\xi=1} = 0.
\]  

(19)

The eigenvalue-eigenfunction problem of the spatial operator \( \mathcal{A} \) under boundary conditions of (19) can be solved analytically utilizing standard techniques. The solution is:

\[
\lambda_j = \frac{\alpha_j^2}{P_e} - \frac{P_e}{4}, \quad \phi_j(\xi) = \beta_j \exp\left(\frac{P_e}{2} \xi\right) \left(\cos(\alpha_j \xi) + \frac{P_e}{2\alpha_j} \sin(\alpha_j \xi)\right)
\]  

(20)

and constant \( \alpha_j \) is obtained from the transcendental equation

\[
P_e \alpha_j = \left(\alpha_j^2 - \left(\frac{P_e}{2}\right)^2\right) \tan(\alpha_j), \quad j = 1, \ldots, \infty
\]

and \( \beta_j \) from \( \beta_j^2 = \int_{0}^{1} \left(\cos(\alpha_j \xi) + \frac{P_e}{2\alpha_j} \sin(\alpha_j \xi)\right)^2 d\xi \).

Defining the Hilbert space with weight \( r(\xi) = \exp(-P_e \xi) \), the system is expressed in the form of (6).

A number of sensing devices are available with the following spatial distribution description when placed at location \( \xi_c, c(\xi_c, \xi) = \frac{1}{\Pi}(\xi_c, \xi) \), where \( \Pi \) is the boxcar function and \( \varepsilon = 0.005 \). The admissible sensor domain is chosen to be the complete system domain \( \Omega_c = [0.01, 0.99] \). A disturbance in the form of a source of \( A \) is assumed to exist with distribution function \( d(\xi) = \frac{1}{0.01} \Pi(\xi - 0.4) \).

<table>
<thead>
<tr>
<th>( P_e )</th>
<th>( N )</th>
<th>( M )</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>( \lambda_3 )</th>
<th>( \lambda_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1.1720</td>
<td>-12.0219</td>
<td>-41.7004</td>
<td>-109.0636</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>-1.3535</td>
<td>-7.2462</td>
<td>-22.1786</td>
<td>-46.8847</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>4</td>
<td>-1.9430</td>
<td>-4.7995</td>
<td>-10.9712</td>
<td>-20.9257</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>6</td>
<td>-3.0219</td>
<td>-4.7670</td>
<td>-8.0706</td>
<td>-13.1639</td>
</tr>
</tbody>
</table>

TABLE I

SYSTEM DIMENSION AND EIGENVALUES FOR \( P_e \) VALUES

To synthesize the sensor network, we initially investigated the effect of Péclet number on the Hilbert representation of the system for \( \Theta = 1, 2, 5, 10 \). We observe in Table I that as \( P_e \) increases, the dimensionality of the dominant system increases rapidly, and to capture the fast transient effect on the sensor network numerous intermediate modes need to be considered. Furthermore, the separation between the slow and the fast dynamics rapidly decreases. This is an expected result, since the increase of the Péclet number marks the transition from a diffusion dominated to an advection dominated regime. The mathematical explanation is that even though the behavior of the system is always dissipative it approaches the behavior of hyperbolic systems. At the limit of \( P_e \to \infty \) the PDE will become first order hyperbolic.

Subsequently, we investigated the effect of \( P_e \) on the sensitivity of a single sensor to the system. In Figure 1(a) we present the sensitivity of a single sensor to the dominant system dynamics. We observe that the best location is always close to the left boundary of the system domain. For \( P_e = 1 \) though most locations lead to “good locations” to place the sensor, as evidenced by the high and almost flat profile of \( \mathbf{H} \). As \( P_e \) increases though we observe that the observability deteriorates fast once away from the left boundary. A reason for this trend is the weight function used in the definition of the Hilbert space. Physically, as we proceed further away from the left boundary, for increasing Péclet numbers it takes a longer time to observe the system dynamics since the diffusive behavior of the system is slowly masked by the advective phenomena, which at the limit of \( P_e \to \infty \) imply that a sensor placed at the center of the domain will have no information of the change of concentration at the left, until the concentration change wave front reaches its location.

Even though the “best” location appears to be the left most when one considers the effect of the fast transients, the conclusions change. In Figure 1(b) we present the spillover spatial observability \( \mathbf{S} \). We observe that the effect of the transients is also maximal at the left boundary of the system. The sensitivity though of the sensor to the fast transient does not present a simple picture. We observe numerous locations where the sensor is insensitive to the fast transients. We also observe that as \( P_e \) increases, \( \mathbf{S} \) becomes ever more significant relative to \( \mathbf{H} \) and it needs to be accounted for. The reason for this is again the behavior of the system approaching the hyperbolic one. We thus stress that when there is an increased \( P_e \) we need to account for the loss of observability and the loss of “robustness” with respect to the fast transients. This
problem is addressed by increasing the number of sensors to better estimate the system states.

Finally, in Figure 1(c) we present the spatial robustness of the system to the disturbance. It is interesting to note that as $Pe$ increases the robustness index $R$ decreases in value. This is due to the fact that as the system behavior approaches the hyperbolic behavior, the sensor becomes more robust to the effects of the disturbance since it slowly looses observability of the system. Another interesting phenomenon is the specific profile of $R$ as a function of the sensor location. We observe that the robustness index does not vary significantly with $\xi$. For small $Pe$ numbers this is expected since the injection by the disturbance source rapidly diffuses to the whole domain, while in the case of high $Pe$ the sensor is not able to observe the disturbance rapidly, thus leading to large robustness.

Based on the intuition gained by the previous analysis we formulated an optimization problem to identify sensor locations. We use the weights $w_s = 2$ and $w_d = 1$ to account for the significant effects of the fast transients. We also employed $\beta_s = 10$ and $\beta_m = 20\%$. The results of the optimization are presented in the following table II. We observe that as $Pe$ increases for the chosen weights the sensors tend to cluster around regions of lower spillover observability.

### Table II

<table>
<thead>
<tr>
<th>$Pe$</th>
<th>$\xi_s$</th>
<th>$\xi_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.500</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>0.811</td>
<td>0.669</td>
</tr>
<tr>
<td>5</td>
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<td>0.549</td>
</tr>
<tr>
<td>10</td>
<td>0.120</td>
<td>0.196</td>
</tr>
<tr>
<td>0.071</td>
<td>0.113</td>
<td>0.278</td>
</tr>
</tbody>
</table>

**VI. Conclusions**

The problem of sensor network synthesis and observer design was considered for a class of spatially distributed dynamic systems with slowly varying parameters, motivated by environmental and meteorological monitoring. The objective of the design is to reconstruct a complete profile of the state in the presence of disturbances and variation of system parameters that lead to radically different system behavior. We addressed this problem in a two-step procedure. First, sensor networks are identified for discrete values of system parameters via the solution of an optimization problem that explicitly accounts for disturbance rejection and sensor sensitivity to fast transients. A sensor-plus-estimator scheduling policy is then presented to switch between different parameter regimes so that the norm of the state estimation error remains below certain acceptable levels.

**REFERENCES**


