Abstract

This article proposes two new classes of imperfect repair models. The (conditional) failure intensity before the first repair is a continuous function of time. The repair effect is characterized by the change induced on the failure intensity before and after failure. In the first class of models, repair effect is expressed by a reduction of failure intensity. In the second class, repair effect is expressed by a reduction of the system virtual age. In each case, several particular cases are studied, which take into account the possibility of a markovian memory property. For almost every model studied, there exists a minimal wear intensity, i.e. a maximal lower bound for failure intensity.

The classification presented here involves existing models and allows the proposition of new ones. The models are compared in terms of wear-out speed. Finally, a numerical statistical study on the quality of the model parameters estimators is presented.

Keywords: Repairable systems reliability; Imperfect repair; Maintenance; Failure intensity; Non-homogeneous poisson process ageing

1. Introduction

Some important industrial systems (like nuclear power plants, planes, trains) come to the end of their planned life, but they seem to be still in normal working conditions. To extend their functioning life, one must justify some reliability requirements. One way to do it is to take into account the effect of repair actions or corrective maintenance. Repair is carried out after a failure and intends to put the system into a state in which it can perform its function again. Modelling the effect of these repair actions is of great practical interest and is the first step in order to be able to assess maintenance efficiency.

The basic assumptions on repair efficiency are known as minimal repair or As Bad As Old (ABAO) and perfect repair or As Good As New (AGAN). In the ABAO case, each repair leaves the system in the same state as it was before failure. In the AGAN case, each repair is perfect and leaves the system as if it were new. Obviously, reality is between these two extreme cases: standard maintenance reduces failure intensity but does not leave the system as good as new. This is sometimes known as imperfect or better-than-minimal repair.

Several models with that type of assumption have already been proposed (see for example a review in Pham and Wang [1]). One of the most famous is the Brown-Proschan model [2], in which system state after repair is AGAN with probability $p$ and ABAO with probability $1 - p$. Another very important class of models is the virtual age models proposed by Kijima [3]. Usually, these models are defined by the conditional distributions of successive interfailure times. For comparison purpose and in view of a statistical analysis, we think, as in Lindqvist [4], that it is more convenient to define a model by the conditional failure intensity of a stochastic point process.

In this article, two new classes of imperfect repair models are proposed, for which repair effect is characterized by the change induced on the failure intensity. Thanks to these classes, some relationships between existing models are discovered and new models are proposed. Section 2 presents the general framework of the study, introducing the concept of minimal wear intensity. Basic ABAO and AGAN cases are presented in Section 3. The reduction of intensity and reduction of age classes of models are defined, respectively, in Sections 4 and 5. Section 6 presents some interesting comparisons between these classes. Finally, a numerical statistical study on the quality of the model parameters estimators is done in Section 7.
2. Assumptions and definitions

Let \( \{T_i\}_{i \geq 1} \) be the successive failure times of a repairable system, starting from \( T_0 = 0 \). Let \( N_t \) be the number of failures observed up to time \( t \). We assume that a repair task is performed after each failure and that repair times are negligible or not taken into account. Then, the failure process is defined equivalently by the random processes \( \{T_i\}_{i \geq 1} \) or \( \{N_t\}_{t \geq 0} \).

The distribution of these processes is completely given by the conditional failure intensity defined as
\[
\forall t \geq 0, \quad \lambda_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P(N_{t+\Delta t} - N_t = 1 | \mathcal{F}_t)
\]
where \( \mathcal{F}_t \) is the past of the failure process at time \( t \), i.e. the set of all events occurred before \( t \). For the sake of simplicity, this conditional failure intensity will be called simply failure intensity in the following.

We assume that before the first failure, the failure intensity is a deterministic and continuous function of time \( \lambda(t) \), called the initial intensity. In addition, the considered system is supposed to wear-out continuously, so the initial intensity is strictly increasing.

For any stochastic point process \( \{E_i\}_{i \geq 0} \), we define \( E_{T_1}^+ \) (resp. \( E_{T_1}^- \)) as the right (resp. left) limit, if it exists, of \( E_i \) when \( t \) tends to \( T_i \).

Fig. 1. NHPP.

For almost every model defined in this article, there exists what we call a minimal wear intensity.

**Definition 1.** For a model with failure intensity \( \lambda_t \), the minimal wear intensity is, if it exists, the deterministic function \( \lambda_{\text{min}}(t) \) defined as
\[
\forall t \in \mathbb{R}^+, \quad P(\lambda_t \geq \lambda_{\text{min}}(t)) = 1
\]
(2)
\[
\forall \epsilon > 0, \forall t \in \mathbb{R}^+, \quad P(\lambda_t \leq \lambda_{\text{min}}(t) + \epsilon) > 0
\]
(3)
Eq. (2) means that \( \lambda_t \) is greater than \( \lambda_{\text{min}}(t) \) and Eq. (3) means that \( \lambda_t \) can be as close as possible to \( \lambda_{\text{min}}(t) \). The minimal wear intensity can be viewed as the maximal lower bound for failure intensity. The wear-out of all system with failure intensity \( \lambda_t \) is greater than that of a system with failure intensity \( \lambda_{\text{min}}(t) \).

3. Basic models

3.1. The non-homogeneous Poisson process

The most simple model is obtained by assuming that repair leaves the system as it was before failure. The system is said to be ABAO. The corresponding random process is the Non-Homogeneous Poisson Process (NHPP) and its failure intensity is a continuous function of time
\[
\lambda_t = \lambda(t)
\]
(4)
If the failure intensity is a constant, \( \forall t \lambda(t) = \lambda \), then the times between failures \( X_i \) are independent and exponentially distributed with parameter \( \lambda \), so there is no wear-out. The failure process is then called an Homogeneous Poisson Process (HPP).

The most usual NHPP is the Power Law Process (PLP), with a failure intensity defined as a power of time
\[
\lambda(t) = \alpha \beta^b t^{b-1}, \quad \alpha > 0, \beta > 0
\]
(5)
Fig. 1 is a plot of the failure intensity of a PLP with \( \alpha = 1 \) and \( \beta = 3 \). Failure times are indicated with stars on the time axis. In the following, all the plots are done in the case where the initial intensity is that of this PLP: \( \lambda(t) = 3t^2 \).

3.2. The renewal process

Another simple assumption is that repair is perfect that is to say, each repair restores the system as if it were brand new. The system is said to be AGAN. Then, the times between failures are independent and have the same distribution. The failure process is called a Renewal Process (RP), and its
failure intensity is of the form (Fig. 2):
\[ \lambda_t = \lambda(t - T_{N(t)}) \]  
(6)

4. Reduction of intensity models

We consider in this section models assuming that the effect of repair is a reduction of the failure intensity. In this article, we only consider arithmetic reduction of intensity, but other types of reduction such as geometric reduction can be taken into account. The basic idea of the Arithmetic Reduction of Intensity model (ARI) is to consider that

H1. Each repair action reduces the failure intensity of an amount proportional to the current failure intensity assuming that repair reduces the failure intensity of an amount depending on the past of the failure process

\[ \forall i \geq 1, \quad \lambda_{T_i} = \lambda_{T_{i-1}} - S(i, T_1, \ldots, T_i) \]  
(7)

H2. After failure, the wear-out speed is the same as before failure. Then, between two failures, the failure intensity is vertically parallel to the initial intensity.

These assumptions lead to a failure intensity such that

\[ \lambda_t = \lambda(t) - \sum_{i=1}^{N(t)} S(i, T_1, \ldots, T_i) \]  
(8)

For all sequence of real numbers \( \{a_i\}_{i \geq 0} \), we set \( \sum_{i=0}^{l} a_i = 0 \) if \( k > l \). Then for \( N_t = 0 \), \( \lambda_t = \lambda(t) \).

4.1. The ARI\(_\infty\) model

A mere idea proposed by Chan and Shaw [5], consists in assuming that repair reduces the failure intensity of an amount proportional to the current failure intensity

\[ \lambda_{T_i} = \lambda_{T_{i-1}} - \rho \lambda_{T_{i-1}} \]  
(9)

The corresponding model will be called here the Arithmetic Reduction of Intensity with Infinite memory (ARI\(_\infty\)) model. The justification of the term ‘Infinite memory’ will be given later.

The ARI\(_\infty\) failure intensity is

\[ \lambda_t = \lambda(t) - \rho \sum_{j=0}^{N(t)-1} (1 - \rho^j) \lambda(T_{N(t)-j}) \]  
(10)

The proof of Eq. (10) is done by a recurrence on the number \( N_t \) of observed failures at time \( t \).

For \( N_t = 0 \), Eq. (10) becomes \( \lambda_t = \lambda(t) \), which is the first assumption stated in Section 2.

Let us assume that Eq. (10) is true for each \( t \) such that \( N_t \leq i \), that is to say for all \( t < T_i \). We have to prove that Eq. (10) is true for \( N_t = i + 1 \) or \( t \in [T_{i+1}, T_{i+2}] \). We have:

\[ \lambda_{T_{i+1}} = (1 - \rho) \lambda_{T_{i+1}} - \rho \sum_{j=0}^{i} (1 - \rho^j) \lambda(T_{i-j}) \]

\[ \lambda_{T_{i+1}} = \lambda(T_{i+1}) - \rho \sum_{j=0}^{i} (1 - \rho^j) \lambda(T_{i-j}) \]

which is the result expressed by Eq. (10). Assumption H2 proves that Eq. (10) is also true for \( t \) in \([T_{i+1}, T_{i+2}]\). Then Eq. (10) is proved.

Fig. 3 shows a possible failure intensity of the ARI\(_\infty\) model. The initial intensity is drawn in dashed line.

4.2. The ARI\(_1\) model

Another idea is to consider that repair actions cannot reduce the global wear of the system, but only the relative wear since the last repair. This assumption leads to

\[ \lambda_{T_i} = \lambda_{T_{i-1}} - \rho(\lambda_{T_{i-1}} - \lambda_{T_{i-2}}) \]  
(11)

The corresponding model will be called the Arithmetic Reduction of Intensity with memory one (ARI\(_1\)) model. Its failure intensity is

\[ \lambda_t = \lambda(t) - \rho \lambda(T_{N(t)}) \]  
(12)

The proof of Eq. (12) is similar to the proof of Eq. (10), with

\[ \lambda_{T_{i+1}} = (1 - \rho) \lambda_{T_{i+1}} + \rho \lambda_{T_{i-1}} \]

\[ \lambda_{T_{i+1}} = (1 - \rho) \lambda(T_{i+1}) + \rho(1 - \rho) \lambda(T_i) \]

\[ \lambda_{T_{i+1}} = (1 - \rho) \lambda(T_{i+1}) = \lambda(T_{i+1}) - \rho \lambda(T_{i+1}) \]
The Arithmetic Reduction of Intensity model

Definition 2. The Arithmetic Reduction of Intensity model with memory \( m \) (ARI\(_ m \)) is defined by the failure intensity

\[
\lambda_i(t) = \lambda(t) - \rho \sum_{j=0}^{\text{Min}(m-1, N_i - 1)} (1 - \rho)^j \lambda(T_{N_i-j})
\]

(14)

Now we can explain why our models are said to be with ‘memory \( m \)’, ‘infinite memory’ or ‘memory one’. In fact, the ‘memory’ is a kind of markovian property: it is the maximal number of previous failure times that are involved in the failure intensity.

Property 1. The minimal wear intensity of the ARI\(_ m \) model is

\[
\lambda_{\text{min}}(t) = (1 - \rho)^m \lambda(t)
\]

(15)

Proof. Since \( \lambda \) is an increasing function: \( \forall j \leq N_i - 1, \lambda(T_{N_i-j}) \leq \lambda(t) \)

Then,

\[
\lambda_{\text{min}}(t) = \lambda(t) - \rho \sum_{j=0}^{\text{Min}(m-1, N_i - 1)} (1 - \rho)^j \lambda(t)
\]

\[
\geq \lambda(t) - \rho \sum_{j=0}^{m-1} (1 - \rho)^j = (1 - \rho)^m \lambda(t)
\]

(16)

So, \( P(\lambda_i \geq (1 - \rho)^m \lambda(t)) = 1 \).

Now we have to prove the second part of the definition of the minimal wear intensity. Let \( t \in \mathbb{R}^+_\ast \) (the result is obvious when \( t = 0 \)). The idea of the proof is that, at time \( t \), failure intensity is excessively close to \( (1 - \rho)^m \lambda(t) \) if there have been \( m \) failures between \( t - \epsilon \) and \( t \) for a small \( \epsilon \).

As \( \lambda \) is an increasing function and \( 0 \leq \rho \leq 1 \), for all \( \epsilon \in \mathbb{R}^+_\ast \) we have

\[
P(\lambda_i \leq \lambda(t) - (1 - (1 - \rho)^m) \lambda(t - \epsilon))
\]

\[
\geq P(\lambda_i \leq \lambda(t) - \rho \sum_{j=0}^{m-1} (1 - \rho)^j \lambda(t - \epsilon))
\]

\[
\geq P(\sum_{j=0}^{m-1} (1 - \rho)^j \lambda(T_{N_i-j}) \geq m - 1) \cap [N_i \geq m]
\]

\[
\geq P([T_{N_i} \leq \ldots \leq T_{N_i+m-1} \leq t-\epsilon] \cap [N_i \geq m])
\]

\[
P(\lambda_{\text{min}} < e)
\]

(17)

In addition, because \( \lambda \) is continuous

\[
\forall \epsilon' \in \mathbb{R}^+_\ast, \exists \eta \in \mathbb{R}^+_\ast, \forall \epsilon \in [0, \eta], \lambda(t) - (1 - \rho)^m \lambda(t - \epsilon) \leq \epsilon'
\]

or \( \lambda(t) - \epsilon' \leq \lambda(t - \epsilon) \)

So \( \lambda(t) - (1 - (1 - \rho)^m) \lambda(t - \epsilon) \leq \lambda(t) - (1 - (1 - \rho)^m) (\lambda(t) - \epsilon') \).

By using Eq. (17) we deduce that

\[
P(\lambda_i \leq \lambda(t) - (1 - (1 - \rho)^m) \lambda(t - \epsilon')) > 0
\]

or \( P(\lambda_i \leq (1 - \rho)^m \lambda(t) + (1 - (1 - \rho)^m) \epsilon') > 0 \)

By letting \( \epsilon = (1 - (1 - \rho)^m) \epsilon' \), we obtain for all \( \epsilon \in \mathbb{R}^+_\ast \)

\[
P(\lambda_i \leq (1 - \rho)^m \lambda(t) + \epsilon) > 0
\]

So finally, \( P(\lambda_i \leq \lambda_{\text{min}}(t) + \epsilon) > 0 \) and the property is proved.

According to Eq. (16), for \( 0 < \rho < 1 \), \( \lambda \) and \( \lambda_{\text{min}}(t) \) can be equal only if \( N_i \geq m \) and \( \forall j \in [N_i - m + 1, B] \).
1, ..., Nt), \( \lambda(T_j) = \lambda(t) \). When \( \lambda \) is strictly increasing, this can happen only when \( m = 1 \). Fig. 5 shows the failure intensity of an ARI\(_2\) model.

An important point is that repair efficiency is measured by the value of \( r \):

- \( 0 < r < 1 \): efficient repair
- \( r = 1 \): optimal repair; failure intensity is put back to zero (but repair effect is different from the AGAN situation)
- \( r = 0 \): useless repair (ABAO)
- \( r < 0 \): harmful repair

Then, assessing repair efficiency in this class of models is estimating \( r \).

5. Reduction of age models

The principle of this class of models is to consider that repair rejuvenates the system such that its intensity at time \( t \) is equal to the initial intensity at time \( A_t \), where \( A_t < t \).

The real age of a system is its functioning time \( t \). Then we define the virtual age of a system as a positive function of its real age, possibly depending on past failures: \( A_t = A(t; N_0, T_1, ..., T_{N_t}) \).

A reduction of age model has a failure intensity which is a function of its virtual age: \( \lambda_t = \lambda(A_t) \). The idea that repair actions reduce the age of the system is the basis of Kijima’s virtual age models [3].

Properties of \( A_t \) are the same as those of \( \lambda_t \) for arithmetic reduction of intensity models when the initial intensity is \( \lambda(t) = t \). So we can build reduction of age models by analogy with reduction of intensity models. For example, the Arithmetic Reduction of Age model (ARA) is defined by a virtual age such that (Fig. 6):

\[
A_t = t - \sum_{i=1}^{N_t} S(i, T_1, ..., T_i) \quad (18)
\]

The failure intensity is then

\[
\lambda_t = \lambda \left( t - \sum_{i=1}^{N_t} S(i, T_1, ..., T_i) \right) \quad (19)
\]

Between two consecutive failures, the failure intensity of an ARA model is horizontally parallel to the initial intensity.

5.1. The ARA\(_{\infty}\) model

The ARA\(_{\infty}\) assumption is that repair reduces the virtual age of the system of an amount proportional to its age just before repair

\[
A_T = A_{T_0} - \rho A_{T_0} \quad (20)
\]

Then, by analogy with the ARI\(_{\infty}\) model, the failure intensity of the Arithmetic Reduction of Age model with infinite memory (ARA\(_{\infty}\)) is

\[
\lambda_t = \lambda \left( t - \rho \sum_{j=0}^{N_t-1} (1 - \rho)^j T_{N_t-j} \right) \quad (21)
\]
The minimal wear intensity is equal to zero. Fig. 7 shows the failure intensity of this model.

This model appears to be the same as the one introduced by Brown et al. [6].

5.2. The ARA1 model

Malik [7] has introduced a model which can be understood as an ARA model for which repair effect reduces the supplement of age of the system since the last failure from 

\[ AT_i - (1 - \rho)[AT_i - AT_i] \]

\[ AT_i = AT_i - (1 - \rho)[AT_i - AT_i] = AT_i - \rho[AT_i - AT_i] \]

(22)

In our classification, this model can be called an Arithmetic Reduction of Age model with memory one (ARA1), and by analogy with ARI1 model, its failure intensity is

\[ \lambda_i = \lambda(t - \rho T_N) \]

(23)

This model appears to be the same as the Kijima et al. model [8]. Moreover, using this model, Shin et al. [9] have developed an optimal preventive maintenance policy.

The minimal wear intensity of the model is equal to \( \lambda(t - \rho T_N) \). Just after maintenance, the minimal wear intensity and the failure intensity are equal (Fig. 8).

5.3. The ARA_m model

By analogy with the ARI_m model, the ARA_m class of models can be defined

**Definition 3.** The Arithmetic Reduction of Age model with memory \( m \) (ARA_m) is defined by the failure intensity

\[ \lambda_i = \lambda(t - \rho \sum_{j=0}^{\text{Min}(m-1,N_i-1)} (1 - \rho)^j T_{N_i-j}) \]

(24)

Property 2. The minimal wear intensity of the ARA_m model is

\[ \lambda_{\text{min}}(t) = \lambda((1 - \rho)^m t) \]

(25)

The proof of Property 2 is very similar to that of Property 1.

An ARA_m failure intensity is represented in Fig. 9. Repair efficiency is measured by the value of \( \rho \)

- \( 0 < \rho < 1 \): efficient repair
- \( \rho = 1 \): optimal repair (AGAN)
- \( \rho = 0 \): useless repair (ABAO)
- \( \rho < 0 \): harmful repair

Here again, assessing repair efficiency is estimating \( \rho \). This model is particularly interesting since the AGAN, ABAO and Malik models appear to be particular cases.

6. Some comparisons between the proposed models

6.1. ARI and ARA models

ARI_1 and ARA_1 models are constructed under quite similar assumptions: repair actions reduce either failure intensity or virtual age. In addition, in both models, repair efficiency is characterized by parameter \( \rho \). So it is interesting to compare these models for the same \( \rho \). It happens to depend on the initial intensity convexity:

**Property 3.** If \( \lambda \) is convex (resp. concave), for the same parameter \( \rho \in [0,1] \), the minimal wear intensity of the ARI_1 model is greater (resp. less) than that of the ARA_1 model (cf. Fig. 10, resp. Fig. 11).

**Proof.** Since \( \lambda \) is convex and \( \rho \in [0,1] \), \( \forall t \geq 0, \lambda'(t) \geq \lambda'((1 - \rho)t) \).
By integrating the previous inequality on \([0,t]\) we obtain
\[
(1 - \rho)\lambda(t) \geq \lambda((1 - \rho)t)
\]
And the property is proved for \(\lambda\) convex. A similar proof holds for \(\lambda\) concave. □

So ARI and ARA models with the same parameters are not comparable. But we can find repair efficiency coefficients such that these models have quite similar wear-out:

**Property 4.** When the initial intensity is that of a Power Law Process (PLP : \(\lambda(t) = \alpha \beta t^{-\alpha-1}\)), there exists two parameters \(\rho_{I1}\) and \(\rho_{A1}\) such that the ARI\(_1\) model with parameters \((\alpha, \beta, \rho_{I1})\) and the ARA\(_1\) model with parameters \((\alpha, \beta, \rho_{A1})\) have the same minimal wear intensity (Figs. 12 and 13):

\[
\rho_{I1} = 1 - (1 - \rho_{A1})^{\beta^{-1}}
\]

The proof is immediate since

\[
(1 - \rho_{I1})\alpha \beta t^{-\alpha-1} = \alpha \beta [(1 - \rho_{A1})t]^{-\alpha-1} \Rightarrow 1 - (1 - \rho_{A1})^{\beta^{-1}}
\]

In this case, if \(\lambda\) is convex (resp. concave) the ARI model (resp. ARA) tends to have a greater wear-out speed than the ARA (resp. ARI) model (Fig. 12, resp. Fig. 13). It can be explained by the fact that, in the ARI model, the failure intensity is vertically parallel to the initial intensity \(\lambda\), whereas in the ARA model, both intensities are horizontally parallel.

Similar results exist for memory greater than one.

6.2. Memory of ARI and ARA models

An obvious result is that, for the same initial intensity and coefficient \(\rho \in [0, 1]\), the greater the memory is, the smaller the minimal wear intensity of the ARI (resp. ARA) model is.

**Property 5.** If the initial intensity is that of a PLP, there exists two parameters \(\rho_{m_{I1}}\) and \(\rho_{m_{A1}}\) such that the ARI\(_{m_{I1}}\) (resp. ARA\(_{m_{A1}}\)) model with parameters \((\alpha, \beta, \rho_{m_{I1}})\) and the ARA\(_{m_{A1}}\) model with parameters \((\alpha, \beta, \rho_{m_{A1}})\) have the same minimal wear intensity (Figs. 12 and 13).

Similar results exist for memory greater than one.

Fig. 10. Thin line: ARI\(_1\) thick line: ARA\(_1\) with \(\beta = 3, \rho = 0.5\).

Fig. 12. Thin line: ARI\(_1\) thick line: ARA\(_1\) with \(\beta = 3, \rho_{I1} = 0.75, \rho_{A1} = 0.5\).

Fig. 11. Thin line: ARI\(_1\) thick line: ARA\(_1\) with \(\beta = 1.5, \rho = 0.5\).

Fig. 13. Thin line: ARI\(_1\) thick line: ARA\(_1\) with \(\beta = 1.5, \rho_{I1} = 0.29, \rho_{A1} = 0.5\).
ARA_m) model with parameters (α, β, ρ_m) and the ARI_m (resp. ARA_m) model with parameters (α, β, ρ_m) have the same minimal wear intensity (Fig. 14, resp. Fig. 15)

\[ ρ_m = 1 - (1 - ρ_m)^{m/n_m} \]  

(27)

This means that there exists repair efficiency coefficients such that ARI (resp. ARA) models with different bounded memories have quite similar wear-out.

7. Numerical statistical study of the maximum likelihood estimators

Only a few work has been done on parametric statistical inference in repair models: Lim [10] for the Brown–Proshan model, Shin et al. [9] for the Malik model, Yun-Choung [11] for the Brown et al. model [6]. The few quantity of these works can be explained by the complexity of the likelihood function. For example, the log-likelihood function of a ARI_m model with a PLP initial intensity is defined by the following expression:

\[
\ln L(α, β, ρ; n, t_1, . . . , t_n) = n \ln α + n \ln β + (β - 1) \ln t_1 + \sum_{i=1}^{m-1} \ln \left( β_{i+1} - ρ \sum_{j=1}^{i} (1 - ρ)^{i-j} β_{j}^{-1} \right) \\
+ \sum_{i=m}^{n-1} \ln \left( β_{i+1} - ρ \sum_{j=i-m+1}^{i} (1 - ρ)^{i-j} β_{j}^{-1} \right) \\
- α \left[ t_{i}^{β} - βρ \sum_{i=1}^{m-1} \sum_{j=i-m+1}^{i} (1 - ρ)^{i-j} t_{j}^{β-1} (t_{i+1} - t_{i}) \right] \\
+ \sum_{i=m}^{n-1} \sum_{j=i-m+1}^{i} (1 - ρ)^{i-j} t_{j}^{β-1} (t_{i+1} - t_{i}) \right]
\]

A theoretical study of the properties of maximum likelihood estimators seems to be quite difficult. That is why only a simulation study has been done in this article.

We have estimated the bias and variance of the maximum likelihood estimators \( \hat{α}_n, \hat{β}_n, \hat{ρ}_n \) of the parameters of all models using 5000 simulations of the failure process when the initial intensity is that of a PLP process. In a first step, \( α, β, \) and \( ρ \) are simultaneously estimated. In a second step, \( α \) and \( β \) are supposed to be known and only \( ρ \) is estimated. All the following simulation results are presented with plots for which \( α = 1 \) and \( β = 3 \).

7.1. Simultaneous estimation of \( α, β \) and \( ρ \)

We have compared the quality of the estimators for ARI_m and ARA_m models with the same parameters \( α, β \) and \( ρ \), for \( α = 1, β \in \{2, 3\}, ρ \in \{0.2, 0.5, 0.9\}, m \in \{1, 2, 3, \infty\} \) and \( n \in \{5, 10, 20, 40, 60, 80, 100\} \).

Fig. 16. MSE(\( α_n \)) vs. \( n, ρ = 0.5 \).
of an estimator $\hat{\theta}_n$ of $\theta \in \{\alpha, \beta, \rho\}$ is assessed by the empirical Mean Squared Error (MSE) defined as

$$\text{MSE}(\hat{\theta}_n) = s^2_{\hat{\theta}_n} + (m_{\hat{\theta}_n} - \theta)^2$$

where $m_{\hat{\theta}_n}$ is the empirical mean of $\hat{\theta}_n$ and $s^2_{\hat{\theta}_n}$ its empirical variance. Figs. 16–18 represent the MSE of the parameters estimators vs. $n$, for $\rho = 0.5$, which is an average value of repair efficiency.

However, taking the same value of $\alpha$, $\beta$ and $\rho$, makes that models with very different wear-out are compared. For example, Fig. 19 represents the failure intensity of $\text{ARI}_1$ and $\text{ARA}_3$ models with the same parameters $\alpha = 1$, $\beta = 3$ and $\rho = 0.5$.

Then, in order to compare the quality of the estimators, models with similar wear-out properties should be considered. We have chosen to compare the estimators for models having the same minimal wear intensity. According to Properties 4 and 5, $\text{ARI}_m$ and $\text{ARA}_m$ models must have the same parameters $\alpha$ and $\beta$, but a parameter $\rho$ depending on the model and on the memory. However, with this method, models with infinite memory cannot be considered. In addition, models with the same minimal wear intensity have truly comparable wear-out properties only if their memories are not too much different.

The MSE of the parameters estimators are compared for two very different repair efficiencies:

**Case 1.** Figs. 20, 22 and 24 represent the MSE of $\hat{\alpha}_n$, $\hat{\beta}_n$ and $\hat{\rho}_n$ for $\text{ARI}_m$ and $\text{ARA}_m$ models with the same minimal wear intensity. 

<table>
<thead>
<tr>
<th>Case</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\rho$</th>
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</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>0.9</td>
<td>0.684</td>
<td>0.536</td>
</tr>
<tr>
<td>Case 2</td>
<td>0.19</td>
<td>0.1</td>
<td>0.0678</td>
</tr>
</tbody>
</table>

Table 1: Values of $\rho$ for Case 1 and Case 2

**Fig. 17.** MSE($\hat{\rho}_n$) vs. $n$, $\rho = 0.5$.

**Fig. 18.** MSE($\hat{\beta}_n$) vs. $n$, $\rho = 0.5$.

**Fig. 19.** Thin line: $\text{ARI}_1$, thick line: $\text{ARA}_3$, with $\alpha = 1$, $\beta = 3$ and $\rho = 0.5$.

**Fig. 20.** Case 1: MSE($\hat{\alpha}_n$) vs. $n$. 

**Fig. 21.** Case 2: MSE($\hat{\alpha}_n$) vs. $n$. 

**Fig. 22.** Case 1: MSE($\hat{\beta}_n$) vs. $n$, $\rho = 0.5$. 

**Fig. 23.** Case 2: MSE($\hat{\beta}_n$) vs. $n$, $\rho = 0.5$. 

**Fig. 24.** Case 1: MSE($\hat{\rho}_n$) vs. $n$, $\rho = 0.5$.

**Fig. 25.** Case 2: MSE($\hat{\rho}_n$) vs. $n$, $\rho = 0.5$. 

wear intensity, corresponding to a good repair efficiency: \( \lambda_{\text{min}}(t) = 0.3t^2 \).

Case 2. Figs 21, 23 and 25 represent the different estimators MSE for a rather bad repair efficiency: \( \lambda_{\text{min}}(t) = 2.43t^2 \).

The corresponding values of \( \rho \) for ARI\(_m\) and ARA\(_m\) models are given in Table 1.

Figs. 20–25 show that the estimators of \( \alpha \) and \( \beta \) are clearly better in the ARI models than in the ARA models. For small samples, the estimators \( \hat{\alpha}_n \) and \( \hat{\beta}_n \) in ARA models are very bad. For large samples, in case of good repair efficiency, the estimator of \( \alpha \) is also bad for ARA models. The effect of memory on the quality of the estimators of \( \alpha \) and \( \beta \) seems to be significant essentially on ARA models. These conclusions seem globally still true regarding Figs. 16 and 17, except that ARA\(_1\) and ARI\(_1\) models provide the worst estimators of \( \alpha \). On this first set of plots, we notice also that the ARI\(_{m}\) model seems to provide the best
estimators, but that can be due to the fact that, in these figures, models have not comparable wear-out.

For the estimator of $r$, conclusions are not so sharp. Figs. 18, 24, 25, show that ARI$_1$ and ARA$_1$ models provide largely the worst estimators of $r$, and the best estimation in all cases is obtained for the ARA$_3$ model.

Finally, it is clear that the estimations in all models are better in Case 1 than in Case 2, that is to say when repair action is the most efficient.

Consequently, from a statistical point of view, ARI$_m$ models are more interesting than ARA$_m$ models and models with memory one are worse than the others. So if we take into account that the more important the memory is, the more complicated the failure intensity is, we can say that ARI$_2$ or ARI$_3$ models are the best models with respect to their statistical properties.

In order to show the influence of the value of $r$ on the quality of the estimators, Figs. 26–28 represent, for a ARI$_2$ model, the quality of the parameters estimators vs. $r$. Plots are quite similar for others ARA$_m$ and ARI$_m$ models.

These figures show that the greater $r$ is, the better the estimators of $b$ and $r$ are, that is to say ARI and ARA models provide the best estimators when repair actions are significantly efficient. When $r$ is close to one, the slight increasing of the MSE of $\hat{r}_n$ is an edge effect due to the fact that $r$ is necessarily smaller than one (Figs. 26–28). Finally, the value of $r$ does not really influence the estimation of $\alpha$.

7.2. Estimation of $r$ when and are known

When both parameters $\alpha$ and $\beta$ of the PLP initial intensity are known, ARA and ARI models provide very good estimators of the repair efficiency $r$, even if the number of failures is very small. For example, Fig. 29 represents the MSE of $\hat{r}_n$ for ARI$_3$ and ARA$_3$ models, for several values of $\rho$. It can be seen that the MSE values are much smaller than those obtained when $\alpha$ and $\beta$ are
unknown. Here again, the quality of the estimators grows with $\rho$.

8. Conclusion

This article proposes a classification of repair models in two classes: the reduction of intensity models and the reduction of age models. This classification involves existing models and allows the proposition of new ones. The parameters of the models are easily understandable: initial intensity represents the wear-out of the system without repair, and parameter $\rho$ represents repair efficiency.

Many further work is needed. The probabilistic properties of the proposed models have to be investigated. The properties of the parameters estimators have to be theoretically studied. In order to check models validity, some goodness-of-fit tests should be developed. At last, preventive maintenance should be included in these models.

References