On robust solutions to uncertain monotone linear complementarity problems (LCPs) and their variants

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Abstract—Variational inequality and complementarity problems have found utility in modeling a range of optimization and equilibrium problems arising in engineering, economics, and the sciences. Yet, while there have been tremendous growth in addressing uncertainty in optimization, far less progress has been seen in the context of variational inequality problems, exceptions being the efforts to solve variational inequality problems with expectation-valued maps [1], [2]. Yet, in many instances, the goal lies in obtaining solutions that are robust to uncertainty. While the fields of robust optimization and control theory have made deep inroads into developing tractable schemes for resolving such concerns, there has been little progress in the context of variational problems. In what we believe is amongst the very first efforts to comprehensively address such problems in a distribution-free environment, we present an avenue for obtaining robust solutions to uncertain monotone affine complementarity problems defined over the nonnegative orthant. We begin with and mainly focus on showing that robust solutions to such problems can be tractably obtained through the solution of a single convex program. Importantly, we discuss how these results can be extended to account for uncertainty in the associated sets by generalizing the results to uncertain affine variational inequality problems defined over uncertain polyhedral sets.

I. INTRODUCTION

The fields of robust control [3] and optimization theory [4] have grown immensely over the last two decades in an effort and are guided by the desire to provide solutions robust to parametric uncertainty. To provide a context for our discussion, we begin by defining a convex optimization problem

$$\min_{x \in X} f(x; u), \quad (\text{UOpt}(u))$$

where $X \subseteq \mathbb{R}^n$, $u \in U$, $f : X \times U \to \mathbb{R}$ is a convex function in $x$ for every $u \in U$. The resulting collection of uncertain optimization problems is given by the following set:

$$\left\{ \min_{x \in X} f(x; u) \right\}_{u \in U}.$$

Given such a set of problems, one avenue for defining a robust solution to this collection of uncertain problems is given by the solution to the following worst case problem:

$$\min_{x \in X} \max_{u \in U} f(x; u). \quad (\text{ROpt})$$

Robust optimization has grown into an established field and there has been particular interest in deriving tractable robust counterparts to (ROpt); in particular, can one formulate a single convex optimization problem whose solution lies in the set of solutions of (ROpt). Such questions have been investigated in linear, quadratic, and in more general convex regimes [4], [5], while more recent efforts have considered integer programming problems [6].

A particularly important class of problems that includes convex optimization problems is that of variational inequality problems [7]. Recall that a variational inequality problem $VI(X, F)$ requires an $x \in X$ such that

$$(y - x)^T F(x) \geq 0, \quad \forall y \in X,$$  

where $F : X \to \mathbb{R}^n$. Moreover when $X$ is a cone, it is known [7] that $VI(X, F)$ is equivalent to the complementarity problem $CP(X, F)$, that requires an $x$ such that

$$x \geq 0, F(x) \in X^*$$

where $X^*$ denotes the dual cone defined as $X^* \triangleq \{ y : y^T x \geq 0, x \in X \}$.

The resulting collection of uncertain variational inequality problems is given by the following:

$$\{ VI(X, F(\bullet); u) \}_{u \in U} \quad (\text{UVI}(X, F, U))$$

In this paper, we focus on instances where $F(x; u) \triangleq M(u)x + q(u)$ where $M(u) \in \mathbb{R}^{n \times n}$, $q(u) \in \mathbb{R}^n$, and $X \subseteq \mathbb{R}^+$ and the resulting affine variational inequality problem is equivalent to the linear complementarity problem:

$$0 \leq x \perp M(u)x + q(u) \geq 0.$$  

Additionally, we consider generalizations to uncertain affine variational inequality problems defined over uncertain polyhedral sets and non-monotone problems.

Before proceeding, we briefly touch upon earlier efforts in addressing this class of problems. In particular, much

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of the prior work has focused on the minimization of the expected residual function [10]–[12]. It may be recalled that the residual function \( h \) for VI\((X, F)\) is a nonnegative function on a closed set \( D \supseteq X \) such that \( h(x) = 0 \) if and only if \( x \) solves VI\((X, F)\). Given such a random map \( F(x; \xi) \) where \( \xi : \Omega \rightarrow \mathbb{R}^d, F : X \times \mathbb{R}^d \rightarrow \mathbb{R}^n \), and an associated probability space \((\Omega, \mathcal{F}, \mathbb{P})\), the expected residual minimization problem (ERM) problem utilizes the associated residual function \( f(x; \xi) \) and is given by the following:

\[
\min_{x \in X} \mathbb{E}[f(x; \xi)] . \tag{ERM}
\]

Such avenues have derived such solutions for both monotone as well as more general stochastic variational inequality problems but are complicated by several challenges:

(i) First, such avenues necessitate the availability of a probability distribution \( \mathbb{P} \).

(ii) Second, the expected residual minimization problem, given by (ERM), leads to a possibly nonconvex stochastic optimization problem and much of the research has focused on providing estimators of local solutions to such problems.

(iii) Third, this approach focuses on minimizing the average or expected residual and may be less capable of providing solutions that minimize worst-case residuals unless one employs risk-based variants.

In the spirit of robust approaches employed for the resolution of a range of optimization and control-theoretic problems, we consider an avenue that requires an uncertainty set \( \mathcal{U} \). An alternate not considered here that of immense importance is the scenario-based approach [13], [14]. Furthermore, rather than minimizing the expected residual function, we consider the minimization of the worst-case residual over this uncertainty set. Specifically, we make the following contributions:

(a) First, in the context of stochastic linear complementarity problems with asymmetric positive semidefinite matrices, we show that the robust counterpart is a single convex optimization problem under varying assumptions on the associated uncertainty set. Surprisingly, direct applications of findings from robust quadratic optimization are not always possible and we consider the development of new uncertainty sets.

(b) Second, this result can be generalized to the regime of more general monotone complementarity problems as well as uncertain affine variational inequality problems over uncertain polyhedral sets. We further examine an approach for deriving bounds associated with more general uncertain variational inequality problems with Lipschitz continuous maps by solving merely convex quadratic programs and a set of tractable variational inequality problems.

The remainder of this paper is organized as follows. In Section II, we motivate our study through two applications and provide an instance of a monotone complementarity problem with arbitrarily high price of robustness. In Section III, we discuss the tractable resolution of uncertain monotone LCPs and address some generalizations in Section IV. We provide concluding remarks in Section V.

II. MOTIVATING EXAMPLES AND APPLICATIONS

In this section, we begin by providing an example of an uncertain linear complementarity problem and proceed to discuss two applications that motivate the study of uncertain variational inequality problems.

a) A class of uncertain LCPs: Consider the simple LCP, denoted by eLCP(u):

\[
0 \leq \begin{pmatrix} x \\ y \end{pmatrix} \perp \begin{pmatrix} M & 0 \\ 0 & S(\xi, \eta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -q_x \\ q(u) \end{pmatrix} \geq 0, \tag{2}
\]

where \( M = \left(I - \frac{1}{(n+1)}ee^T\right) \in \mathbb{R}^{n \times n}, S(\xi, \eta) = \xi S_1 + \eta S_2, x \in \mathbb{R}^n, y \in \mathbb{R}^n, q_x \in \mathbb{R}_+^n, q(u) = uw, u \in \mathcal{U}_u \subseteq [0, 1]. \) Furthermore, \( S_1 = nI + ee_ee^T \) and \( S_2 = ee^T + ee^T. \) \( e \) denotes the column of ones, \( e_n = (1, \ldots, n)^T \) and \( \mathcal{U}_\xi \triangleq \{ \xi = (\xi, \eta) : \xi + \eta \leq 1, \xi, \eta \geq 0 \}. \) We begin by noting that a solution to the upper system

\[
0 \leq x + Mx - q_x \geq 0
\]

is uniquely defined by \( x^* = (I + ee^T)q_x \geq 0. \) The lower system requires solving the following equation:

\[
0 \leq y + S(\xi, \eta)y + q(u) \geq 0.
\]

Since \( S(\xi, \eta) \geq 0 \) and \( q(u) \geq 0, \) it follows that \( v \equiv 0 \) is a solution for all \( u \in \mathcal{U}_u \) and \( (\xi, \eta) \in \mathcal{U}_\xi. \) However, if \( \xi = \eta = u = 0, \) then any nonnegative \( y \) is also a solution implying that there is a ray of solutions. Our focus lies in obtaining solutions that minimize the worst-case residual defined as follows:

\[
\max_{u \in \mathcal{U}_u, (\xi, \eta) \in \mathcal{U}_\xi} \max_{y \in \mathbb{R}^n} \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} M & 0 \\ 0 & S(\xi, \eta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -q_x \\ q(u) \end{pmatrix}
\]

We use this setting to distinguish between a non-robust solution and a robust solution. A non-robust solution. Suppose the realization of \( u, \xi, \eta \) is such that \( u = 0, \xi = 0, \eta = 0 \) and the resulting solution is given by \( (x^*, y_1) \) where \( y_1 \geq 0. \) Consequently, the worst-case residual is given by

\[
\max_{u \in \mathcal{U}_u, (\xi, \eta) \in \mathcal{U}_\xi} \max_{y \in \mathbb{R}^n} \left( y^T S(\xi, \eta)y_1 + uy^T e \right).
\]

which could be arbitrarily high since \( y_1 \) is any nonnegative vector. In effect, a non-robust solution chosen under a single realization can have large worst-case residual.

A robust solution. The robust solution of this problem is given by \( (x^*, 0) \) and achieves a zero worst-case residual.

b) Uncertain traffic equilibrium problems: A static traffic equilibrium model [7], [9] captures equilibrating (or steady-state) flows in a traffic network in which a collection of selfish users attempt to minimize travel costs. Here, we present a path-based formulation where \( \mathcal{N} \) denotes the network while \( A \) represents the associated set of edges. Further, let \( \mathcal{O} \) and \( \mathcal{D} \) denote the set of origin and destination nodes, respectively while the set of origin-destination (OD) pairs is given by \( \mathcal{W} \subseteq \mathcal{O} \times \mathcal{D}. \) Let \( \mathcal{P}_w \) denote the set of paths...
connecting each \( w \in W \) and \( P = (N_w \mid w \in W) \). Let \( h_p \) denote the flow on path \( p \in P \) while \( C_p(h; u) \), the associated (uncertain) travel cost on \( p \), is a function of the entire vector of flows \( h \equiv (h_p) \) and the uncertainty \( u \in U \). Let \( d_w(v; u) \) represent the uncertain travel demand between O-D pair \( w \) and is a function of \( v \equiv (v_y) \), the vector of minimum travel costs between any OD pair, and the uncertainty \( u \in U \). Based on Wardrop user equilibrium principle, users choose a minimum cost path between each O-D pair:

\[
0 \leq h_p \perp v_w - C_p(h) \geq 0, \quad \forall w \in W, p \in P_w. \tag{3}
\]

Additionally, the travel costs are related to demand satisfaction through this problem:

\[
0 \leq v_w \perp \sum_{p \in P_w} h_p - d_w(v) \geq 0, \quad \forall w \in W. \tag{4}
\]

The static traffic user equilibrium problem requires a pair \((h, u)\) satisfying (3) and (4), compactly stated as the following complementarity problem:

\[
0 \leq \frac{h}{v} \perp \left( \frac{C(h; u) - B^T v}{B h - d(v; u)} \right) \geq 0, \tag{5}
\]

where \( C(h; u) = (C_p(h; u) \mid p \in P) \), \( d(v; u) = (d_w(v; u) \mid w \in W) \) and \( B \) is the (OD pair, path)-incidence matrix \((b_{wp})\) defined as follows:

\[
b_{wp} \triangleq \begin{cases} 1 & \text{if } p \in P_w \\ 0 & \text{otherwise.} \end{cases}
\]

The collection of problems (5) for every \( u \in U \) represents an uncertain traffic equilibrium problem and we desire an equilibrium \((h, v)\) that is robust to uncertainty.

c) Uncertain Nash-Cournot games: Nash-Cournot models for competitive behavior find application in a variety of settings, including the context of networked power markets [15]. We describe an instance of a single node \( N \)-player Nash-Cournot game in which \( N \) players compete in the market for a single good. Suppose player \( i \)'s uncertain linear cost function given by \( c_i(u)x_i \) where \( x_i \) is her production level decision. Furthermore, the \( i \)th player’s capacity is denoted by \( c_i \). We assume that sales of the good are priced using an (uncertain) price function dependent on aggregate sales \( X \) and denoted by \( p(X; u) \) where \( u \in U \). We restrict our attention to settings where this price function is affine and defined as follows: \( p(X; u) \triangleq a(u) - b(u)X \) where \( a(u), b(u) > 0 \), \( X \triangleq \sum_{i=1}^N x_i \) and \( u \in U \). The \( i \)th agent’s problem is given by the following:

\[
\min_{x_i} \ (c_i(u)x_i - p(X; u)x_i) \quad \text{(Player } x_i \text{)}
\]

subject to

\[
\begin{align*}
x_i &\leq c_i, \\
x_i &\geq 0.
\end{align*}
\]

The sufficient equilibrium conditions of the Nash-Cournot game are given by the concatenation of the resulting optimality conditions:

\[
\begin{align*}
0 &\leq x_i \perp b(u)(X + x_i) + \lambda_i + c_i(u) - a(u) \geq 0, \quad \forall i, \\
0 &\leq \lambda_i \perp c_i - x_i \geq 0, \quad \forall i.
\end{align*}
\]

This can be more compactly stated as the following LCP:

\[
0 \leq z \perp M(u) + q(u) \geq 0, \tag{6}
\]

where

\[
M(u) \triangleq \begin{pmatrix} b(u)(I + ee^T) & 0 \\ -I & 0 \end{pmatrix}, \quad q(u) \triangleq \begin{pmatrix} e(u) - a(u)e \\ \text{cap} \end{pmatrix},
\]

\( e \) denotes the column of ones, \( \text{cap} \) is the column of capacities, and \( I \) represents the identity matrix. The collection of problems (6) for every \( u \in U \) represents an uncertain Nash-Cournot equilibrium problem and we desire an equilibrium decision \((x, \lambda)\) that is robust to uncertainty.

III. UNCERTAIN MONOTONE LCPs

In this section, we develop tractable robust counterparts for the uncertain linear complementarity problem under various assumptions on the uncertainty sets. Throughout this section, we assume that the set \( X \) is a cone and the residual function is defined by the gap function:

\[
f(x, u) = \theta_{\text{gap}}(x, u) \triangleq \sup_{y \in X} F(x, u)^T(x-y), \quad x \in D \supseteq X.
\]

Much of the efforts in the resolution of uncertain variational inequality problems has considered the minimization of the expected residual; Instead, we pursue a strategy that has defined the field of robust optimization in that we consider the minimization of the worst-case residual over a prescribed uncertainty set. In its original form, such a problem is relatively challenging nonsmooth semi-infinite optimization problem. Yet, it can be shown that these problems are shown to be equivalent to tractable convex programs.

Recall from [7], the gap function is defined as follows when \( X \) is a cone:

\[
\theta_{\text{gap}}(x, u) \triangleq \begin{cases} F(x, u)^T x & \text{if } F(x, u) \in X^*, \\
+\infty & \text{otherwise.}
\end{cases}
\]

It follows that (ROpt) can be recast as follows:

\[
\min_{u \in U} \max_{x \in D} F(x, u)^T x \quad \text{subject to} \quad F(x, u) \in X^*, \quad \forall u \in U, \quad x \in X. \tag{7}
\]

Before proceeding, it is worth noting that the robust formulation attempts to find a solution that minimizes the maximal (worst-case) value taken by \( F(x, u)^T x \) over the set of solutions that are feasible for every \( u \in U \). In fact, the following relationship holds between the optimization problem (7) and the original uncertain complementarity problem.

**Lemma 1:** Consider the problem given by (7). Then \( x \in X \) solves \( X \ni x \perp F(x, u) \in X^* \) for all \( u \in U \), if and only if \( x \) is a solution of (7) with optimal value zero.

Next we define \( X \equiv \mathbb{R}_+^n \) and \( F(x, u) \triangleq M(u)x + q(u) \), where \( M(u) \in \mathbb{R}^{n \times n}, q(u) \in \mathbb{R}^n \), and \( u \in U \). Then (7) may be rewritten as follows:

\[
\min_{x \geq 0} \max_{u \in U} x^T (M(u)x + q(u)) \quad \text{subject to} \quad \min_{u \in U} M_u(x) + q_i(u) \geq 0, \forall i. \tag{8}
\]

Unfortunately, (8) appears to be a relatively challenging problem to solve. However, under certain assumptions on
$U$, we may reformulate this problem as a tractable convex program and its global minimizer provides a robust solution to the uncertain LCP. The remainder of this section is broken up into two subsections. In the first subsection, we consider a setting when merely $q$ is uncertain and consider the more case in the second subsection.

A. Uncertainty in $q$

In this section, $q$ is subject to uncertainty while $M(u) \triangleq M$ for every $u \in U$. Then the problem can be reduced to obtaining a robust solution to an uncertain quadratic program (cf. [4]). Define $q(u)$ as $q(u) : = q_0 + \sum_{i=1}^L u_i q_i$, $u \in U$ and (8) can be reformulated as follows:

$$\min_{x \geq 0} \quad x^T(Mx + q_0) + \max_{u \in U} \sum_{i=1}^L u_i(x^T q_i)$$

subject to $\quad M_i x + q_0 + \min_{u \in U} \sum_{i=1}^L u_i(q_i) \geq 0, \forall i$. (9)

We begin by considering three types of uncertainty sets: $U_1, U_\infty$, or $U_2$, where

$$U_1 \triangleq \{ u : ||u||_1 \leq 1 \}, U_2 \triangleq \{ u : ||u||_2 \leq 1 \},$$

and $U_\infty \triangleq \{ u : ||u||_\infty \leq 1 \}$. (10)

Our first result is proved next and its proof is inspired by Examples 1.3.2 and 1.3.3 from [4].

Proposition 1: Consider the uncertain LCP given by (9) where $M$ is a positive semidefinite matrix. Let $U$ be defined as $U_1, U_2$ or $U_\infty$ (see (10)). Then (9) may be reformulated as a convex program.

Proof: The proof is done by leveraging Example 1.3.2 and 1.3.3 in [4].

Next, we present a more general case where the uncertainty set is captured by a more general convex set. Specifically, $U := U_c$, where

$$U_c \triangleq \{ u \in \mathbb{R}^L : \exists \nu \in \mathbb{R}^k : Pu + Q\nu + p \in K \subseteq \mathbb{R}^N \},$$

(11) $K$ is a closed convex pointed cone, $P, Q$ are given matrices, and $p$ is a given vector.

Theorem 1 (Tractability when $U := U_c$): Consider the uncertain LCP given by (9) where $M$ is a positive semidefinite matrix. Let $U := U_c$, where $U_c$ is defined as (11). Suppose one of the following holds:

(i) $K$ is a polyhedral cone;

(ii) $K$ is a convex cone and the following holds:

$$\exists (\bar{u}, \bar{\nu}) \text{ such that } P\bar{u} + Q\bar{\nu} + p \in \text{int}(K).$$

(12)

Then the robust counterpart of (9) is a tractable convex program.

Proof: The proof is omitted and follows from Theorem 1.3.4 and Proposition 6.2.1 from [4].

Remark: If $K$ is the nonnegative orthant, the perturbation set is a polyhedron given $Q = 0$. Box uncertainty (using the $|| \cdot ||_1$ norm) and diamond uncertainty (using the $|| \cdot ||_\infty$ norm) are both included in this general case. If $K$ is a second-order cone, one special case of the perturbation set is a ball (using the $|| \cdot ||_2$). Under both circumstances, the problem is tractable. Notice that nonnegative orthants and Lorentz cones are self-dual. When $K$ is $\mathbb{R}^N_+$, the above problem is a convex quadratic program (QP) while if $K$ is $L^N$, the above problem is a convex quadratically constrained quadratic program (QCQP).

B. Uncertainty in $M$ and $q$

Next, we consider the setting where $M$ is uncertain. In this setting, there are far more challenges and a direct application of the results from robust quadratic programming is not possible. We consider two specific avenues distinguished by the choice of the uncertainty sets assumed. We assume that $q$ is deterministic and reformulate (8) as follows:

$$\min_{x \geq 0} \quad t$$

subject to $\quad x^T(M(u)x + q) \leq t, \forall u \in U,$

$$M(u)x + q \geq 0, \forall u \in U.$$ (13)

We assume that $M$ takes on the following form:

$$M \triangleq M_0 + \sum_{k=1}^K u_k M_k, \quad M_0 \geq 0, \quad M_k \geq 0, \quad \forall k.$$ (14)

Note that such uncertainty sets do not appear to have been studied to the best of our knowledge.

Proposition 2 (Tractability for uncertain $M$): Consider the problem (13). Then this problem has a tractable robust counterpart when $M$ is defined by (14) and $U$ is defined as either $\{ u : ||u||_\infty \leq 1, u \geq 0 \}$ or $\{ u : ||u||_1 \leq 1, u \geq 0 \}$.

If the uncertainty set that prescribes $M(u)$ is unrelated to that producing $q(u)$, then we may address each term individually, as in the prior subsections.

Next, we extend the realm of applicability of the tractability result to accommodate perturbation sets that are more general than (14) but necessitate symmetric matrices. Specifically, we employ an uncertainty set that relies on computing the Cholesky factorization of $M$, defined next as adopted in [5]:

$$U_A \triangleq \{ (M, q) \mid M = A^T A,$$

$$A = \sum_{i=1}^L \xi_i A_i + A_0, q = q_0 + \sum_{i=1}^L \xi_i q_i, ||\xi||_2 \leq 1 \}.$$ (15)

Consequently, $q \in U_1$ and $M \in U_2$ can be combined to allow for a single uncertainty set $(M, q) \in U_A$ and (8) may be recast as follows:

$$\min_{x \geq 0} \quad t$$

subject to $\quad x^T(Mx + q) \leq t, \forall (M, q) \in U_A,$

$$Mx + q \geq 0, \forall (M, q) \in U_A.$$ (16)

If $z = (x; t)$, the first constraint is equivalent to:

$$z^T M'z + z^T q' \leq 0, \forall (M', q') \in U'_A,$$ (17)

$$U_A' \triangleq \{ (M', q') \mid M' \triangleq \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} ,$$

$$q' \triangleq \begin{pmatrix} q \\ -1 \end{pmatrix} , (M, q) \in U_A \}.$$ (18)
Lemma 2: [5, Theorem 2.3] Consider the constraint (17). Then the tractable counterpart of this constraint is given by:

$$\begin{pmatrix}
-q_0^T x + t - \tau & -q_1^T x & \cdots & -q_L^T x \\
-q_1^T x & \tau & \cdots & \\
\vdots & \ddots & \ddots & \\
-q_L^T x & \cdots & \tau & (A_0 x)^T \\
(A_0 x)^T & (A_1 x)^T & \cdots & (A_L x)^T
\end{pmatrix} \geq 0.
$$

However, it is somewhat more challenging constructing a robust counterpart of the second constraint.

$$(Mx + q) \geq 0, \quad \forall (M, q) \in \mathcal{U}_A. \quad (19)$$

In fact, this is the key departure from the result provided in [5]. We proceed to show that (19) can be managed in a tractable fashion. We begin by rewriting $\mathcal{U}_A$ in terms of $A_0, q_0$ and $A_l, q_l, l = 1, \ldots, L$ for purposes of clarity:

$$\mathcal{U}_A \triangleq \{(M, q) \mid M = A_0^T A_0 + \sum_{l=1}^L (A_l^T A_0 + A_0^T A_l) \xi_l + \sum_{l<m} (A_l^T A_m + A_m^T A_l) \xi_l \xi_m + \sum_{l=1}^L A_l^T A_l \xi_l^2, \quad q = q_0 + \sum_{l=1}^L q_l \xi_l, \quad \|\xi\|_2 \leq 1\}.$$  

(20)

Following [4], we obtain the result below:

Theorem 2: Consider the problem (19). This problem has a tractable robust counterpart.

Proof: First notice that obtaining a feasible solution of (19) requires solving the $i$th optimization problem where $i = 1, \ldots, L$:

$$\min \sum_{l=1}^L \left( [A_l^T A_0 + A_0^T A_l] \xi_l + \sum_{l<m} (A_l^T A_m + A_m^T A_l) \xi_l \xi_m + \sum_{l=1}^L A_l^T A_l \xi_l^2 \right), \quad \|\xi\|_2 \leq 1.$$

(21)

We may rewrite this problem as follows:

$$\min \left\{ b_i(x)^T \xi + \xi^T C_i(x) \xi : \|\xi\|_2 \leq 1 \right\}, \quad (22)$$

where $b_i(x) \in \mathbb{R}^L, C_i(x) \in \mathbb{R}^{L \times L}$ are all linear functions of $x$. By defining $\hat{\xi}, M_i(x)$ and $Z$ as

$$\hat{\xi} \triangleq \begin{pmatrix} \xi \\ \xi^T \end{pmatrix}, \quad M_i(x) \triangleq \left[ \frac{1}{2} b_i(x)^T C_i(x) \right],$$

$$Z \triangleq \left\{ \begin{pmatrix} \xi \\ \xi^T \end{pmatrix} : \|\xi\|_2 \leq 1 \right\},$$

the problem (22) is equivalent to:

$$\min \left\{ \langle \hat{\xi}, M_i(x) \rangle : \hat{\xi} \in Z \right\}. \quad (24)$$

where $A B = Tr(A^T B)$. Since the objective function is linear, we may extend the feasible region $Z$ to its convex hull $\hat{Z} = \text{conv} \{ \hat{Z} \}$. By Lemma 14.3.7 from [4], we have

$$\hat{Z} = \left\{ \begin{pmatrix} \hat{\xi} \\ w \end{pmatrix} \in \mathbb{S}^{L+1} \mid \begin{pmatrix} 1 & w^T \\ w & 0 \end{pmatrix} \succeq 0, Tr(W) \leq 1 \right\}. \quad (25)$$

Using variable replacement, (24) is equivalent to semidefinite program:

$$\min_{x \in \mathbb{S}_+^{L+1}} \langle X, M_i(x) \rangle$$

subject to $\langle X, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \rangle \leq 1$, $\langle X, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle = 1$, $\langle X, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle \leq M_i(x), y_{i,1} \leq 0$. \hfill (26)

Remark: When $\|u\|_\infty \leq 1$ or $\|u\|_1 \leq 1$ in (17), Theorem 2 does not hold. What we may do is enlarge the uncertainty set to get a tractable robust counterpart. In the case of $\|u\|_\infty \leq 1$, [4, Lemma 14.3.9] provides a semidefinite representable set containing $\hat{Z}$. On the other hand, we may enlarge $\|u\|_\infty \leq 1$ or $\|u\|_1 \leq 1$ to their circumscribed spheres to get tractable robust counterparts of (19).

IV. GENERALIZATIONS

In this section, we consider several generalizations to the uncertain monotone linear complementarity problem.

A. Uncertain affine monotone polyhedral VIs

Two shortcomings immediately come to the fore when considering the model (8). First, the set $X$ is a cone and second, the set is deterministic in that it is uncorrupted by uncertainty. In this subsection, we show that examining uncertain polyhedral sets can also be managed within the same framework. Specifically, consider an uncertain affine variational inequality problem over a polyhedral set of the form given by (1) wherein $X(u)$ and $F(x, u)$ are defined as

$$X(u) \triangleq \{ x : A(u)x \geq b(u), x \geq 0 \}, \quad (27)$$

$$F(x, u) \triangleq M(u)x + q(u), \quad (27)$$

where $A(u) = A_0 + \sum_{l=1}^L u_l A_l, b(u) = b_0 + \sum_{l=1}^L u_l b_l, \|u\|_2 \leq 1$. From [7], it follows that $x$ solves VI($X(u), F(\bullet, u)$) if and only if $(x, \lambda)$ solves the following complementarity problem:

$$0 \leq x \perp M(u)x - A(u)^T \lambda + q(u) \geq 0$$

$$0 \leq \lambda \perp A(u)x - b(u) \geq 0. \quad (28)$$
This can be more compactly stated as the following monotone linear complementarity problem:
\[
0 \leq z^\top B(u)z + d(u) \geq 0,
\]
where \( B(u) \equiv \begin{pmatrix} M(u) & -A(u) \quad 0 \\ A(u) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) and \( d(u) \equiv \begin{pmatrix} q(u) \\ -b(u) \end{pmatrix} \).

It is relatively easy to see that \( B(u) \) is a positive semidefinite matrix since \( z^\top B(u)z = \frac{1}{2} x^\top (M(u) + M(u)^\top) x \geq 0 \) if \( M(u) \) is a positive semidefinite matrix. Given such a complementarity problem, two possible approaches may be adopted. When \( b(u) \) is a deterministic vector and under the assumption that \( B(u) \) abides by the uncertainty set specified by (1-4), then a tractable robust counterpart of the robust VI is available by using Prop. 2. We also believe that an alternate approach that requires a symmetric \( M(u) \) may be possible through an extension of Theorem 2 to address this setting. This is left as the subject of future work.

B. General uncertain VIs

Next, we consider a general uncertain VI, as specified by (1). In such a setting, we consider the computation of a robust solution when the map \( F(x, u) \) is a \( P \)-map for every \( u \in \mathcal{U} \) (see [7]). In this subsection, we consider regimes where \( \|u\| < \infty \). Rather than the gap function, we consider the residual associated with the natural map of VI \((X, F(\cdot, u))\). Recall that \( F_X^{nat}(x) \) is defined as \( F_X^{nat}(x) \equiv x - \Pi_X(x - F(x)) \). Furthermore, we have that \( F_X^{nat}(x) = 0 \) if and only if \( x \) solves VI \((X, F)\). In effect, \( x \) is a solution of VI \((X, F)\) if and only if \( x \) is a minimizer of \( \min_{x \in X} \|F_X^{nat}(x, u)\|^2 \).

We consider a solution to the following problem:
\[
\min_{x \in X} \max_{u \in \mathcal{U}} \|F_X^{nat}(x, u)\|. \tag{30}
\]

**Lemma 3:** \( X \) is a convex set. Consider the residual function \( \|F_X^{nat}(x, u)\|^2 \). If \( x(u) \) denotes the solution of VI \((X, F(\cdot, u))\) and \( L(u) \) is the Lipschitz constant associated with \( F(\cdot, u) \), then \( \|F_X^{nat}(x, u)\| \leq \|2 + L(u)\|\|x - x(u)\| \).

Consequently, we may now consider the following problem:
\[
\min_{x \in X} \max_{u \in \mathcal{U}} (2 + L(u))\|x - x(u)\|, \tag{31}
\]
where \( x(u) = \text{SOL}(X, F(\cdot, u)) \). This can be viewed as a center-location problem (a class of facility location problems) which can be reformulated as a convex quadratic program when \( X \) is a polyhedral set.

**Proposition 3:** Consider the problem given by (31). This problem can be reformulated as a convex quadratic program.

**Remark:** In effect, the computation of a bound requires solving \( K \) tractable variational inequality problems and 1 convex quadratic program where \( \|u\| = K \). Note that complementarity problems with \( P \)-maps may be tractably solved.

V. CONCLUDING REMARKS

In this paper, we consider the resolution of finite-dimensional monotone complementarity problems corrupted by uncertainty. Much of the available avenues rely on the availability of a probability distribution and the solution of stochastic nonconvex programs. Instead, we consider an avenue that relies on the availability of an uncertainty set. By leveraging findings from robust convex programming, we show that uncertain monotone linear complementarity problems can be tractably resolved as a single convex program. Note that when both the matrix \( M(u) \) and \( q(u) \) are uncertain and related, such avenues necessitate defining both new uncertainty sets and enlarging existing sets in an effort to derive robust counterparts. Notably, such statements can be seen to accommodate uncertain affine variational inequality problems over uncertain polyhedral sets. When the maps lose monotonicity, we show that in some instances, bounds may be available by solving a set of tractable problems.

**References**


