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On fuzzy bipolar soft sets, their algebraic structures and applications

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Abstract. We have defined fuzzy bipolar soft sets and basic operations of union, intersection and complementation for fuzzy bipolar soft sets. The algebraic properties of fuzzy bipolar soft sets are discussed. The concept of bipolar fuzzy soft set is also given and the equivalence of both structures is established. An application of fuzzy bipolar soft sets in decision making problems is presented with the help of an example.

Keywords: Bipolarity, bipolar fuzzy sets, soft sets, extended union, extended intersection, restricted union, restricted intersection

1. Introduction

While we talk about the modeling of real world problems which are ranging from engineering to medical and medical to social fields, we come across with the presence of uncertainty in data. L. A. Zadeh [21] was the first one to introduce the theory of fuzzy sets that yielded a whole field of fuzzy mathematics. The nature of data is an important factor in the process of developing mathematical models in various fields like engineering, life sciences, pattern recognition, neural networks, artificial intelligence, behavioral and social sciences. There are also some other factors which may affect our considerations related to the nature of data and an obvious one is the bipolarity of data. It is evidently observed that every information about a particular phenomenon has two aspects i.e. presence of a property or its absence [5]. There are models that are developed through the tools (e.g. bipolar fuzzy sets [8, 9]) in which a positive measure has been used to approximate the presence of a particular attribute and a negative measure is used to approximate the degree of absence of that same attribute. There is always a possibility of gray areas where we get uncertain to decide whether a phenomenon possesses a property or not. Some other theories which are capable of handling these kinds of situations include intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets etc [4, 7].

Theory of soft sets was introduced by Molodstov in 1999 [15]. The purpose of the novel concept was to remove the inadequacy of parameterization tool in previously defined theories of fuzzy Mathematics. Although the theory of rough sets [10, 16] addresses the issue of parameterization and the hybrid structure such as fuzzy rough sets can also be utilized for incorporating the fuzziness of data but the addition of any further factor such as bipolarity of information makes it too complicated to use. On the other hand, the absence of any restrictions while making approximations for a given object in soft sets establishes this theory as more handy, convenient and easily applicable in practice. Since the introduction of the theory of soft sets in 1999, a lot of work has been done so far. We can find the studies on structure as well as on the applications of soft sets in various fields [1–3, 6, 11–13, 17–20].

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In this paper, we have initiated a concept of fuzzy bipolar soft sets. The idea is generated with the motivation of bipolarity of parameters and then the fuzziness of data comes into play. We have considered a set of parameters and its negative set i.e. the absence of these parameters and denote this set by "not set", for each parameter$e$, not $e = \neg e$ is the absence of e. A fuzzy bipolar soft set is defined with the help of two mappings, one for approximating the degree of fuzziness of the positivity or presence of a certain parameter in the objects of initial universal set and the other one is to approximate the relative degree of fuzziness of the negativity or absence of the same parameter. In this way, we have combined these three concepts of bipolarity, fuzziness and parameterization and thus it is shown through examples that we have found a very easy to use way of modeling the phenomena where all these three factors are involved. To move further, we have defined the basic algebra for the fuzzy bipolar soft sets and discussed their algebraic properties in detail. It is also shown that the collection of fuzzy bipolar soft sets forms a stone algebra. At the end, an application of fuzzy bipolar soft sets in the decision making problems is presented along with the algorithm.

2. Preliminaries

Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice with least element 0 and maximum element 1. An involution μ on L is a mapping $\mu : L \to L$ such that $\mu(\mu(x)) = x$, $\mu(0) = 1$ and $\mu(1) = 0$. A bounded lattice is called distributive if the distributive laws hold with respect to $\vee$ and $\wedge$. If De Morgan's laws hold for a bounded distributive lattice having an involution μ, then it is called De Morgan algebra. Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice and $x \in L$, then an element x^* is called a pseudo complement of x, if $x \wedge x^* = 0$ and $y \leq x^*$ whenever $x \wedge y = 0$. If every element has a pseudo complement then L is pseudo complemented. The equation $x^* \vee x^{**} = 1$ is called Stone's identity. A Stone algebra is a pseudo complemented distributive lattice satisfying Stone's identity.

Now we define fuzzy sets. Let X be a given set.

Definition 1. [21]. A fuzzy subset of X is a function from X into the unit closed interval $[0, 1]$. The set of all fuzzy subsets of X is called the fuzzy power set of X, and is denoted by $FP(X)$.

Definition 2. [21] Let $\mu, \nu \in FP(X)$. If $\mu(x) \leq \nu(x)$ for all $x \in X$, then μ is said to be contained in ν, and we write $\mu \subseteq \nu$(or $\nu \supseteq \mu$).

Clearly, the inclusion relation $\subseteq$ is a partial order on $FP(X)$.

Definition 3. [21] Let $\mu, \nu \in FP(X)$. Then $\mu \vee \nu$ and $\mu \wedge \nu$ are fuzzy subsets of X, defined as follows:
For all $x \in X$,

$$(\mu \vee \nu)\ (x) = \mu\,(x) \vee \nu\,(x)\ ,$$

$$(\mu \wedge \nu)\ (x) = \mu\,(x) \wedge \nu\,(x)\ .$$

The fuzzy subsets $\mu \vee \nu$ and $\mu \wedge \nu$ are called the union and intersection of μ and ν, respectively.

Definition 4. [21] Two fuzzy subsets of X are denoted by $\emptyset$ and X which map every element of onto 0 and 1 respectively. We call $\emptyset$ as the empty set or null fuzzy subset and X as the whole fuzzy subset of X.

Definition 5. [8] A bipolar fuzzy set μ in X is defined as:

$$\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) :\ \ x \in X\right\}$$

where $\mu^P : X \to [0, 1]$ and $\mu^N : X \to [-1, 0]$ are mappings. The positive membership degree $\mu^P(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set

$$\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) :\ \ x \in X\right\}$$

and the negative membership degree $\mu^N(x)$ denotes the satisfaction degree of x to some implicit counter-property of

$$\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) :\ \ x \in X\right\}.$$

if $\mu^P(x) \neq 0$ and $\mu^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for

$$\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) :\ \ x \in X\right\}.$$

if $\mu^P(x) = 0$ and $\mu^N(x) \neq 0$, it is the situation that x does not satisfy the property of

$$\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) :\ \ x \in X\right\},$$

but somewhat satisfies the counter-property of

$$\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) :\ \ x \in X\right\}.$$

it is possible for an element x to be $\mu^P(x) \neq 0$ and $\mu^N(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain. For the sake of simplicity, we shall write $\mu = \left(\mu^P,\ \mu^N\right)$ for the bipolar fuzzy set

$$\mu = \left\{\left(x,\ \mu^P\,(x)\,,\ \mu^N\,(x)\right)\ :\ x \in X\right\}$$

3. Fuzzy bipolar soft sets

Let U be an initial universe and E be a set of parameters. Let $FP(X)$ denotes the collection of all fuzzy subsets of U and A, B, C are non-empty subsets of E. Now, we define

Definition 6. A triplet (F, G, A) is called a fuzzy bipolar soft set over U, where F and G are mappings, given by $F \ : \ A \rightarrow FP(U)$ and $G \ : \neg A \rightarrow FP(U)$ such that

$$0 \leq (F(e))(x) + (G(\neg e))(x) \leq 1$$

for all $e \in A$.

In other words, a fuzzy bipolar soft set over U gives two parameterized families of subsets of the universe U and the condition

$$0 \leq (F(e))(x) + (G(\neg e))(x) \leq 1$$

for all $e \in A$, is imposed as a consistency constraint. For each $e \in A$, $F(e)$ and $G(\neg e)$ are regarded as the set of e -approximate elements of the fuzzy bipolar soft set (F, G, A).

Definition 7. For a fuzzy bipolar soft set (F, G, A) over U, we define a fuzzy soft set $(H_{(F,G)}, A)$ over U for the approximation of the degree of hesitation in (F, G, A) as follows:

$H_{(F,G)} \ : \ A \rightarrow FP(U)$ defined by

$$(H_{(F,G)}(e))(x) = 1 - (F(e))(x) - (G(\neg e))(x)$$

for all $x \in U, e \in A$. Clearly, $(H_{(F,G)}, A)$ approximates the lack of knowledge about the objects of U while considering the presence or absence of a particular parameter of A.

Definition 8. For two fuzzy bipolar soft sets (F, G, A) and (F_1, G_1, A) over a universe U, we say that (F, G, A) is a fuzzy bipolar soft subset of (F_1, G_1, A), if,

1. $A \subseteq B$ and
   $F(e) \subseteq F_1(e)$ and $G_1(\neg e) \subseteq G(\neg e)$ for all $e \in A$.

This relationship is denoted by $(F, G, A) \ \tilde{\subseteq} \ (F_1, G_1, A)$. Similarly (F, G, A) is said to be a fuzzy bipolar soft superset of (F_1, G_1, A), if (F_1, G_1, A) is a fuzzy bipolar soft subset of (F, G, A). We denote it by $(F, G, A)\tilde{\supseteq}(F_1, G_1, A)$.

Definition 9. Two fuzzy bipolar soft sets (F, G, A) and (F_1, G_1, A) over a universe U are said to be equal if (F, G, A) is a fuzzy bipolar soft subset of (F_1, G_1, A) and (F_1, G_1, A) is a fuzzy bipolar soft subset of (F, G, A).

Definition 10. The complement of a fuzzy bipolar soft set (F, G, A) is denoted by $(F, G, A)^c$ and defined by $(F, G, A)^c = (F^c, G^c, A)$ where F^c and G^c are mappings given by $F^c(e) = G(\neg e)$ and $G^c(\neg e) = F(e)$ for all $e \in A$.

Definition 11. A fuzzy bipolar soft set over U is said to be a relative null fuzzy bipolar soft set, denoted by (Φ, U, A) if for all $e \in A$, $\Phi(e) = \emptyset$ and $U(\neg e) = U$, for all $e \in A$.

Definition 12. A fuzzy bipolar soft set over U is said to be a relative absolute fuzzy bipolar soft set, denoted by (Φ, U, A), if for all $e \in A$, $U(e) = U$ and $\Phi(\neg e) = \emptyset$, for all $e \in A$.

Definition 13. If (F, G, A) and (F_1, G_1, B) are two fuzzy bipolar soft sets over U then "(F, G, A) and (F_1, G_1, B)" denoted by $(F, G, A) \wedge (F_1, G_1, B)$ is defined by $(F, G, A) \wedge (F_1, G_1, B) = (H, I, A \times B)$ where $H(a, b) = F(a) \wedge F_1(b)$ and $I(\neg a, \neg b) = G(\neg a) \vee G_1(\neg b)$ for all $(a, b) \in A \times B$.

Definition 14. If (F, G, A) and (F_1, G_1, B) are two fuzzy bipolar soft sets over U then "(F, G, A) or (F_1, G_1, B)" denoted by $(F, G, A) \vee (F_1, G_1, B)$ is defined by $(F, G, A) \vee (F_1, G_1, B) = (H, I, A \times B)$ where $H(a, b) = F(a) \vee F_1(b)$ and $I(\neg a, \neg b) = G(\neg a) \wedge G_1(\neg b)$ for all $(a, b) \in A \times B$.

Proposition 1. If (F, G, A) and (F_1, G_1, B) are two fuzzy bipolar soft sets over U then

1. $((F, G, A) \vee (F_1, G_1, B))^c = (F, G, A)^c \wedge (F_1, G_1, B)^c$
2. $((F, G, A) \wedge (F_1, G_1, B))^c = (F, G, A)^c \vee (F_1, G_1, B)^c$

Proof: Straightforward.

Definition 15. Extended Union of two fuzzy bipolar soft sets (F, G, A) and (F_1, G_1, B) over the common universe U is the fuzzy bipolar soft set (H, I, C) over U where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & if\ e \in A - B \\ F_1(e) & if\ e \in B - A \\ F(e) \vee F_1(e) & if\ e \in A \cap B \end{cases}$$

$$I(\neg e)=\begin{cases} G(\neg e) & if\ e\in(\neg A)-(\neg B)\\ G_1(\neg e) & if\ e\in(\neg B)-(\neg A)\\ G(\neg e)\wedge G_1(\neg e) & if\ e\in(\neg A)\cap(\neg B)\end{cases}$$

we denote it by $(F,\ G,\ A)\tilde{\cup}(F_1,\ G_1,\ B)=(H,\ I,\ C)$.

Definition 16. Extended Intersection of two fuzzy bipolar soft sets $(F,\ G,\ A)$ and $(F_1,\ G_1,\ B)$ over the common universe U is the fuzzy bipolar soft set $(H,\ I,\ C)$ over U where $C=A\cup B$ and for all $e\in C$,

$$H(e)=\begin{cases} F(e) & if\ e\in A-B\\ F_1(e) & if\ e\in B-A\\ F(e)\wedge F_1(e) & if\ e\in A\cap B\end{cases}$$

$$I(\neg e)=\begin{cases} G(\neg e) & if\ e\in(\neg A)-(\neg B)\\ G_1(\neg e) & if\ e\in(\neg B)-(\neg A)\\ G(\neg e)\vee G_1(\neg e) & if\ e\in(\neg A)\cap(\neg B)\end{cases}$$

we denote it by $(F,\ G,\ A)\tilde{\cap}(F_1,\ G_1,\ B)=(H,\ I,\ C)$.

Definition 17. Restricted Union of two fuzzy bipolar soft sets $(F,\ G,\ A)$ and $(F_1,\ G_1,\ B)$ over the common universe U is the fuzzy bipolar soft set $(H,\ I,\ C)$, where $C=A\cap B$ is non-empty and for all $e\in C$

$H(e)=F(e)\vee G(e)$ and $I(\neg e)=F_1(\neg e)\wedge G_1(\neg e)$. We denote it by $(F,\ G,\ A)\cup_R(F_1,\ G_1,\ B)=(H,\ I,\ C)$.

Definition 18. Restricted Intersection of two fuzzy bipolar soft sets $(F,\ G,\ A)$ and $(F_1,\ G_1,\ B)$ over the common universe U is the fuzzy bipolar soft set $(H,\ I,\ C)$, where $C=A\cap B$ is non-empty and for all $e\in C$:

$H(e)=F(e)\wedge G(e)$ and $I(\neg e)=F_1(\neg e)\vee G_1(\neg e)$. We denote it by $(F,\ G,\ A)\cap_R(F_1,\ G_1,\ B)=(H,\ I,\ C)$.

Conventionally we assume that $(F,G,A)\cap_R(F_1,\ G_1,\ B)=(\Phi,\ U,\emptyset)=(F,\ G,\ A)\cup_R(F_1,\ G_1,\ B)$ whenever $A\cap B=\emptyset$.

Lemma 1. *Let (F,G,A), (F_1,G_1,B) and $(F_2,\ G_2,\ C)$ be any fuzzy bipolar soft sets over a common universe U. Then the following are true:*

1. $(F,\ G,\ A)\alpha((F_1,\ G_1,\ B)\alpha(F_2,\ G_2,\ C))=((F,\ G,\ A)\alpha(F_1,\ G_1,\ B))\alpha(F_2,\ G_2,\ C)$
2. $(F,\ G,\ A)\alpha(F_1,\ G_1,\ B)=(F,\ G,\ A)\alpha(F_1,G_1,B)$ *for all* $\alpha\in\{\tilde{\cap},\cap_R,\ \tilde{\cup},\cup_R\}$.

Proof. Straightforward.

Lemma 2. *If $(\Phi,\ U,\ A)$ is a null fuzzy bipolar soft set $(U,\ \Phi,\ A)$ an absolute fuzzy bipolar soft set, and $(F,\ G,\ A)$, $(F_1,\ G_1,\ A)$ are fuzzy bipolar soft sets over U. Then*

1. $(F,\ G,\ A)\tilde{\cup}(F_1,\ G_1,\ A)=(F,\ G,\ A)\cup_R(F_1,\ G_1,\ A)$,
2. $(F,\ G,\ A)\tilde{\cap}(F_1,\ G_1,\ A)=(F,\ G,\ A)\cap_R(F_1,\ G_1,\ A)$,
3. $(F,\ G,\ A)\tilde{\cup}(F,\ G,\ A)=(F,\ G,\ A)\cup_R(F,\ G,\ A)\ =(F,\ G,\ A)$,
4. $(F,\ G,\ A)\tilde{\cap}(F,\ G,\ A)=(F,\ G,\ A)\cap_R(F,\ G,\ A)\ =(F,\ G,\ A)$,
5. $(F,\ G,\ A)\tilde{\cup}(\Phi,\ U,\ A)=(F,\ G,\ A)\cup_R(F,\ G,\ A)\ =(F,\ G,\ A)$,
6. $(F,\ G,\ A)\tilde{\cap}(U,\ \Phi,\ A)=(F,\ G,\ A)\cap_R(U,\ \Phi,\ A)\ =(F,\ G,\ A)$.

Proof. Straightforward.

Lemma 3. *Let $(F,\ G,\ A)$ and $(F_1,\ G_1,\ B)$ be two fuzzy bipolar soft sets over a common universe U. Then the following are true:*

1. *$(F,\ G,\ A)\tilde{\cup}(F_1,\ G_1,\ B)$ is the smallest fuzzy bipolar soft set over U which contains both $(F,\ G,\ A)$ and $(F_1,\ G_1,\ B)$.*
2. *$(F,\ G,\ A)\cap_R(F_1,\ G_1,\ B)$ is the largest fuzzy bipolar soft set over U which is contained in both $(F,\ G,\ A)$ and $(F_1,\ G_1,\ B)$.*

Proof. Straightforward.

Lemma 4. Let $(F,\ G,\ A)$ and $(F_1,\ G_1,\ B)$ be two fuzzy bipolar soft sets over a common universe U. Then

1. $((F,\ G,\ A)\tilde{\cup}(F_1,\ G_1,\ B))^c=(F,\ G,\ A)^c\tilde{\cap}(F_1,\ G_1,\ B)^c$,
2. $((F,\ G,\ A)\tilde{\cap}(F_1,\ G_1,\ B))^c=(F,\ G,\ A)^c\tilde{\cup}(F_1,\ G_1,\ B)^c$,
3. $((F,\ G,\ A)\cup_R(F_1,\ G_1,\ B))^c=(F,\ G,\ A)^c\cap_R(F_1,\ G_1,\ B)^c$,
4. $((F,\ G,\ A)\cap_R(F_1,\ G_1,\ B))^c=(F,\ G,\ A)^c\cup_R(F_1,\ G_1,\ B)^c$.

Proof. Straightforward.

Lemma 5. *Let (F,G,A), (F_1,G_1,B) and $(F_2,\ G_2,\ C)$ be any fuzzy bipolar soft sets over a common universe U. Then*

1. $(F, G, A)\alpha((F_1, G_1, B)\beta(F_2, G_2, C)) = ((F, G, A)\alpha(F_1, G_1, B))\beta((F, G, A)\alpha(F_2, G_2, C))$ where $\alpha \neq \beta$, $\alpha \in \{\cap_R, \cup_R\}$ and $\beta \in \{\cap_R, \cup_R, \tilde{\cup}, \tilde{\cap}\}$
2. $(F, G, A)\tilde{\cup}((F_1, G_1, B)\tilde{\cap}(F_2, G_2, C)) \tilde{\supset}((F, G, A)\tilde{\cup}(F_1, G_1, B))\tilde{\cap}((F, G, A)\tilde{\cup}(F_2, G_2, C))$
3. $(F, G, A)\tilde{\cup}((F_1, G_1, B) \cup_R (F_2, G_2, C)) \tilde{\subset}((F, G, A)\tilde{\cup}(F_1, G_1, B)) \cup_R ((F, G, A)\tilde{\cup}(F_2, G_2, C))$
4. $(F, G, A)\tilde{\cup}((F_1, G_1, B) \cap_R (F_2, G_2, C)) = ((F, G, A)\tilde{\cup}(F_1, G_1, B)) \cap_R ((F, G, A)\tilde{\cup}(F_2, G_2, C))$
5. $(F, G, A)\tilde{\cap}((F_1, G_1, B)\tilde{\cup}(F_2, G_2, C)) \tilde{\subset}((F, G, A)\tilde{\cap}(F_1, G_1, B))\tilde{\cup}((F, G, A)\tilde{\cap}(F_2, G_2, C))$
6. $(F, G, A)\tilde{\cap}((F_1, G_1, B) \cup_R (F_2, G_2, C)) = ((F, G, A)\tilde{\cap}(F_1, G_1, B)) \cup_R ((F, G, A)\tilde{\cap}(F_2, G_2, C))$
7. $(F, G, A)\tilde{\cap}((F_1, G_1, B) \cap_R (F_2, G_2, C))\tilde{\supset}((F, G, A)\tilde{\cap}(F_1, G_1, B)) \cap_R ((F, G, A)\tilde{\cap}(F_2, G_2, C))$.

Proof:

1) For any $e \in A \cap (B \cup C)$, we have following three disjoint cases:

(i) If $e \in A \cap (B - C)$, then

$$(F \cap_R (F_1 \tilde{\cup} F_2))(e) = F(e) \wedge F_1(e)$$
$$(G \cap_R (G_1 \tilde{\cup} G_2))(\neg e) = G(\neg e) \vee G_1(\neg e)$$

and

$$((F \cap_R F_1)\tilde{\cup}(F \cap_R F_2))(e) = (F \cap_R F_1)(e) \vee \emptyset = F(e) \wedge F_1(e)$$
$$((G \cap_R G_1)\tilde{\cup}(G \cap_R G_2))(\neg e) = (G \cap_R G_1)(\neg e) \wedge U = G(\neg e) \vee G_1(\neg e).$$

(ii) *If $e \in A \cap (C - B)$, then*

$$(F \cap_R (F_1 \tilde{\cup} F_2))(e) = F(e) \wedge F_2(e)$$
$$(G \cap_R (G_1 \tilde{\cup} G_2))(\neg e) = G(\neg e) \vee G_2(\neg e)$$

and

$$((F \cap_R F_1)\tilde{\cup}(F \cap_R F_2))(e) = \emptyset \vee (F \cap_R F_2)(e) = F(e) \wedge F_2(e)$$
$$((G \cap_R G_1)\tilde{\cup}(G \cap_R G_2))(\neg e) = U \wedge (G \cap_R G_2)(\neg e) = G(\neg e) \vee G_2(\neg e).$$

(iii) *If $e \in A \cap (B \cap C)$, then*

$$(F \cap_R (F_1 \tilde{\cup} F_2))(e) = F(e) \wedge (F_1(e) \vee F_2(e))$$
$$(G \cap_R (G_1 \tilde{\cup} G_2))(\neg e) = G(\neg e) \vee (G_1(\neg e) \wedge G_2(\neg e))$$

and

$$((F \cap_R F_1)\tilde{\cup}(F \cap_R F_2))(e) = (F \cap_R F_1)(e) \vee (F \cap_R F_2)(e) = (F(e) \wedge F_1(e)) \vee (F(e) \wedge F_2(e)) = F(e) \wedge (F_1(e) \vee F_2(e))$$
$$((G \cap_R G_1)\tilde{\cup}(G \cap_R G_2))(\neg e) = (G \cap_R G_1)(\neg e) \wedge (G \cap_R G_2)(\neg e) = (G(\neg e) \vee G_1(\neg e)) \wedge (G(\neg e) \vee G_2(\neg e)) = G(\neg e) \vee (G_1(\neg e) \wedge G_2(\neg e)).$$

thus

$$(F, G, A) \cap_R ((F_1, G_1, B)\tilde{\cup}(F_2, G_2, C)) = ((F, G, A) \cap_R (F_1, G_1, B))\tilde{\cup}((F, G, A) \cap_R (F_2, G_2, C))$$

Similarly, we can check for the remaining parts.

Example 1. Let U be the set of houses under consideration, and E be the set of parameters, $U = \{h_1, h_2, h_3, h_4, h_5\}$ $E = \{e_1, e_2, e_3, e_4, e_5\} = \{$in the green surroundings, cheap, in good repair, furnished, traditional$\}$. Let $\neg E = \{\neg e_1, \neg e_2, \neg e_3, \neg e_4, \neg e_5\} = \{$ in the commercial area, expensive, in bad repair, non-furnished, modern$\}$.

Suppose that $A = \{e_1, e_2, e_3\}$, $B = \{e_2, e_3, e_4\}$ and $C = \{e_3, e_4, e_5\}$. The fuzzy bipolar soft sets (F, G, A), (F_1, G_1, B) and (F_2, G_2, C) describe the requirements of the houses which Mr. X, Mr. Y and Mr. Z are going to buy respectively.

suppose that

$$F(e_1) = \{h_1/0.3, h_2/0.1, h_3/0.3, h_4/0.1, h_5/0.7\},$$
$$F(e_2) = \{h_1/0.1, h_2/0.9, h_3/0.3, h_4/0.8, h_5/0.2\},$$
$$F(e_3) = \{h_1/0.1, h_2/0.3, h_3/0.3, h_4/0.3, h_5/0.8\},$$
$$G(\neg e_1) = \{h_1/0.4, h_2/0.7, h_3/0.7, h_4/0.7, h_5/0.1\},$$
$$G(\neg e_2) = \{h_1/0.8, h_2/0, h_3/0.5, h_4/0.1, h_5/0.6\},$$
$$G(\neg e_3) = \{h_1/0.7, h_2/0.5, h_3/0.7, h_4/0.6, h_5/0.1\},$$

and

$$F_1(e_2) = \{h_1/0.1, h_2/0.3, h_3/0.6, h_4/0.2, h_5/0.3\},$$
$$F_1(e_3) = \{h_1/0.8, h_2/0.9, h_3/0.5, h_4/0.4, h_5/0.2\},$$
$$F_1(e_4) = \{h_1/0.1, h_2/0.4, h_3/0.3, h_4/0.6, h_5/0.9\},$$

$$G_1(\neg e_2) = \{h_1/0.1,\ h_2/0.3,\ h_3/0.3,\ h_4/0.6,\ h_5/0.6\},$$
$$G_1(\neg e_3) = \{h_1/0.1,\ h_2/0,\ h_3/0.3,\ h_4/0.4,\ h_5/0.6\},$$
$$G_1(\neg e_4) = \{h_1/0.9,\ h_2/0.5,\ h_3/0.5,\ h_4/0.3,\ h_5/0.1\}.$$

and

$$F_2(e_3) = \{h_1/0.1,\ h_2/0.2,\ h_3/0.3,\ h_4/0.1,\ h_5/0.1\},$$
$$F_2(e_4) = \{h_1/0.2,\ h_2/0.2,\ h_3/0.3,\ h_4/0.3,\ h_5/0.2\},$$
$$F_2(e_5) = \{h_1/0.1,\ h_2/0.1,\ h_3/0.3,\ h_4/0.5,\ h_5/0.7\},$$
$$G_2(\neg e_3) = \{h_1/0.7,\ h_2/0.7,\ h_3/0.4,\ h_4/0.7,\ h_5/0.4\},$$
$$G_2(\neg e_4) = \{h_1/0.6,\ h_2/0.5,\ h_3/0.6,\ h_4/0.1,\ h_5/0.6\},$$
$$G_2(\neg e_5) = \{h_1/0.3,\ h_2/0.4,\ h_3/0.4,\ h_4/0.3,\ h_5/0.1\}.$$

let

$$(F,\ G,\ A)\,\tilde{\cup}\,((F_1,\ G_1,\ B)\,\tilde{\cap}\,(F_2,\ G_2,\ C)) = (H_1,\ I_1,\ A \cup B \cup C)$$

and

$$((F,\ G,\ A)\,\tilde{\cup}\,(F_1,\ G_1,\ B))\,\tilde{\cap}\,((F,\ G,\ A)\,\tilde{\cup}\,(F_2,\ G_2,\ C)) = (H_2,\ I_2,\ A \cup B \cup C).$$

then

$$H_1(e_1) = \{h_1/0.3,\ h_2/0.1,\ h_3/0.3,\ h_4/0.1,\ h_5/0.7\},$$
$$H_1(e_2) = \{h_1/0.1,\ h_2/0.9,\ h_3/0.6,\ h_4/0.8,\ h_5/0.3\},$$
$$H_1(e_3) = \{h_1/0.1,\ h_2/0.3,\ h_3/0.3,\ h_4/0.3,\ h_5/0.8\},$$
$$H_1(e_4) = \{h_1/0.1,\ h_2/0.2,\ h_3/0.3,\ h_4/0.3,\ h_5/0.2\},$$
$$H_1(e_5) = \{h_1/0.1,\ h_2/0.1,\ h_3/0.3,\ h_4/0.5,\ h_5/0.7\},$$

and

$$I_1(\neg e_1) = \{h_1/0.4,\ h_2/0.7,\ h_3/0.7,\ h_4/0.7,\ h_5/0.1\},$$
$$I_1(\neg e_2) = \{h_1/0.1,\ h_2/0.0,\ h_3/0.3,\ h_4/0.1,\ h_5/0.6\},$$
$$I_1(\neg e_3) = \{h_1/0.7,\ h_2/0.5,\ h_3/0.4,\ h_4/0.6,\ h_5/0.1\},$$
$$I_1(\neg e_4) = \{h_1/0.9,\ h_2/0.5,\ h_3/0.6,\ h_4/0.3,\ h_5/0.6\},$$
$$I_1(\neg e_5) = \{h_1/0.3,\ h_2/0.4,\ h_3/0.4,\ h_4/0.3,\ h_5/0.1\}.$$

also

$$H_2(e_1) = \{h_1/0.3,\ h_2/0.1,\ h_3/0.3,\ h_4/0.1,\ h_5/0.7\},$$
$$H_2(e_2) = \{h_1/0.1,\ h_2/0.9,\ h_3/0.3,\ h_4/0.8,\ h_5/0.2\},$$
$$H_2(e_3) = \{h_1/0.1,\ h_2/0.3,\ h_3/0.3,\ h_4/0.3,\ h_5/0.8\},$$
$$H_2(e_4) = \{h_1/0.1,\ h_2/0.2,\ h_3/0.3,\ h_4/0.3,\ h_5/0.2\},$$
$$H_2(e_5) = \{h_1/0.1,\ h_2/0.1,\ h_3/0.3,\ h_4/0.5,\ h_5/0.7\},$$

and

$$I_2(\neg e_1) = \{h_1/0.4,\ h_2/0.7,\ h_3/0.7,\ h_4/0.7,\ h_5/0.1\},$$
$$I_2(\neg e_2) = \{h_1/0.8,\ h_2/0.0,\ h_3/0.5,\ h_4/0.1,\ h_5/0.6\},$$
$$I_2(\neg e_3) = \{h_1/0.7,\ h_2/0.5,\ h_3/0.4,\ h_4/0.6,\ h_5/0.1\},$$
$$I_2(\neg e_4) = \{h_1/0.9,\ h_2/0.5,\ h_3/0.6,\ h_4/0.3,\ h_5/0.6\},$$
$$I_2(\neg e_5) = \{h_1/0.3,\ h_2/0.4,\ h_3/0.4,\ h_4/0.3,\ h_5/0.1\}.$$

Clearly $H_1(e_2) \neq H_2(e_2)$ and $I_1(\neg e_2) \neq I_2(\neg e_2)$, so that

$$(F,\ G,\ A)\,\tilde{\cup}\,((F_1,\ G_1,\ B)\,\tilde{\cap}\,(F_2,\ G_2,\ C)) \neq ((F,\ G,\ A)\,\tilde{\cup}\,(F_1,\ G_1,\ B))\,\tilde{\cap}\,((F,\ G,\ A)\,\tilde{\cup}\,(F_2,\ G_2,\ C)).$$

now, if we take

$$(F,\ G,\ A)\,\tilde{\cap}\,((F_1,\ G_1,\ B)\,\tilde{\cup}\,(F_2,\ G_2,\ C)) = (H_3,\ I_3,\ A \cup B \cup C)$$

and

$$((F, G, A)\,\tilde{\cap}\,(F_1, G_1, B))\,\tilde{\cup}\,((F, G, A)\,\tilde{\cap}\,(F_2, G_2, C)) = (H_4,\ I_4,\ A \cup B \cup C)$$

then

$$H_3(e_1) = \{h_1/0.3,\ h_2/0.1,\ h_3/0.3,\ h_4/0.1,\ h_5/0.7\},$$
$$H_3(e_2) = \{h_1/0.1,\ h_2/0.3,\ h_3/0.3,\ h_4/0.2,\ h_5/0.2\},$$
$$H_3(e_3) = \{h_1/0.1,\ h_2/0.3,\ h_3/0.3,\ h_4/0.3,\ h_5/0.2\},$$
$$H_3(e_4) = \{h_1/0.2,\ h_2/0.4,\ h_3/0.3,\ h_4/0.6,\ h_5/0.9\},$$
$$H_3(e_5) = \{h_1/0.1,\ h_2/0.1,\ h_3/0.3,\ h_4/0.5,\ h_5/0.7\},$$

and

$$I_3(\neg e_1) = \{h_1/0.4,\ h_2/0.7,\ h_3/0.7,\ h_4/0.7,\ h_5/0.1\},$$
$$I_3(\neg e_2) = \{h_1/0.8,\ h_2/0.3,\ h_3/0.5,\ h_4/0.6,\ h_5/0.6\},$$
$$I_3(\neg e_3) = \{h_1/0.7,\ h_2/0.5,\ h_3/0.7,\ h_4/0.6,\ h_5/0.4\},$$
$$I_3(\neg e_4) = \{h_1/0.6,\ h_2/0.5,\ h_3/0.5,\ h_4/0.1,\ h_5/0.1\},$$
$$I_3(\neg e_5) = \{h_1/0.3,\ h_2/0.4,\ h_3/0.4,\ h_4/0.3,\ h_5/0.1\}.$$

also

$$H_4(e_1) = \{h_1/0.3,\ h_2/0.1,\ h_3/0.3,\ h_4/0.1,\ h_5/0.7\},$$
$$H_4(e_2) = \{h_1/0.1,\ h_2/0.9,\ h_3/0.3,\ h_4/0.8,\ h_5/0.2\},$$
$$H_4(e_3) = \{h_1/0.1,\ h_2/0.3,\ h_3/0.3,\ h_4/0.3,\ h_5/0.2\},$$

$$H_4(e_4) = \{h_1/0.2,\ h_2/0.4,\ h_3/0.3,\ h_4/0.6,\ h_5/0.9\},$$
$$H_4(e_5) = \{h_1/0.1,\ h_2/0.1,\ h_3/0.3,\ h_4/0.5,\ h_5/0.7\},$$

and

$$I_4(\neg e_1) = \{h_1/0.4,\ h_2/0.7,\ h_3/0.7,\ h_4/0.7,\ h_5/0.1\},$$
$$I_4(\neg e_2) = \{h_1/0.8,\ h_2/0.0,\ h_3/0.5,\ h_4/0.1,\ h_5/0.6\},$$
$$I_4(\neg e_3) = \{h_1/0.7,\ h_2/0.5,\ h_3/0.7,\ h_4/0.6,\ h_5/0.4\},$$
$$I_4(\neg e_4) = \{h_1/0.6,\ h_2/0.5,\ h_3/0.5,\ h_4/0.1,\ h_5/0.1\},$$
$$I_4(\neg e_5) = \{h_1/0.3,\ h_2/0.4,\ h_3/0.4,\ h_4/0.3,\ h_5/0.1\}.$$

Clearly, $H_3(e_2) \neq H_4(e_2)$ and $I_3(\neg e_2) \neq I_4(\neg e_2)$, so that

$$(F,\ G,\ A)\,\tilde{\cap}\,((F_1,\ G_1,\ B)\,\tilde{\cup}\,(F_2,\ G_2,\ C)) \neq ((F,\ G,\ A)\,\tilde{\cap}\,(F_1,\ G_1,\ B))\,\tilde{\cup}\,((F,\ G,\ A)\,\tilde{\cap}\,(F_2,\ G_2,\ C)).$$

similarly we can show that

$$(F,\ G,\ A)\,\tilde{\cup}\,((F_1,\ G_1,\ B)\cup_R(F_2,\ G_2,\ C)) \neq ((F,\ G,\ A)\,\tilde{\cup}\,(F_1,\ G_1,\ B))\cup_R((F,\ G,\ A)\,\tilde{\cup}\,(F_2,\ G_2,\ C))$$

and

$$(F,\ G,\ A)\,\tilde{\cap}\,((F_1,\ G_1,\ B)\cap_R(F_2,\ G_2,\ C)) \neq ((F,\ G,\ A)\,\tilde{\cap}\,(F_1,\ G_1,\ B))\cap_R((F,\ G,\ A)\,\tilde{\cup}\,(F_2,\ G_2,\ C))$$

Now we consider the collection of all fuzzy bipolar soft sets over U and denote it by $FBSS(U)^E$ and let us denote its sub collection of all fuzzy bipolar soft sets over U with fixed set of parameters A by $FBSS(U)_A$. We note that this collection is partially ordered by inclusion. We conclude from above results that:

Proposition 2. $(FBSS(U)^E,\ \tilde{\cap},\cup_R)$ and $(FBSS(U)^E,\ \tilde{\cup},\cap_R)$ are distributive lattices and $(FBSS(U)^E,\cup_R,\ \tilde{\cap})$ and $(FBSS(U)^E,\cap_R,\ \tilde{\cup})$ are their duals respectively.

Proof. Follows from above results.

Proposition 3. $(FBSS(U)^E,\cap_R,\ \tilde{\cup})$ is a bounded distributive lattice, with least element $(\Phi,\ U,\emptyset)$ and greatest element $(U,\ \Phi,\ E)$, while $(FBSS(U)^E,\ \tilde{\cup},\cap_R,(U,\ \Phi,\ E),(\Phi,\ U,\emptyset))$ is its dual.

Proof. Follows from above results.

Proposition 4. $(FBSS(U)_A,\cap_R,\ \tilde{\cup}) = (FBSS(U)_A,\ \tilde{\cap},\cup_R)$ is a bounded distributive lattice, with least element $(\Phi,\ U,\ A)$ and greatest element $(U,\ \Phi,\ A)$.

Proof. Follows from above results.

Proposition 5. Let (F, G, A) and (F_1, G_1, A) be two fuzzy bipolar soft sets over a common universe U. Then

1. $((F,\ G,\ A)^c)^c = (F, G, A)$,
2. $(F,\ G,\ A)\tilde{\subseteq}(F_1,\ G_1,\ A)$ *implies* $(F_1,\ G_1,\ A)^c\tilde{\subseteq}(F,\ G,\ A)^c$.

Proof.

1. is straightforward.
2. If $(F,\ G,\ A)\tilde{\subseteq}(F_1,\ G_1,\ A)$ then

$F(e) \subseteq F_1(e)$ and $G_1(\neg e) \subseteq G(\neg e)$ for all $e \in A$ implies that $(G_1,\ F_1,\ A)\tilde{\subseteq}(G,\ F,\ A)$.

Hence $(F_1,\ G_1,\ A)^c\tilde{\subseteq}(F,\ G,\ A)^c$.

Proposition 6. $(FBSS(U)_A,\cap_R,\cup_R,{}^c,(U,\ \Phi,\ A),(\Phi,\ U,\ A))$ is a De Morgan algebra.

Proof. Straightforward.

Definition 19. For a fuzzy bipolar soft set $(F,\ G,\ A)$ over U, we define a fuzzy bipolar soft set over U, which is denoted by $(F,\ G,\ A)^*$ and given by $(F,\ G,\ A)^* = (F^*,\ G^*,\ A)$ where

$$(F^*(e))(u) = \begin{cases} 0 & \text{if } (F(e))(u) \neq 0 \\ 1 & \text{if } (F(e))(u) = 0 \end{cases}$$

and

$$(G^*(e))(\neg u) = \begin{cases} 1 & \text{if } (G(\neg e))(u) \neq 1 \\ 0 & \text{if } (G(\neg e))(u) = 1 \end{cases}$$

for all $u \in U$ and for all $e \in A$.

Theorem 1. *Let $(F,\ G,\ A)$ be a fuzzy bipolar soft set over U, then the following are true:*

1. $(F,\ G,\ A)\cap_R(F,\ G,\ A)^* = (\Phi,\ U,\ A)$,
2. $(F_1,\ G_1,\ A)\tilde{\subseteq}(F,\ G,\ A)^*$ *whenever* $(F,\ G,\ A)\cap_R(F_1,\ G_1,\ A) = (\Phi,\ U,\ A)$,
3. $(F,\ G,\ A)^*\cup_R(F,\ G,\ A)^{**} = (U,\ \Phi,\ A)$.

Thus $(FBSS(U)_A,\cap_R,\cup_R,{}^*,(U,\ \Phi,\ A),(\Phi,\ U,\ A))$ *is a Stone algebra.*

Proof.

(1) Consider $(F,\ G,\ A)\cap_R(F,\ G,\ A)^*$. For any $e \in A$

$$(F \cap_R F^*)(e) = F(e) \wedge F^*(e)$$

and

$$(G \cap_R G^*)(\neg e) = G(\neg e) \vee G^*(\neg e).$$

$$\Rightarrow$$

$$((F \cap_R F^*)(e))(u)$$

$$= \begin{cases} (F(e))(u) \wedge 0 & \text{if } (F(e))(u) \neq 0 \\ 0 \wedge 1 & \text{if } (F(e))(u) = 0 \end{cases}$$

$$= 0$$

and

$$((G \cap_R G^*))(\neg e)(u)$$

$$= \begin{cases} (G(\neg e))(u) \vee 1 & \text{if } (G(\neg e))(u) \neq 1 \\ 1 \vee 0 & \text{if } (G(\neg e))(u) = 1 \end{cases}$$

$$= 1$$

for all $u \in U$.

Thus $(F, G, A) \cap_R (F, G, A)^* = (\Phi, U, A)$.

(2) If $(F, G, A) \cap_R (F_1, G_1, A) = (\Phi, U, A)$, then $(F(e))(u) \wedge (F_1(e))(u) = 0$ *and* $(G(\neg e))(u) \vee (G_1(\neg e))(u) = 1$ *for all* $u \in U$ $e \in A$. *We have two cases here:*

(i) If (F(e))(u) = 0 then

$$(F^*(e))(u) = 1 \geq (F_1(e))(u) \text{ and}$$

(ii) If (F(e))(u) $\neq$ 0 then

$$(F_1(e))(u) = 0 \leq (F^*(e))(u).$$

Thus $(F_1(e))(u) \leq (F^*(e))(u)$ *for all* $u \in U$.

Again there are two cases:

(i) If (G(¬ e))(u) = 1 then

$$(G^*(\neg e))(u) = 0 \leq (G_1(\neg e))(u) \text{ and}$$

(ii) If (G(¬ e))(u) $\neq$ 1 then

$$(G_1(\neg e))(u) = 1 \geq (G^*(\neg e))(u).$$

So $(G^*(\neg e))(u) \leq (G_1(\neg e))(u)$ for all $u \in U$. This implies that

$$F_1(e) \subseteq F^*(e) \text{ and } G^*(\neg e) \subseteq G_1(\neg e)$$

for all $e \in A$.

Therefore, $(F_1, G_1, A) \tilde{\subseteq} (F, G, A)^*$.

(3) Consider $(F, G, A)^* \cup_R (F, G, A)^{**}$. For any $e \in A$

$$(F^* \cup_R F^{**})(e) = F^*(e) \vee F^{**}(e)$$

and

$$(G^* \cup_R G^{**})(\neg e) = G^*(\neg e) \wedge G^{**}(\neg e).$$

$$\Rightarrow$$

$$((F^*(e))(u) \vee (F^{**}(e))(u)$$

$$= \begin{cases} 0 \vee 1 & \text{if } (F(e))(u) \neq 0 \\ 1 \vee 0 & \text{if } (F(e))(u) = 0 \end{cases}$$

$$= 1$$

and

$$((G^*(e))(u) \wedge (G^{**}(e))(u)$$

$$= \begin{cases} 1 \wedge 0 & \text{if } (G(\neg e))(u) \neq 1 \\ 0 \wedge 1 & \text{if } (G(\neg e))(u) = 1 \end{cases}$$

$$= 0$$

for all $u \in U$.

Thus $(F, G, A)^* \cup_R (F, G, A)^{**} = (U, \Phi, A)$.

4. Application of fuzzy bipolar soft sets in a decision making problem

Decision making is an important factor of all scientific professions where experts apply their knowledge in that area to make decisions wisely. We apply the concept of *fuzzy bipolar soft sets* for modeling of a given problem and then we give an algorithm for the choice of optimal object based upon the available sets of information. Let U be the initial universe and E be a set of parameters. We shall adapt the following terminology afterwards:

Definition 20. Let (F, G, E) be a fuzzy bipolar soft set defined over U. A Comparison table for F is a square table in which the number of rows and number of columns are equal, rows and columns both are labeled by the object names $h_1, h_2, h_3, \ldots, h_n$ of the initial universe U, and the entries are t_{ij}, $i, j = 1, 2, ..., n$, given by

$t_{ij} =$ the number of parameters for which the membership value of h_i exceeds or equal to the membership value of h_j

Clearly, $0 \leq t_{ij} \leq k$, and $t_{ii} = k$, for all i, j where k is the number of parameters present in E. Thus t_{ij} indicates a numerical measure, which is an integer. A Comparison table for G is a square table in which the number of rows and number of columns are equal, rows and columns both are labeled by the object names $h_1, h_2, h_3, \ldots, h_n$ of the initial universe U, and the entries are s_{ij}, $i, j = 1, 2, \ldots, n$, given by

$s_{ij} =$ the number of parameters for which the membership value of h_i dominates or equal to the membership value of h_j

Clearly, $0 \leq s_{ij} \leq k$, and $s_{ii} = k$, for all i, j where k is the number of parameters present in E. Thus s_{ij} also indicates a numerical measure, which is an integer.

Definition 21. The positive row sum and column of an object h_i, denoted by r_i and c_i are calculated by using the formulae,

$$r_i = \sum_{j=1}^{n} t_{ij}, \, c_j = \sum_{i=1}^{n} t_{ij},$$

The negative row sum and column sum of an object h_i, denoted by r'_i and c'_j are calculated by using the formulae,

$$r'_i = \sum_{j=1}^{n} s_{ij}, \, c'_j = \sum_{i=1}^{n} s_{ij}.$$

Definition 22. The positive score P_i of object h_i will be given by:

$$P_i = r_i - c_i$$

while the negative score N_i will be given by:

$$N_i = r'_i - c'_i.$$

The final score S_i of object h_i will be given by:

$$S_i = P_i - N_i$$

for all $j = 1, 2, ..., n$.

We wish to find an object from the set of choice parameters A. We are now giving an algorithm for the choice of best object according to the specifications made by observer and recorded data with the help of a fuzzy bipolar soft set.

Algorithm. The algorithm for the selection of the best choice is given as:

(1) Input the fuzzy bipolar soft set (F, G, E).
(2) Input the set of choice parameters $P \subseteq E$ *and find the reduced fuzzy bipolar soft set* (F, G, P).
(3) Compute the comparison tables for functions F *and* G *respectively*
(4) Compute the positive and negative scores for each object.
(5) Compute the final score.
(6) Find k, *for which* $S_k = \max S_i$.
(7) Then h_k *is the optimal choice object. If* k *has more than one values, then any one of* h_k *'s can be chosen.*

Example 2. Let $U = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8\}$ be the set of candidates who have applied for a job position of Office Representative in Customer Care Centre of a company. Let $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$ = Hard Working, Optimism, Enthusiasm, Individualism, Imaginative, Flexibility, Decisiveness, Self-confidence, Politeness and $\neg E = \{\neg e_1, \neg e_2, \neg e_3, \neg e_4, \neg e_5, \neg e_6, \neg e_7, \neg e_8, \neg e_9\}$ = Negligent, Pessimism, Half-hearted, Dependence, Unimaginative, Rigidity, Indecisiveness, Shyness, Harshness. Here the gray area is obviously the moderate form of parameters. Let the fuzzy bipolar soft sets (F, G, E) describes the Personality Analysis of Candidates as:

F	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
m_1	0.5	0.6	0.8	0.7	0.5	0.4	0.3	0.6	0.8
m_2	0.7	0.7	0.8	0.6	0.8	0.9	0.8	0.7	0.5
m_3	0.6	0.8	0.4	0.1	0.6	0.5	0.4	0.5	0.6
m_4	0.7	0.8	0.6	0.7	0.5	0.4	0.6	0.5	0.6
m_5	0.5	0.4	0.5	0.6	0.7	0.7	0.8	0.6	0.7
m_6	0.5	0.4	0.5	0.6	0.3	0.3	0.2	0.4	0.4
m_7	0.4	0.2	0.4	0.6	0.7	0.6	0.5	0.3	0.2
m_8	0.8	0.7	0.8	0.9	0.6	0.5	0.4	0.6	0.7

And

G	$\neg e_1$	$\neg e_2$	$\neg e_3$	$\neg e_4$	$\neg e_5$	$\neg e_6$	$\neg e_7$	$\neg e_8$	$\neg e_9$
m_1	0.3	0.4	0.1	0.2	0.4	0.4	0.7	0.4	0.1
m_2	0.2	0.1	0.1	0.3	0.2	0.2	0.1	0.2	0.4
m_3	0.4	0.2	0.5	0.6	0.3	0.3	0.5	0.3	0.4
m_4	0.1	0.1	0.3	0.2	0.4	0.3	0.3	0.4	0.3
m_5	0.3	0.5	0.4	0.3	0.1	0.2	0.2	0.4	0.2
m_6	0.5	0.5	0.3	0.3	0.6	0.5	0.8	0.4	0.5
m_7	0.4	0.7	0.6	0.2	0.2	0.2	0.4	0.6	0.8
m_8	0.2	0.1	0.1	0.1	0.3	0.3	0.4	0.3	0.2

(1) Input the fuzzy bipolar soft set (F, G, E).
(2) Input the set of choice parameters $P = \{e_1, e_3, e_4, e_5, e_7, e_8\} \subseteq E$ and find

the reduced fuzzy bipolar soft set (F, G, P) given as:

F	e_1	e_3	e_4	e_5	e_7	e_8
m_1	0.5	0.8	0.7	0.5	0.3	0.6
m_2	0.7	0.8	0.6	0.8	0.8	0.7
m_3	0.6	0.4	0.1	0.6	0.4	0.5
m_4	0.7	0.6	0.7	0.5	0.6	0.5
m_5	0.5	0.5	0.6	0.7	0.8	0.6
m_6	0.5	0.5	0.6	0.3	0.2	0.4
m_7	0.4	0.4	0.6	0.7	0.5	0.3
m_8	0.8	0.8	0.9	0.6	0.4	0.6

G	$\neg e_1$	$\neg e_3$	$\neg e_4$	$\neg e_5$	$\neg e_7$	$\neg e_8$
m_1	0.3	0.1	0.2	0.4	0.7	0.4
m_2	0.2	0.1	0.3	0.2	0.1	0.2
m_3	0.4	0.5	0.6	0.3	0.5	0.3
m_4	0.1	0.3	0.2	0.4	0.3	0.4
m_5	0.3	0.4	0.3	0.1	0.2	0.4
m_6	0.5	0.3	0.3	0.6	0.8	0.4
m_7	0.4	0.6	0.2	0.2	0.4	0.6
m_8	0.2	0.1	0.1	0.3	0.4	0.3

(3) Compute the comparison tables for functions F and G respectively.

(4) Compute the positive and negative scores for each object as given by Table 3 and 4.

(5) Compute the final score given by Table 5.

From Table 5 we find $k = 5$.

Thus m_5 is the best candidate for the position. In case that m_5 can not join the position m_2 may be selected.

5. Bipolar fuzzy soft sets

Let U be an initial universe and E be a set of parameters. Let $BFP(U)$ denotes the set of all bipolar fuzzy sets of U and A, B, C be non-empty subsets of E.

Definition 23. A pair (F, A) is called a bipolar fuzzy soft set over U, where F is a mapping given by $F : A \to BFP(U)$.

Thus a bipolar fuzzy soft set over U gives a parameterized family of bipolar fuzzy subsets of the universe U. For any $e \in A$,

$F(e) = \{(x, \mu^P_{F(e)}, \mu^N_{F(e)}) : x \in U\}$ where $\mu^P_{F(e)} : U \to [0, 1]$ and $\mu^N_{F(e)} : U \to [-1, 0]$ are mappings.

Before proceeding to the further development of theory of bipolar fuzzy soft sets, we give the following interpretations:

Table 1

F	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8
m_1	6	2	3	4	4	6	4	2
m_2	5	6	6	5	6	6	6	4
m_3	3	0	6	2	1	4	3	2
m_4	4	2	5	6	3	6	5	1
m_5	4	2	5	3	6	6	6	3
m_6	1	1	2	0	3	6	4	0
m_7	2	1	4	1	2	3	6	2
m_8	6	3	6	5	4	6	4	6

Table 2

G	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8
m_1	6	2	3	4	4	6	4	1
m_2	5	6	6	4	5	5	5	5
m_3	3	0	6	2	1	4	3	2
m_4	4	2	4	6	4	6	5	2
m_5	4	1	5	3	6	5	5	2
m_6	1	2	2	2	3	6	2	0
m_7	2	2	4	2	2	4	6	2
m_8	6	2	6	4	4	6	5	6

Table 3

	Row sum: r_i	Column sum: c_i	Positive score: P_i
m_1	31	31	0
m_2	44	17	27
m_3	21	37	−16
m_4	32	26	6
m_5	35	29	6
m_6	17	43	−26
m_7	21	38	−17
m_8	40	20	20

Table 4

	Row sum: r'_i	Column sum: c'_i	Negative score: N_i
m_1	30	32	2
m_2	41	17	24
m_3	21	36	−15
m_4	33	27	6
m_5	31	29	2
m_6	18	42	−24
m_7	25	35	−10
m_8	39	20	19

Proposition 7. Let (F, G, A) and (F_1, A) be the fuzzy bipolar and bipolar fuzzy soft sets defined over U respectively. Then (F, G, A) and (F_1, A) are equivalent.

Proof. Let (F, G, A) be a given fuzzy bipolar soft set defined over U. We define a bipolar fuzzy soft set (F_1, A) over U as:

Table 5

	Final Score
m_1	−2
m_2	3
m_3	−1
m_4	0
m_5	4
m_6	−2
m_7	−7
m_8	1

$F_1(e) = \{(x,\ F(e), -G(\neg e)):\ \ x \in U\}$ where $-G(\neg e)(x) = -(G(\neg e)(x))$ for all $e \in A$.

Conversely assume that we are given a bipolar fuzzy soft set (F_1, A) over U. We can define a fuzzy bipolar soft set $(F,\ G,\ A)$ over U in the following manner:

$F(e) = \mu^P_{F_1(e)},\ G(\neg e) = -\mu^N_{F_1(e)}$ for all $e \in A$.

Thus both definitions are equivalent and may be used interchangeably.

Consider the following example:

Example 3. Let $U = \{m_1,\ m_2,\ m_3,\ m_4,\ m_5\}$ be the set of candidates who have applied for a job position of Office Representative in Customer Care Centre of a company. Let $E = \{e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6,\ e_7\}$={Hard Working, Optimism, Enthusiasm, Individualism, Imaginative, Decisiveness, Self-confidence} and $\neg E = \{\neg e_1, \neg e_2, \neg e_3, \neg e_4, \neg e_5, \neg e_6, \neg e_7\}$={Negligent, Pessimism, Half-hearted, Dependence, Unimaginative, Indecisiveness, Shyness}. Here the gray area is obviously the moderate form of parameters. Let the fuzzy bipolar soft sets $(F,\ G,\ E)$ describes the Personality Analysis of Candidates as:

$$F(e_1) = \{m_1/0.5,\ m_2/0.7,\ m_3/0.6,\ m_4/0.7\},$$
$$F(e_2) = \{m_1/0.6,\ m_2/0.7,\ m_3/0.8,\ m_4/0.8\},$$
$$F(e_3) = \{m_1/0.8,\ m_2/0.8,\ m_3/0.4,\ m_4/0.6\},$$
$$F(e_4) = \{m_1/0.7,\ m_2/0.6,\ m_3/0.1,\ m_4/0.7\},$$
$$F(e_5) = \{m_1/0.5,\ m_2/0.8,\ m_3/0.6,\ m_4/0.5\},$$
$$F(e_6) = \{m_1/0.4,\ m_2/0.9,\ m_3/0.5,\ m_4/0.4\},$$
$$F(e_7) = \{m_1/0.3,\ m_2/0.8,\ m_3/0.4,\ m_4/0.6\},$$

and

$$G(\neg e_1) = \{m_1/0.3,\ m_2/0.2,\ m_3/0.4,\ m_4/0.1\},$$
$$G(\neg e_2) = \{m_1/0.4,\ m_2/0.1,\ m_3/0.2,\ m_4/0.1\},$$
$$G(\neg e_3) = \{m_1/0,\ m_2/0.1,\ m_3/0.5,\ m_4/0.3\},$$
$$G(\neg e_4) = \{m_1/0.2,\ m_2/0.3,\ m_3/0.6,\ m_4/0.2\},$$
$$G(\neg e_5) = \{m_1/0.4,\ m_2/0.2,\ m_3/0.3,\ m_4/0.4\},$$
$$G(\neg e_6) = \{m_1/0.4,\ m_2/0.2,\ m_3/0.3,\ m_4/0.3\},$$
$$G(\neg e_7) = \{m_1/0.7,\ m_2/0.1,\ m_3/0.5,\ m_4/0.3\}.$$

Now let's see the corresponding bipolar fuzzy soft set:

$$F_1(e_1) = \{(m_1,\ 0.5,\ -0.3),\ (m_2,\ 0.7,\ -0.2),\ (m_3,\ 0.6,\ -0.4),\ (m_4,\ 0.7,\ -0.1)\},$$
$$F_1(e_2) = \{(m_1,\ 0.6,\ -0.4),\ (m_2,\ 0.7,\ -0.1),\ (m_3,\ 0.8,\ -0.2),\ (m_4,\ 0.8,\ -0.1)\},$$
$$F_1(e_3) = \{(m_1,\ 0.8,\ -0),\ (m_2,\ 0.8,\ -0.1),\ (m_3,\ 0.4,\ -0.5),\ (m_4,\ 0.6,\ -0.3)\},$$
$$F_1(e_4) = \{(m_1,\ 0.7,\ -0.2),\ (m_2,\ 0.6,\ -0.3),\ (m_3,\ 0.1,\ -0.6),\ (m_4,\ 0.7,\ -0.2)\},$$
$$F_1(e_5) = \{(m_1,\ 0.5,\ -0.4),\ (m_2,\ 0.8,\ -0.2),\ (m_3,\ 0.6,\ -0.3),\ (m_4,\ 0.5,\ -0.4)\},$$
$$F_1(e_6) = \{(m_1,\ 0.4,\ -0.4),\ (m_2,\ 0.9,\ -0.2),\ (m_3,\ 0.5,\ -0.3),\ (m_4,\ 0.4,\ -0.3)\},$$
$$F_1(e_7) = \{(m_1,\ 0.3,\ -0.7),\ (m_2,\ 0.8,\ -0.1),\ (m_3,\ 0.4,\ -0.5),\ (m_4,\ 0.6,\ -0.3)\}.$$

It is clear that fuzzy bipolar soft set depicts the information in a better and comprehensive way than bipolar fuzzy soft set. For example, if we read the data of candidate m_1 with fuzzy bipolar soft set $(F,\ G,\ E)$ then he is having 0.6 fuzzy value for optimism and 0.4 fuzzy value for pessimism and if we use the bipolar fuzzy soft set $(F_1,\ E)$ then m_1 is having 0.6 fuzzy value for optimism and −0.4 shows the degree where m_1 is lacking optimism.

6. Conclusion

Our approach in this paper combines the bipolarity, fuzziness and parameterization for defining the fuzzy bipolar soft sets. The idea of fuzzy bipolarity of soft sets has been given. We have also given the definition of bipolar fuzzy soft sets in which the parameterization is done through a single mapping from the set of parameters to the collection of all bipolar fuzzy sets of initial universal set. We have shown through a formation that the two ideas actually coincide with each other and the fuzzy bipolar soft set is similar in working as bipolar fuzzy soft set. Both definitions are equivalent but it is easier and straightforward to model the phenomenon using fuzzy bipolar soft sets because it is a more logical and suitable approach according to the nature of the modeling problems. Future research may be done to explore further aspects of this newly defined structure. Modeling of supported physical phenomenon is our next goal. Another prospective direction is to study the topological structure and similarity measures of fuzzy bipolar soft sets in order to explore for a solid foundation of the research work and development of working methodologies.

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