How to construct left-continuous triangular norms—state of the art\textsuperscript{\r}

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Abstract

Left-continuity of triangular norms is the characteristic property to make it a residuated lattice. Nowadays residuated lattices are subjects of intense investigation in the fields of universal algebra and nonclassical logic. The recently known construction methods resulting in left-continuous triangular norms are surveyed in this paper.

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1. Introduction

Triangular norms (t-norms) play a crucial role in several fields of mathematics and artificial intelligence. For an exhaustive overview on t-norms we refer to [24]. Recently an increasing interest of left-continuous t-norm-based theories can be observed (see e.g. [3,6–10,22]). The condition of left-continuity is a frequently cited property and plays a central role in all the fields that use t-norms. The role of left-continuous t-norms with strong associated negations is even more relevant, since then the negation, which is associated to the t-norm is an involution, and hence one can define a t-conorm via the de Morgan rule. In spite of their significance, the knowledge about left-continuous t-norms was rather poor for a long time; there were no results in the literature where left-continuous t-norms stood as the focus of interest. Moreover, until 1995 there were no known examples for left-continuous t-norms, except for the standard class of continuous t-norms. Continuous t-norms have become well understood from the famous and widely cited paper of Ling, as ordinal sums of continuous Archimedean t-norms [26] and have been used in several applications. The poor
knowledge about left-continuous t-norms on one hand and the good understanding of continuous t-norms on the other hand result in the use of continuous t-norms when left-continuity would be sufficient in theory. This very much restricts the freedom of choice when the proper operation has to be found in the mathematical setting in question. In other words, this makes modeling, e.g., in probabilistic metric spaces, in game theory, in the theory of non-additive measures and integrals, in the theory of measure-free conditioning, in fuzzy set theory, in fuzzy logic, in fuzzy control, in preference modeling and decision analysis, and in artificial intelligence much less flexible.

In this paper we discuss in detail the presently existing construction methods which result in left-continuous triangular norms. The methods are (together with their sources):

- annihilation [2,4,15],
- ordinal sum of t-subnorms [12,14],
- rotation construction [11,16,17],
- rotation–annihilation construction [16,18],
- embedding method [9,21].

An infinite number of left-continuous t-norms can be generated with these constructions (and with their combinations), which provides a tremendously wide spectrum of choice for e.g. logical and set theoretical connectives in non-classical logic and in fuzzy theory.

2. Preliminaries

A t-norm is a binary operation $T$ (that is, a function $T : [0, 1]^2 \to [0, 1]$) such that for all $x, y, z \in [0, 1]$ the following four axioms (T1)—(T4) are satisfied:

1. **Symmetry** $T(x, y) = T(y, x),$
2. **Associativity** $T(x, T(y, z)) = T(T(x, y), z),$
3. **Monotonicity** $T(x, y) \leq T(x, z)$ whenever $y \leq z,$
4. **Boundary condition** $T(x, 1) = x,$
5. **Boundary condition** $T(x, 0) = 0,$
6. **Range condition** $T(x, y) \leq \min(x, y).$

As we will see later, an essential role is played by t-subnorms in the construction of left-continuous t-norms:

**Definition 1** (Jenei [18]). A triangular subnorm (t-subnorm) is a function $T : [0, 1]^2 \to [0, 1]$ such that for all $x, y, z \in [0, 1]$ axioms (T1), (T2), (T3) and (T4$''$) are satisfied.

Any t-norm is a t-subnorm. We say that a t-subnorm $T$ has zero divisors if there are $x, y \in ]0, 1]$ such that $T(x, y) = 0$. A t-subnorm is said to be continuous resp. left-continuous if it is continuous resp. left-continuous as a two-place function.

One can define t-(sub)norms on any $[a, b] \subset \mathbb{R}$ ($a < b$) and get the notion of a t-subnorm on $[a, b]$. Then for any t-(sub)norm $T$, the function $T_{[a,b]} : [a, b] \times [a, b] \to [a, b]$ defined by $T_{[a,b]}(a, b) = a + (b - a) T((x - a)/(b - a), (y - a)/(b - a))$ is a t-(sub)norm on $[a, b]$. If $T_{[a,b]}$ is a t-(sub)norm on
Fig. 1. Minimum $T_M$ (left), product $T_P$ (center) and Lukasiewicz t-norms $T_L$ (right).

$[a, b]$ then the function $T : [0, 1] \times [0, 1] \to [0, 1]$ defined by

$$T(a, b) = \frac{T_{[a,b]}(a + x(b - a), a + y(b - a)) - a}{b - a}$$

is a t-(sub)norm. Call $T_{[a,b]}$ the linear transformation of $T$ into $[a,b]$. Similarly, call $T$ the linear transformation of $T_{[a,b]}$ into $[0,1]$.

**Example 1.** Non-trivial examples of t-subnorms are e.g. $T_P(x,y) = x \land y$, $T_L(x,y) = \max(0,x + y - 1 - \varepsilon)$, when $\varepsilon$ is a fixed real from $[0,1]$ (see Fig. 3) with the exception of $T_{L_0}$ and $T_P$, which are just $T_L$, the Lukasiewicz t-norm and the product $T_P$, t-norm, respectively. Fig. 1 presents the three basic continuous t-norms.

**Example 2.** We remark that the construction in [13, Theorem 2] (see as well [24], and Proposition 11 in [23]) produces t-subnorms if the boundary of the resulting t-norm is not redefined (in the formula which can be found in the cited reference). Moreover, if one starts with a left-continuous t-(sub)norm, then the just mentioned construction (again without the separate definition on the boundary) produces a left-continuous t-subnorm.

A negation [29] $N$ is a non-increasing function on $[0,1]$ with boundary conditions $N(0) = 1$ and $N(1) = 0$. A negation is called strong if $N$ is an involution, that is, if in addition $N(N(x)) = x$ holds for all $x \in [0,1]$. A negation is strong if and only if its graph is invariant w.r.t. the reflection at the median (given by $y = x$). A strong negation is automatically a strictly decreasing, continuous function and hence it has exactly one fixed point.

Let $T : [0, 1]^2 \to [0, 1]$ be a function satisfying (T1) and (T3). The implication $I_T : [0, 1]^2 \to [0, 1]$ generated by $T$ is given by $I_T(x,y) = \sup\{t \in [0,1] \mid T(x,t) \leq y\}$. If $T$ is left-continuous then $I_T$ is called the residual implication generated by $T$.

For a left-continuous t-subnorm $T : [0, 1]^2 \to [0, 1]$ define $N_T(x) = I_T(x,0)$ for $x \in [0,1]$. If $N_T$ is a negation (this holds e.g. if $T$ satisfies (T4); that is $N_T$ is always a negation if $T$ is a t-norm) then $N_T$ is called the associated negation of $T$. 
Let $T : [0, 1]^2 \to [0, 1]$ be a function satisfying (T3), and let $N$ be a strong negation. We say that $T$ admits the \textit{rotation invariance property} \cite{19} with respect to $N$ or \textit{rotation invariant} w.r.t. $N$ if for all $x, y, z \in [0, 1]$ we have $T(x,y) \leq z \iff T(y,N(z)) \leq N(x)$.

3. Annihilation

The nilpotent minimum t-norm $T_{\text{M}_0}$ is introduced in \cite{4} in such a way that the values of the minimum t-norm are replaced by 0 under the negation $1-x$. More formally, for $x, y \in [0, 1]$ let

$$T_{\text{M}_0}(x, y) = \begin{cases} 0 & \text{if } y \leq 1-x, \\ \min(x, y) & \text{otherwise}. \end{cases} \tag{1}$$

For a visualization, see Fig. 2, and compare with the picture of $T_{\text{M}}$. It is observed that the same construction works for any strong negation instead of the standard one $1-x$, and that the construction does not result in a t-norm (in fact, the associativity property is violated) if the minimum t-norm is replaced by the product t-norm.

Motivated by this observation the concept of $N$-annihilation ($N$ being any strong negation) is investigated in \cite{15} and a characterization of those continuous t-norms where the annihilated operator is a t-norm is given as follows:

Let $T$ be a t-norm and $N$ be a strong negation. Define the binary operation $T_{(N)}$ (called the $N$-annihilation of $T$) as follows:

$$T_{(N)} : [0, 1] \times [0, 1] \to [0, 1];$$

$$T_{(N)}(x, y) = \begin{cases} 0 & \text{if } x \leq N(y), \\ T(x, y) & \text{otherwise}. \end{cases} \tag{2}$$

\textbf{Fig. 2.} The nilpotent minimum $T_{\text{M}_0}$ (left), a continuous t-norm (center) and its annihilation $T_J$ which is defined in (3) (right).
Theorem 1. For any strong negation $N$ and continuous t-norm $T$, $T_{(N)}$ is a t-norm if and only if $T_{(N)}$ is isomorphic either to $T_{M_b}$ or to $T_L$ or to

$$T_J(x, y) = \begin{cases} 
0 & \text{if } x \leq 1 - y, \\
\frac{1}{3} + x + y - 1 & \text{if } x, y \in \left[\frac{1}{3}, \frac{2}{3}\right] \text{ and } x > 1 - y,
\end{cases} \text{ otherwise. (3)}$$

In this way a new family of left-continuous t-norms with the additional property of strongness of their associated negation is introduced. For a visualization, see Fig. 2 (right). Let me offer a more "philosophical" reformulation of the results of [15], which is based on the idea of "level curves": For any $c \in [0, 1]$ call the one-place function $f_c(x) := I_T(x, c), x \in [c, 1]$ the $c$-level curve of the continuous t-norm $T$. Because of the continuity of $T$ we can infer $T(x, y) = c$ if we have $f_c(x) = y$, this explains the name "level curve". Therefore, the $c$-level curve is a part (in fact it is the "upper border") of the $c$-level set $\{(x, y) \in [0, 1]^2 \mid T(x, y) = c\}$. Further, we say that the negation $N$ "cuts" a $c$-level curve, if there exist $x, y \in [0, 1]$ such that $f_c(x) < N(x)$ and $f_c(y) > N(y)$. In other words, the graph of the $c$-level curve is not entirely in the upper closed subdomain of $[0, 1]$ which is determined by the graph of $N$.

By using this terminology, we can reformulate the results of [15] as follows: If any of the $c$-level curves is cut by $N$, then in general $T$ loses its associativity via $N$-annihilation (that is, $T_{(N)}$ is not associative). The only exception is if in the "remaining part" of the $c$-level curve $T$ coincides with the minimum t-norm. More formally, if a $c$-level curve is cut by $N$ then we should have $T(x, y) = \min(x, y)$ whenever $f_c(x) = y$ and $x > N(y)$.

In [2] the authors generalize the results of [15] by considering any left-continuous t-norm $T$ and any negation $N$ (i.e., not necessarily strong ones). They characterize those left-continuous t-norms $T$, for which $T_{(N)}$ is a t-norm. They provide a more detailed description for the case when $T$ is a continuous t-norm: Instead of quoting this rather formal result we remark that its equivalent reformulation is just the above-presented "philosophical" description (written in italic).

Another generalization of the nilpotent minimum t-norm can be found in [24] (see Propositions 3.63 and 3.64): Let $A$ be a subset of $[0, 1]^2$ which is symmetric with respect to the line $y = x$, for which $A \cup \{(0) \times [0, 1]\} \cup \{(0, 1) \times \{0\}\}$ is closed, and which has the following property: if $(x, y) \in A$ then $[0, x] \times [0, y] \subseteq A$. Replace the values of the minimum t-norm in $(x, y)$ by 0 when $(x, y) \in A$. Then the obtained operation is a left-continuous t-norm. By taking into account the above-mentioned "philosophical" description it is easy to see that this is a very particular case of the general result in [2].

4. Ordinal sums of t-subnorms

Ordinal sum theorems have been adapted from the field of algebra into the field of t-norms by many authors (in the below-described form first in [5]) and has been widely spread in the literature with the following formulation under the name ordinal sum theorem for t-norms.

Theorem 2. Suppose that $\{[a_i, b_i]\}_{i \in K} (a_i < b_i)$ is a countable family of non-overlapping, closed subintervals of $[0, 1]$, denoted by $\mathcal{I}$. With each $[a_i, b_i] \in \mathcal{I}$ associate a t-norm $T_i$. Let $T$ be
function defined on \([0,1]^2\) by

\[
T(x, y) = \begin{cases} 
    a_m + (b_m - a_m)T_m \left( \frac{x - a_m}{b_m - a_m}, \frac{y - a_m}{b_m - a_m} \right) & \text{if } (x, y) \in [a_m, b_m]^2, \\
    \min(x, y) & \text{otherwise}.
\end{cases}
\]

Then \(T\) is a t-norm and called ordinal sum of \(\{([a_i, b_i], T_i)\}_{i \in K}\) and each \(T_i\) is called a summand.

It follows from the Mostert-Shields theorem [27] that any continuous t-norm is an ordinal sum of continuous Archimedean t-norms. Continuous Archimedean t-norms are isomorphic to either the product t-norm, or to the Łukasiewicz t-norm. See Fig. 2 (center) for an example: The continuous t-norm \(T\) is defined by a product summand on \([0,0.05,0.3]\) and a Łukasiewicz summand on \([\frac{1}{3}, \frac{2}{3}]\), that is, \(T = \{([0.05, 0.3], T_P), ([\frac{1}{3}, \frac{2}{3}], T_L)\}\).

It is observed in [14] that the well-known ordinal sum theorem of t-norms can be generalized by using t-subnorms as summands. It is not difficult to see, that any further generalization, which still results in t-norms or t-subnorms is impossible, as it was pointed out in the survey paper [25].

**Theorem 3** (Ordinal sum theorem for t-subnorms). Suppose that \(\{[a_i, b_i]\}_{i \in K}\) \((a_i < b_i)\) is a countable family of non-overlapping, closed subintervals of \([0,1]\), denoted by \(\mathcal{I}\). With each \([a_i, b_i] \in \mathcal{I}\) associate a t-subnorm \(T_i\) where for each \([a_i, b_i], [a_j, b_j] \in \mathcal{I}\) with \(b_i = a_j\) and with zero divisors in \(T_j\) we have that \(T_i\) is a t-norm. Let \(T\) be a function defined on \([0,1]^2\) by

\[
T(x, y) = \begin{cases} 
    a_m + (b_m - a_m)T_m \left( \frac{x - a_m}{b_m - a_m}, \frac{y - a_m}{b_m - a_m} \right) & \text{if } (x, y) \in [a_m, b_m]^2, \\
    \min(x, y) & \text{otherwise}.
\end{cases}
\]

Then \(T\) is a t-subnorm and called the ordinal sum of \(\{([a_i, b_i], T_i)\}_{i \in K}\) and each \(T_i\) is called a summand.

**Theorem 4** (Generalized ordinal sum theorem for t-norms). Suppose that \(\{[a_i, b_i]\}_{i \in K}\) \((a_i < b_i)\) is a countable family of non-overlapping, closed subintervals of \([0,1]\), denoted by \(\mathcal{I}\). With each \([a_i, b_i] \in \mathcal{I}\) associate a t-subnorm \(T_i\) where for each \([a_i, b_i], [a_j, b_j] \in \mathcal{I}\) with \(b_i = a_j\) and with zero divisors in \(T_j\) we have that \(T_i\) is a t-norm, and if \([a_i, 1] \in \mathcal{I}\) then we have that \(T_i\) is a t-norm. Let \(T\) be a function defined on \([0,1]^2\) by (5). Then \(T\) is a t-norm.

It is not difficult to see that \(T\) defined by (5) is left-continuous if and only if all \(T_i\)’s are left-continuous.

**Example 3.** Let \(T\) be a function defined on \([0,1]^2\) by (5) based on \(\{([0,0.5], T_{P_{0.5}}), ([0.5,1], T_L)\}\). Then \(T\) is not associative, hence it is neither a t-norm nor a t-subnorm. The ordinal sum \(\{([0,0.5], T_L), ([0.5,1], T_{P_{0.5}})\}\) is not a t-norm only a t-subnorm since the “last” summand \(T_{P_{0.5}}\) is not a t-norm.

The ordinal sum \(\{([0.2,0.5], T_P), ([0.5,0.8], T_{L_{0.8}})\}\) is a t-norm. Indeed, since \(T_{L_{0.8}}\) has zero divisors
the only thing we need to verify that the summand which is just below it (that is, $T_P$) is a t-norm. Fig. 3 (right) visualizes the latest two ordinal sums.

5. Rotation

The rotation method is introduced in [17] and a characterization theorem is given in [11]. As in the ordinal sum theorem for t-subnorms, we remark, that it is not possible to provide any further generalization of the method (which still produces t-norms or t-subnorms). The method produces left-continuous (but not continuous) t-norms which have strong associated negations from any left-continuous t-norm $T_1$ which either has no zero divisors or all the zero values of its graph are in a sub-square of the unit square (see Fig. 4). The construction of t-subnorms is as well possible, see Remark 1.

**Theorem 5.** Let $N$ be a strong negation, $t$ its unique fixed point and $T$ be a left-continuous t-norm. Let $T_1$ be the linear transformation of $T$ into $[t, 1]$, $I^+ = [t, 1]$ and $I^- = [0, t]$. Define $T_{Rot}$ and $I_{Rot}$ (of types $[0, 1] \times [0, 1] \rightarrow [0, 1]$) by

$$T_{Rot}(x, y) = \begin{cases} 
T_1(x, y) & \text{if } x, y \in I^+, \\
N(I_{T_1}(x, N(y))) & \text{if } (x, y) \in I^+ \times I^-,
N(I_{T_1}(y, N(x))) & \text{if } (x, y) \in I^- \times I^+,
0 & \text{if } x, y \in I^-.
\end{cases} \quad (6)$$

$$I_{Rot}(x, y) = \begin{cases} 
I_{T_1}(x, y) & \text{if } x, y \in I^+, \\
N(T_1(x, N(y))) & \text{if } (x, y) \in I^+ \times I^-,
1 & \text{if } (x, y) \in I^- \times I^+,
I_{T_1}(N(y), N(x)) & \text{if } x, y \in I^-.
\end{cases} \quad (7)$$

$T_{Rot}$ is a left-continuous t-norm if and only if either

C1. $T$ has no zero divisors or
C2. there exists $c \in [0, 1]$ such that for any zero divisor $x$ of $T$ we have $I_T(x, 0) = c$. 

Fig. 3. $T_{P_{0.5}}$ and $T_{L_{0.4}}$ (left). A t-subnorm and a t-norm, which are ordinal sums of t-subnorms (right). See Examples 1 and 3.
Remark 1. If $T$ is a left-continuous t-norm in Theorem 5 then $T_{\text{Rot}}$ is a left-continuous t-norm, which is rotation invariant w.r.t. $N$ and the residual implication generated by $T_{\text{Rot}}$ is given by (7).

Remark 2. Recently, it is observed that even left-continuous uninorms can be rotated. Then the resulting operation is a left-continuous uninorm. By taking its underlying t-norm, this procedure results in another method for constructing left-continuous t-norms. See [20] for the details.

Example 4. In Fig. 5 the rotation of the minimum t-norm (the rotated units are highlighted) and the rotation of the product t-norm can be seen on the left. Observe that the nilpotent minimum t-norm is nothing else but the rotation of the minimum t-norm. On the right, the first t-norm is $\{([0,0.4],0), ([0.2,0.9],T_P)\}$ (0 stands for the t-subnorm with 0 values only), the second t-norm is its rotation. In Fig. 6 the rotations of ordinal sums can be seen. On the left-hand side the ordinal sum has two summands: the Łukasiewicz t-norm and the product t-norm. On the right-hand side the summand is the rotation of the product.
6. Rotation–annihilation

The rotation–annihilation method was introduced in [18]. It produces left-continuous (but not continuous) t-norms which have strong associated negations from a pair of connectives, as it is given in the following definition. Again, we remark, that it is not possible to provide any further generalization of the method (which still produces t-norms or t-subnorms).

**Definition 2** (Jenei [15]). Let $N$ be a strong negation and $t$ be its unique fixed point. Let $d \in ]t, 1]$. Then $N_d : [0, 1] \rightarrow [0, 1]$ defined by

$$N_d(x) = \frac{N(x \cdot (d - N(d))) + N(d) - N(d)}{d - N(d)}$$

is a strong negation. Call $N_d$ the zoomed $d$-negation of $N$.

**Definition 3.** Let $N$ be a strong negation, $t$ its unique fixed point, $\in ]t, 1]$ and $N_d$ be the zoomed $d$-negation of $N$. Let $T_1$ be a left-continuous $t$-subnorm.

i. If $T_1$ has no zero divisors then let $T_2$ be a left-continuous $t$-subnorm which admits the rotation invariance property w.r.t. $N_d$. Further, let $I^- = [0, N(d)], I^0 = [N(d), d]$ and $I^+ = ]d, 1]$.

ii. If $T_1$ has zero divisors then let $T_2$ be a left-continuous $t$-norm which admits the rotation invariance property w.r.t. $N_d$ (it is equivalent to saying that $T_2$ is a left-continuous $t$-norm with strong associated negation equal with $N_d$, see [19]). Further, let $I^- = [0, N(d)], I^0 = ]N(d), d[ \text{ and } I^+ = ]d, 1]$.

Let $T_3$ be the linear transformation of $T_1$ into $[d, 1]$, $T_4$ be the linear transformation of $T_2$ into $[N(d), d]$ and $T_5 : [N(d), d]^2 \rightarrow [N(d), d]$ be the annihilation of $T_4$ given by

$$T_5(x, y) = \begin{cases} 
0 & \text{if } x, y \in [N(d), d] \text{ and } x \leq N(y), \\
T_4(x, y) & \text{if } x, y \in [N(d), d] \text{ and } x > N(y).
\end{cases}$$
Define \( T_{RA} : [0, 1] \times [0, 1] \rightarrow [0, 1] \) by

\[
T_{RA}(x, y) = \begin{cases} 
T_3(x, y) & \text{if } x, y \in I^+, \\
N(T_3(x, N(y))) & \text{if } x \in I^+, y \in I^-, \\
N(T_3(x, N(x))) & \text{if } x \in I^-, y \in I^+, \\
0 & \text{if } x, y \in I^-, \\
T_5(x, y) & \text{if } x, y \in I^0, \\
y & \text{if } x \in I^+ \text{ and } y \in I^0, \\
x & \text{if } x \in I^0 \text{ and } y \in I^+, \\
0 & \text{if } x \in I^- \text{ and } y \in I^0, \\
0 & \text{if } x \in I^0 \text{ and } y \in I^-.
\end{cases}
\] (8)

Call \( T_{RA} \) the \( N \)-d-rotation–annihilation of \( T_1 \) and \( T_2 \). If \( N(x) = 1 - x \) (the standard negation) then call \( T_{RA} \) the \( d \)-rotation–annihilation of \( T_1 \) and \( T_2 \).

**Theorem 6** (Rotation–annihilation). Let \( N \) be a strong negation, \( t \) its unique fixed point, \( d \in ]t, 1[ \) and \( T_1 \) be a left-continuous \( t \)-norm. Take \( T_2 \), depending on the zero divisors of \( T_1 \), as it is taken in Definition 3 and let \( T_{RA} \) be the \( N \)-d-rotation–annihilation of \( T_1 \) and \( T_2 \).

Finally, define \( I_{T_{RA}} : [0, 1] \times [0, 1] \rightarrow [0, 1] \) by

\[
I_{T_{RA}}(x, y) = \begin{cases} 
I_3(x, y) & \text{if } x, y \in I^+, \\
N(I_3(x, N(y))) & \text{if } x \in I^+, y \in I^-, \\
1 & \text{if } x \in I^-, y \in I^+, \\
I_5(N(y), N(x)) & \text{if } x, y \in I^-, \\
I_3(x, y) & \text{if } x, y \in I^0, \\
y & \text{if } x \in I^+ \text{ and } y \in I^0, \\
x & \text{if } x \in I^0 \text{ and } y \in I^+, \\
N(x) & \text{if } x \in I^0 \text{ and } y \in I^-, \\
1 & \text{if } x \in I^- \text{ and } y \in I^0, \\
1 & \text{if } x \in I^0 \text{ and } y \in I^-.
\end{cases}
\] (9)

Then \( T_{RA} \) is a left-continuous \( t \)-norm, its associated negation is \( N \), it is rotation invariant w.r.t. \( N \), and the residual implication generated by \( T_{RA} \) is given by (9).

Fig. 7 explains geometrically the rotation–annihilation construction. The linear transformations of \( T_1 \) (the minimum \( t \)-norm) and \( T_2 \) (the Łukasiewicz \( t \)-norm) are on the left. Then the minimum is rotated, the Łukasiewicz \( t \)-norm is annihilated, and the rest is defined by either the minimum or by 0 (center), and the result is \( T_J \) (right).

**Remark 3.** If \( T_1 \) is left-continuous \( t \)-subnorm in Theorem 6 then \( T_{RA} \) is a left continuous \( t \)-subnorm, which is rotation invariant w.r.t. \( N \) and the residual implication generated by \( T_{RA} \) is given by (9).

**Example 5.** In Fig. 8 (left) the rotation–annihilation of the Łukasiewicz \( t \)-norm and the Łukasiewicz \( t \)-norm is presented. In Fig. 8 (right) the rotation–annihilation of the Łukasiewicz \( t \)-norm and \( T_2 \) is presented, where \( T_2 \) is the rotation of the product \( t \)-norm. In Fig. 9 (left) the rotation–annihilation
Fig. 7. Geometrical explanation of the rotation–annihilation construction.

Fig. 8. T-norms generated by the rotation–annihilation construction, see Example 5.

Fig. 9. Other t-norms generated by the rotation–annihilation construction, see Example 5.
of $T_1$ and $T_2$ is presented, where $T_1$ is an ordinal sum defined by a Łukasiewicz t-norm and a product t-norm and $T_2$ is the rotation of the product. In Fig. 9 (right) the rotation–annihilation of the product t-norm and $T_2$ is presented, where $T_2$ is $T_{L_{0.5}}$, the rotation invariant t-subnorm given in Section 2. In Fig. 10 a t-norm is presented, which is the result of consecutive applications of the rotation–annihilation, the ordinal sum and the rotation constructions. The t-norm from the right-hand side of Fig. 9 is used as a summand, then the obtained ordinal sum is rotated. One can continue by iterating these steps to achieve more and more difficult graphs.

7. Embedding method

7.1. Completions of left-continuous monoids

Let $\mathcal{D} = \langle D, \star, \leq, 1 \rangle$ be a commutative totally ordered integral monoid. We say that $\star$ is a left-continuous operation on $\mathcal{D}$ if whenever $X \subseteq D$ and $\sup(X)$ exists in $D$, then for every $y \in D$ one has: $y \star \sup(X) = \sup\{ y \star x : x \in X \}$. In this case we speak of left-continuous totally ordered commutative integral monoid. Any residuated monoid is left-continuous, the converse is not true in general.

Theorem 7. We can embed any countable left-continuous totally and densely ordered commutative integral monoid $\mathcal{D}$ into a left-continuous t-norm.

(i) Clearly $\mathcal{D}$ is order isomorphic to $\mathbb{Q} \cap [0, 1]$. Let $h$ be any order isomorphism from $\mathcal{D}$ onto $\mathbb{Q} \cap [0, 1]$, and let $\circ$ be defined on $\mathbb{Q} \cap [0, 1]$ by $x \circ y = h^{-1}(h(x) \star h^{-1}(y))$. Then $h$ preserves the monoidal operation, i.e., it is an isomorphism from $\mathcal{D}$ into $\langle \mathbb{Q} \cap [0, 1], \circ, \leq, 1 \rangle$, as desired.

(ii) Define for $\alpha, \beta \in [0, 1]$, $T_4(\alpha, \beta) = \sup\{ h(d \star e) : h(d) \leq \alpha$ and $h(e) \leq \beta \}$. Then $T_4$ is a left-continuous t-norm on $[0, 1]$ which extends $\circ$. Hence $h$ is an embedding of $\mathcal{D}$ into $\langle [0, 1], T_4, \leq, 1 \rangle$, and using the density of $\mathbb{Q}$ in $\mathbb{R}$, we see that $h$ preserves suprema and infima. This ends the construction.
Definition 4. We say that the t-norm $T_4$ defined in the proof of Theorem 7 is the completion of the monoidal operation $\star$.

Theorem 8. We can embed any countable, totally ordered, commutative integral monoid into a left-continuous t-norm.

Indeed, let $\mathcal{D}$ be a countable, totally ordered, commutative integral monoid (not necessarily densely ordered and left-continuous). Then we can embed it into a densely and totally ordered left-continuous commutative integral monoid $\mathcal{M}$ with minimum in the following way:

- If $\mathcal{D}$ has no minimum, then add a new element $m$ to be the minimum of the lattice reduct of $\mathcal{D}$ and extend the operation to $m$ in the obvious way (so that it becomes the zero of the multiplication).
- The domain of $\mathcal{M}$ is $\{(m,1)\} \cup \{(a,q) : a \in \mathcal{D} - \{m\}, q \in \mathbb{Q} \cap (0,1]\}$.
- The order is the lexicographic order.
- The monoidal operation $\circ$ is defined by $(a,q) \circ (b,r) = \min\{(a,q),(b,r)\}$ if $a \ast b = \min\{a,b\}$, and $(a,q) \circ (b,r) = (a \ast b, 1)$ otherwise.

That $\mathcal{M}$ defined in this way is a commutative linearly and densely ordered integral monoid is proved as in [22]. Left-continuity is due to the fact that if $\lim_{n \to \infty} (a_n, q_n) = (a,q)$, then for almost all $n$, $a_n = a$, and $\lim_{n \to \infty} q_n = q$. An application of Theorem 7 ends the construction.

7.2. Embedding finite lexicographical products

As a very special case of the method in Section 7.1 we shall we embed commutative, residuated integral $\ell$-monoids of $\bigtimes_{i=1}^{\infty} \mathbb{N}$ into $[0,1]$. Contrary to the general case, where the proofs did not result in formulas for the obtained t-norm (since a formula for the order isomorphism $h$, which is used in Theorem 7, cannot be provided in general) here we can construct the order isomorphism and thus we can obtain formulas (and 3D figures) for the resulted t-norms. Let $a \in ]0,1[, j \in \mathbb{N}$. Denote $\varphi_{a,j}(x)$ the order preserving linear isomorphism between $[a^{j+1},a^j]$ and $]0,1]$. That is, let

$$\varphi_{a,j}(x) = \frac{x - a^{j+1}}{a^j - a^{j+1}}.$$

Then

$$\varphi_{a,j}^{-1}(x) = a^{j+1} + (a^j - a^{j+1})x.$$

In case $j = 0$ denote $\varphi_{a,0}$ (the order preserving linear isomorphism between $[a,1]$ and $]0,1]$) by $\varphi_a$ for sake of simplicity.

Theorem 9. ($T_k$) Let $k \in \mathbb{N}$ and $T$ be any t-norm without zero divisors. Let $\oplus_i$ be a commutative, disjunctive $\ell$-monoid on $\mathbb{N}$ with zero 0 for $1 \leq i \leq k$.

1. Let $\varepsilon > 0$ be an infinitesimal, $X_k = \{1 - \sum_{i=1}^{k+1} n_i \cdot \varepsilon^i | n_i \in \mathbb{N} \ (1 \leq i \leq k), \ n_{k+1} \in \mathbb{R}^+\}$. Fix arbitrarily $0 = x_0 < x_1 < x_2 < \cdots < x_k < x_{k+1} < 1$ and let $a_i = \varphi_{x_{i-1}}(x_i)$ $(1 \leq i \leq k + 1)$. Define a binary
operation \( \oplus_T \) on \( \mathbb{R}^+ \) by
\[
\forall r \in \mathbb{R}^+, s = \log_{a_{k+1}}(T(a_{k+1}^r, a_{k+1}^{s'})),
\]
and a binary operation \( T \) on \( X_k \) by
\[
T \left( \left( 1 - \sum_{i=1}^{k+1} n_i \varepsilon^d \right), \left( 1 - \sum_{i=1}^{k+1} m_i \varepsilon^e \right) \right) = 1 - \left( \sum_{i=1}^{k} (n_i \oplus m_i) \varepsilon^e \right) - (n_{k+1} \oplus_T m_{k+1}) \varepsilon^{k+1}.
\]
Denote \( \phi_0 = id_{[0,1]}, \phi_{n_1,n_2,\ldots,n_k} = \varphi_{a_{1},a_{1}}^{-1} \circ \varphi_{a_{2},a_{2}}^{-1} \cdots \circ \varphi_{a_{k},a_{k}}^{-1}, \) and define \( \eta_k : X_k \rightarrow ]0,1] \) by
\[
\eta_k \left( 1 - \sum_{i=1}^{k+1} n_i \cdot \varepsilon^d \right) = \phi_{n_1,n_2,\ldots,n_k}(a_{k+1}^{n_{k+1}}).
\]
Then \( \eta_k \) is an order-preserving bijection from \( X_k \) to \([0,1] \). Finally, define a binary operation \( T_{\kappa} \) on \([0,1] \) by
\[
T_{\kappa}(x,y) = \eta_k(T(\eta_k^{-1}(x), \eta_k^{-1}(y))).
\]

2. For \( x \in [0,1] \) set \( n_{k,0}(x) = 0, \ x_{k,0} = x \). Define recursively
\[
n_{k,i}(x) = \lfloor \log_{a_i}(x_{k,i-1}) \rfloor,
\]
\[
x_{k,i} = \varphi a_i \left( \frac{x_{k,i-1}}{a_i^{n_{k,i}(x)}} \right)
\]
for \( i \leq k \) and let \( n_{k,k+1}(x) = \log_{a_{k+1}}(x_{k+1}) \). Define a binary operation \( T_{\kappa} \) on \([0,1] \) by
\[
T_{\kappa}(x,y) = \varphi_{n_{k,1}(x) \oplus n_{k,2}(y), n_{k,2}(x) \oplus n_{k,3}(y), \ldots, n_{k,k}(x) \oplus n_{k,k}(y)}(a_{k+1}^{n_{k+1}(x) \oplus_T m_{k+1}(y)}).
\]

3. Let \( T(0) = T \). For \( 1 \leq i \leq k \) define binary operations \( T_{\kappa} \) on \([0,1] \) recursively by
\[
T_{\kappa}(x,y) = \varphi^{-1}_{a_{k+1-i},n_{k,1}(x) \oplus k+1-n_{k,1}(y)} \left( T_{\kappa}(x,y) \right) \left( \varphi a_{k+1-i} \left( \frac{x}{a_{k+1-i}^{n_{k,1}(x)}} \right), \varphi a_{k+1-i} \left( \frac{y}{a_{k+1-i}^{n_{k,1}(y)}} \right) \right)
\]

(i) The three definitions for \( T_{\kappa} \) given in 1, 2 and 3, are equivalent.
(ii) \( T_{\kappa} \) is a t-norm without zero divisors.
(iii) \( T_{\kappa} \) is left-continuous if and only if so does \( T \).
(iv) \( T_{\kappa} \) is strictly increasing if and only if so does \( T \).
(v) By using the definition in 3, we have that \( T_{\kappa} \mid [0,1] \) is order-isomorphic to \( T_{\kappa-1} \).

Remark 4. If \( T \) is a left-continuous t-subnorm, or if 0 is not necessarily zero of \( \oplus_i \)'s and we suppose \( 0 \oplus 0 = 0 \) only then everything holds true but the boundary condition of the resulted t-norm is violated. Then we obtain t-subnorms.
Remark 5. As far as we can see Theorem 1 cannot be extended so that $T$ is a t-norm with zero divisors without losing either the associativity or the left-continuity (that is the residuated nature) of the resulted structure.

Corollary 1 ($T_{(\oplus)}$). Let $T$ be any t-norm without zero divisors, $\oplus$ be any commutative, disjunctive $\ell$-monoid on $\mathbb{N}$ with zero $0$, $a \in [0,1[$, and $n(x) = [\log_a(x)]$. The binary operation $T_{(\oplus)}$ on $]0,1]$ given by

$$T_{(\oplus)}(x, y) = a^{n(x)\oplus n(y)} \left( T\left( \frac{x}{a^{n(x)}}, \frac{y}{a^{n(y)}} \right) \right) = a^{n(x)\oplus n(y)} \left( a + (1 - a) \cdot T\left( \frac{x}{a^{n(x)}}, \frac{y}{1 - a} \right) \right) = a^{[\log_a(x)]\oplus[\log_a(y)]} \left( a + (1 - a) \cdot T\left( \frac{x}{a^{[\log_a(x)]}}, \frac{y}{1 - a} \right) \right)$$

is a t-norm without zero divisors. In addition, $T_{(\oplus)}$ is left-continuous (resp. strictly increasing on $]0,1[^2$) if and only if so does $T$, and $T_{(\oplus)[a,1]}$ is order-isomorphic to $T$.

Remark 6. It is clear from the recursive description of $T_{(k)}$ (see Eq. (13)) that consecutive applications of Corollary 1 can result in all the t-norms, which can be generated by Theorem 9. Moreover, we see that $T_{(\oplus^{k\cdots\oplus 1})} = (T_{(\oplus^{k\cdots\oplus 1})})_{(\oplus^{k\cdots\oplus 1})}$.

7.3. Embedding infinite lexicographical products

As another very special case of the method in Section 7.1, we embed commutative, residuated integral $\ell$-monoids of $\times_{i=1}^{\infty} \mathbb{N}$ into $[0,1]$. Like in Section 7.2, we can formulate the order isomorphism, and hence we shall obtain formulas (and 3D figures) for the resulted t-norms.

Theorem 10 ($T_{(\infty)}$). For $i \in \mathbb{N}$ let $\oplus_i$ be a commutative, disjunctive $\ell$-monoid on $\mathbb{N}$ with zero $0$.

1. Let $\varepsilon > 0$ be an infinitesimal, $X = \{1 - \sum_{i=1}^{\infty} n_i \varepsilon^i | n_i \in \mathbb{N}, i \in \mathbb{N}\}$. Fix arbitrarily $0 = x_0 < x_1 < x_2 < \cdots < x_i < \cdots < 1$ such that $\lim_{i=1}^{\infty} x_i = 1$ and let $a_i = \frac{1}{\log_a(x_i)}(a_i)(i \in \mathbb{N})$.

Define a binary operation $T$ on $X$ by

$$T \left( \left( 1 - \sum_{i=1}^{\infty} n_i \varepsilon^i \right), \left( 1 - \sum_{i=1}^{\infty} m_i \varepsilon^i \right) \right) = 1 - \left( \sum_{i=1}^{\infty} (n_i \oplus m_i) \varepsilon^i \right).$$

Define $X_k, \phi_{n_1, n_2, \ldots, n_k}$ and $\eta_k$ as in Theorem 9. Let $\eta_0 : X \to ]0,1]$ be given by

$$\eta_\infty \left( 1 - \sum_{i=1}^{\infty} n_i \varepsilon^i \right) = \lim_{k=1}^{\infty} \eta_k \left( 1 - \sum_{i=1}^{k+1} n_i \varepsilon^i \right).$$

1 Here sum may be understood formally, think e.g. the vectors with countably infinite integer coordinates $(n_1, n_2, \ldots)$ equipped with the lexicographical order.
Then $\eta_\infty$ is an order-preserving bijection from $X$ to $[0,1]$. Finally, define a binary operation $T_{(\infty)}$ on $[0,1]$ by

$$T_{(\infty)}(x,y) = \eta_\infty(T(\eta_\infty^{-1}(x),\eta_\infty^{-1}(y))).$$

(15)

2. For $x \in [0,1]$ set $n_0(x) = 0$, $x_0 = x$. Define recursively

$$n_i(x) = \log_{a_i}(x_{i-1}),$$

$$x_i = \varphi_{a_i}\left(x_{i-1}\right)$$

for $i \in \mathbb{N}$, $i > 0$. Define $\phi_{n_i} = \lim_{k=1}^{\infty} \phi_{n_1,n_2,\ldots,n_k}$, and a binary operation $T_{(\infty)}$ on $[0,1]$ by

$$T_{(\infty)}(x,y) = \phi_{\eta_\infty^{-1}(x),\eta_\infty^{-1}(y)}(1).$$

(16)

3. Let $T$ be an arbitrary left-continuous t-norm, and define $T_{(i)}$ for $i \in \mathbb{N}$ by (13). Let $T_{(\infty)}$ be a binary operation on $[0,1]$ given by

$$T_{(\infty)}(x,y) = \lim_{i=1}^{\infty} T_{(i)}(x,y).$$

(17)

i. The three definitions for $T_{(\infty)}$ given in 1–3 are equivalent.

ii. $T_{(\infty)}$ is a strictly increasing, left-continuous t-norm without zero divisors.

iii. If $\oplus_i = \oplus_1$ for $i \in \mathbb{N}$, $i > 0$ then $T_{(\infty)}|_{[0,1]}$ is order-isomorphic to $T_{(\infty)}$.

Remark 7. It is clear from (17) that consecutive applications of Corollary 1 together with pointwise limit can result in all the t-norms, which can be generated by Theorem 10.

Motivated by Theorems 9 and 10 we shall present further examples together with their 3D plots (Fig. 11). We will use the notations introduced until here without making reference to them; but instead of the short notation $T_{(k)}$ sometimes we shall use $T_{(\oplus_k,\ldots,\oplus_1)}$. We remark that the t-norm $(T_P)_{(+)}$ was discovered independently by Hájek in [9].
Example 6. Let $T_M$ stands for the minimum operation on $[0,1]$. Define an ordinal sum with one Łukasiewicz summand as follows:

$$T_{os}(x,y) = \begin{cases} 
\frac{2}{9} + \frac{5}{9} \max \left(0, \frac{x-\frac{2}{9}}{\frac{5}{9}} + \frac{y-\frac{2}{9}}{\frac{5}{9}} - 1\right), \\
\min(x,y) & \text{otherwise.}
\end{cases}$$

For the 3D plots of $(T_M)_{(+)}$ and $(T_{os})_{(+)}$ see Fig. 12.

Example 7. Let the operation $\oplus_\chi$ on $\mathbb{N}$ be given by $x \oplus_\chi y = (x - 1) \cdot (y - 1) + 1$. The graphs of $(T_P)_{(\oplus)}$ and $(T_P)_{(\oplus, \oplus)}$ are presented in Figs. 13 and 14.

Example 8. For the sake of completeness we remark that the left-continuous t-norm which is introduced by Smutná [28] (motivated by the original idea of Budinčević and Kurilić [1]) can be constructed by Theorem 10, see [21].
Finally, we remark, that all the t-norms in this section can be rotated, can be used in the rotation–annihilation construction, and so on.

8. Conclusion

The presently existing methods that result in left-continuous t-norms are summarized in this paper. Some of them have the additional advantage that the associated negation of the resulting t-norm is strong, which may be useful in logical applications. By using these methods (consecutive combination of them is as well possible) an infinite number of new left-continuous t-norms can be generated. The resulting operations may be of interest to researchers in algebra, probabilistic metric spaces, non-classical measures and integrals, non-classical logics, fuzzy set theory and its applications.

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