The chromatic index of multigraphs of order at most 10

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Abstract

The maximum of the maximum degree and the ‘odd set quotients’ provides a well-known lower bound \( \phi(G) \) for the chromatic index of a multigraph \( G \). Plantholt proved that if \( G \) is a multigraph of order at most 8, its chromatic index equals \( \phi(G) \) and that if \( G \) is a multigraph of order 10, the chromatic index of \( G \) cannot exceed \( \phi(G) + 1 \). We identify those multigraphs \( G \) of order 9 and 10 whose chromatic index equals \( \phi(G) + 1 \), thus completing the determination of the chromatic index of all multigraphs of order at most 10.

1. Introduction

We refer the reader to [1] or [3] for all terminology and notation that is not defined in this paper.

Let \( G \) be a multigraph with vertex set \( V(G) \) and edge set \( E(G) \). The chromatic index of \( G \) (denoted by \( \chi'(G) \)) is the minimum number of colors that are required to color the edges of \( G \) in such a way that no two adjacent edges are assigned the same color. Thus, \( \chi'(G) \) is the minimum number of matchings of \( G \) that are required to cover \( E(G) \). Clearly, the maximum degree of \( G \) (denoted by \( \Delta(G) \)) is a lower bound for \( \chi'(G) \). Another lower bound for \( \chi'(G) \) can be derived as follows. We first note that if \( H \) is a multigraph of odd order at least 3 then, \( \chi'(H) \geq \lceil \frac{|E(H)|}{\frac{1}{2}(|V(H)|-1)} \rceil \) since any matching in \( H \) contains at most \( \frac{1}{2}(|V(H)|-1) \) edges. We denote this lower bound on \( \chi'(H) \) by \( t(H) \). Now, for \( S \subseteq V(G) \), denote by \( \langle S \rangle \) the subgraph of \( G \) induced by the vertices in \( S \).

Define \( \Gamma(G) \) by

\[
\Gamma(G) = \max \{ t(\langle S \rangle) : S \subseteq V(G), |S| \geq 3, |S| \text{ odd} \}.
\]

Clearly, \( [\Gamma(G)] \) provides another lower bound for \( \chi'(G) \).

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Combining the two lower bounds, $\Delta(G)$ and $\lceil \Gamma(G) \rceil$ for $\chi'(G)$ we get an improved lower bound $\phi(G)$ for $\chi'(G)$. We have that

$$\chi'(G) \geq \phi(G) = \max\{\Delta(G), \lceil \Gamma(G) \rceil\}.$$ 

Goldberg [2] and Seymour [6] independently conjectured that this improved lower bound is quite tight, in the following sense (Goldberg’s conjecture is somewhat stronger than the one stated here).

**Conjecture 1** (Goldberg [2] and Seymour [6]). For any multigraph $G$, $\chi'(G) \leq \max\{\Delta(G) + 1, \lceil \Gamma(G) \rceil\}$.

We find it convenient to work with the following slightly weaker form of Conjecture 1 that also appeared in [6].

**Conjecture 2** (Seymour [6]). For any multigraph $G$, $\chi'(G) \leq \phi(G) + 1$.

We follow the terminology that was introduced by Seymour and call a multigraph $G$ an $r$-graph if $G$ is $r$-regular and $\Gamma(G) \leq r$. Hence, note that if $G$ is an $r$-graph then $\phi(G) = r$. Plantholt [5] used properties of $r$-graphs to prove that if $G$ is a multigraph of order $n \leq 8$, then $\chi'(G) = \phi(G)$, and that if $G$ is a multigraph of order 9 or 10, then $\chi'(G) \leq \phi(G) + 1$, thus verifying Conjecture 2 for all multigraphs of order at most 10. Plantholt [5] also proved that if $G$ is an $r$-graph of order 10, then unless $G$ is the Petersen graph, there exists a 1-factor $F$ of $G$ such that $G - F$ is an $(r - 1)$-graph.

In this paper we use these results by Plantholt to classify multigraphs of order 9 and 10 according to whether $\chi'(G) = \phi(G)$ or $\chi'(G) = \phi(G) + 1$.

For a multigraph $G$, we write $H \subseteq G$ to mean that $H$ is a subgraph of $G$, and for $v \in V(G)$, we denote by $G - v$ the multigraph obtained from $G$ by deleting the vertex $v$ and all edges incident on $v$. The simple graph underlying $G$, denoted by $G^S$, is the graph obtained from $G$ by replacing all the multiple edges of $G$ by single edges. We denote the Petersen graph by $P$ and define $P^*$ to be the set of all multigraphs $G$ of order 10 that satisfy the following three properties.

1. $G$ is an $r$-graph.
2. $G^S$ is isomorphic to $P$.
3. There exists a 5-cycle in $G$ with an odd number of edges (including multiple edges).

Note that for any multigraph $G \in P^*$, since $G$ has order 10, Plantholt’s [5] theorem implies that $\chi'(G) \leq \phi(G) + 1$. In fact, for any multigraph $G \in P^*$ we must have that $\chi'(G) = \phi(G) + 1 = \Delta(G) + 1$ because if $\chi'(G) = \phi(G) = \Delta(G)$; then the edges of $G$ can be partitioned into 1-factors of $G$, which in turn is impossible because each 1-factor of $G$ must use an even number of edges from the 5-cycle in condition (3) of the definition of $P^*$. In this paper we show that in a sense $P^*$ describes all multigraphs $G$ of order at most 10 for which $\chi'(G) = \phi(G) + 1$. 

In the next section we will prove Theorems 1 and 2 stated below. Theorem 1 is the main result in this paper and Theorem 2 will follow from Theorem 1, thus giving a classification of multigraphs of order at most 10 according to whether $\chi'(G)=\phi(G)$ or $\chi'(G)=\phi(G)+1$.

**Theorem 1.** Let $G$ be an $r$-regular multigraph of order 10. Then,

$$\chi'(G)=\begin{cases} 
\phi(G)+1 & \text{if } G \in P^*, \\
\phi(G) & \text{otherwise}.
\end{cases}$$

**Theorem 2.** Let $G$ be a multigraph of order at most 10. Then,

$$\chi'(G)=\begin{cases} 
\phi(G)+1 & \text{if there exists } G' \in P^* \text{ and } v \in V(G') \text{ such that } \\
G' - v \subseteq G \subseteq G', \\
\phi(G) & \text{otherwise}.
\end{cases}$$

2. Background results

We state some results that will be used to prove Theorems 1 and 2.

The following properties of the Petersen graph (denoted by $P$) are easily verified.

**Proposition 2.1.** There are six different 1-factors in $P$ and each 1-factor contains 0 or 2 edges of each of the 5-cycles of $P$.

**Proposition 2.2.** Let $u, v$ be nonadjacent vertices of $P$. Then $P + uv$ contains one 3-cycle and two 4-cycles. These cycles are edge-disjoint, except for the edge $uv$.

For a multigraph $G$ and for $S \subseteq V(G)$, we denote by $\overline{S}$ the complement of $S$. If $t(\langle S \rangle) = \Delta(G)$ then we say that the subgraph $\langle S \rangle$ is full and if $t(\langle S \rangle) > \Delta(G)$ then we say that the subgraph $\langle S \rangle$ is overfull. The following proposition is easily verified.

**Proposition 2.3.** Let $G$ be a regular multigraph of even order $n$ and $S$ an odd cardinality subset of $V(G)$ with $1 < |S| < (n - 1)$. If $\langle S \rangle$ is full then $\langle \overline{S} \rangle$ is full. If $\langle S \rangle$ is overfull then $\langle \overline{S} \rangle$ is overfull.

The multigraph obtained from $G$ by shrinking $S$, denoted by $G_S$, is defined to be the multigraph with vertex set $V(G_S) = (V(G) - S) \cup \{s^*\}$ and edge set $E(G_S) = E(G - S) \cup \{(u,s^*) : u \in \overline{S} \text{ and } (u,v) \in E(G) \text{ for some } v \in S\}$, where the multiplicity of the edge $(u,s^*)$ equals the number of edges in $G$ from vertex $u$ to vertices in $S$. Theorem 3 [4–6] below states that shrinking a full subset of vertices of an $r$-graph produces an $r$-graph and Theorem 4 [6] below relates the chromatic index of an $r$-graph $G$ to the chromatic indices of the $r$-graph obtained by shrinking a full subset $S$ of vertices and the $r$-graph obtained by shrinking $\overline{S}$.
Theorem 3 (Marcotte [4], Plantholt [5] and Seymour [6]). Let $G$ be an $r$-graph and $S$ a nonempty, proper subset of $V(G)$. If $(S)$ and (hence) $(\bar{S})$ are full subgraphs of $G$ then, $G_S$ and $G_{\bar{S}}$ are $r$-graphs.

Theorem 4 (Seymour [6]). Let $G$ be an $r$-graph and $S$ a nonempty, proper subset of $V(G)$. If $(S)$ and (hence) $(\bar{S})$ are full subgraphs of $G$ then, $\chi'(G) \leq \max\{\chi'(G_S), \chi'(G_{\bar{S}})\}$

The following theorem of Plantholt [5] verifies Conjecture 2 for multigraphs of order at most 10.

Theorem 5 (Plantholt [5]). Let $G$ be any multigraph of order $n$. If $n \leq 8$, $\chi'(G) = \phi(G)$ and if $n \leq 10$, $\chi'(G) \leq \phi(G) + 1$.

Now, suppose that $G$ is an $r$-graph of order 10, $S$ is a subset of $V(G)$, with $1 < |S| < 9$, and $(S)$ and (hence) $(\bar{S})$ are full subgraphs of $G$. Then, by Theorems 3–5 we have that

$$\chi'(G) \leq \max\{\chi'(G_S), \chi'(G_{\bar{S}})\} = \max\{\phi(G_S), \phi(G_{\bar{S}})\} \leq r = \phi(G).$$

Hence, we have the following useful corollary.

Corollary 2.1. Let $G$ be an $r$-graph of order 10 and $S$ a subset of $V(G)$ with $1 < |S| < 9$. If $(S)$ and (hence) $(\bar{S})$ are full subgraphs of $G$, then $\chi'(G) = \phi(G)$.

The following theorem [6] asserts that every $r$-graph contains a 1-factor.

Theorem 6 (Seymour [6]). Every $r$-graph contains a 1-factor.

The following theorem of Plantholt [5] states that for $r$-graphs of order 10, Theorem 6 can be considerably strengthened unless $G$ is the Petersen graph.

Theorem 7 (Plantholt [5]). Let $G$ be an $r$-graph of order 10. Then, unless $G$ is the Petersen graph, $G$ contains a 1-factor $F$ such that $G - F$ is an $(r - 1)$-graph.

The following theorem of Seymour [6] states that in order to prove Conjecture 2, it suffices to prove it for $r$-graphs.

Theorem 8 (Seymour [6]). Let $G$ be any multigraph of order $n$ and let $r = \phi(G)$. Then, $G$ is contained in an $r$-graph of order $n$ if $n$ is even, and $G$ is contained in an $r$-graph of order $(n + 1)$ if $n$ is odd.

3. Proof of the main result

We are now ready to prove our main result.
Recall that we denoted the Petersen graph by $P$ and defined $P^*$ to be the set of all multigraphs $G$ of order 10 that satisfy the following three properties.

1. $G$ is an $r$-graph.
2. $G^S$ is isomorphic to $P^*$.
3. There exists a 5-cycle in $G$ with an odd number of edges (including multiple edges).

Let $G$ be an $r$-graph of order 10. If there exists a subset $S$ of $V(G)$ of odd cardinality, with $1 < |S| < (n - 1)$ such that $(S)$ and (hence) $(\bar{S})$ are full subgraphs of $G$, then we will say that $G$ is shrinkable; otherwise, we will say that $G$ is non-shrinkable. Note that by Corollary 2.1, if $G$ is a shrinkable $r$-graph of order 10 then $\chi'(G) = \phi(G)$.

**Theorem 1.** Let $G$ be an $r$-regular multigraph of order 10. Then,

$$\chi'(G) = \begin{cases} 
\phi(G) + 1 & \text{if } G \in P^*, \\
\phi(G) & \text{otherwise}.
\end{cases}$$

**Proof.** If $G \in P^*$ then since $G$ has order 10, Theorem 5 implies that $\chi'(G) \leq \phi(G) + 1$. Since $G$ is an $r$-regular multigraph with $\phi(G) = r$, if $\chi'(G) = \phi(G)$, the edges of $G$ can be partitioned into 1-factors of $G$. But this is impossible since by Proposition 2.1, each 1-factor of $G$ contains 0 or 2 edges of the 5-cycle in $G$ that has been assumed to contain an odd number of edges (Property (3) in the definition of $P^*$). Hence, if $G \in P^*$ we have that $\chi'(G) = \phi(G) + 1$.

Now, suppose that $G \notin P^*$. We will prove that $\chi'(G) = \phi(G) = r$ in each of the following two cases.

**Case 1:** $G$ is an $r$-graph. Since $G \notin P^*$, Subcases 1(a)–(c) below cover all possibilities for $G$.

- **Subcase 1(a):** $G$ does not contain $P$ as a subgraph. Theorem 7 can be applied repeatedly to obtain a partition of the edges of $G$ into 1-factors, thus proving that $\chi'(G) = \phi(G) = r$.
- **Subcase 1(b):** $G^S$ is isomorphic to $P$ and $G$ does not contain a 5-cycle with an odd number of edges. Again, Theorem 7 can be applied repeatedly to obtain a partition of the edges of $G$ into 1-factors, thus proving that $\chi'(G) = \phi(G) = r$.
- **Subcase 1(c):** $G$ contains $P$ as a subgraph but $G^S$ is not isomorphic to $P$. Theorem 7 implies that there exists a 1-factor $F$ of $G$ such that $(G - F)$ is an $(r - 1)$-graph. By Corollary 2.1 we may assume that both $G$ and $(G - F)$ are non-shrinkable. Also, if $(G - F)^S$ is not isomorphic to $P$, repeated application of Theorem 7 either gives a partition of the edges of $G$ into 1-factors, thus proving that $\chi'(G) = \phi(G) = r$ or we have $k$ 1-factors, $M_1, M_2, \ldots, M_k$ of $G$ such that $G' = (G - M_1 - M_2 - \cdots - M_k)$ is an $(r - k)$-graph and $G'^S$ is isomorphic to $P$.

To summarize, we may assume without loss of generality that $G$ is a non-shrinkable $r$-graph of order 10 that contains $P$ but $G^S$ is not isomorphic to $P$, and there is a 1-factor $F$ of $G$ such that $(G - F)$ is a non-shrinkable $(r - 1)$-graph with $(G - F)^S$ isomorphic to $P$. At this point we first prove the following claim.
Claim. Let $F_1, F_2, \ldots, F_6$ be the six 1-factors of $P$ (isomorphic to $(G-F)^S$) considered as subgraphs of $G$. Then, for some $i$, $1 \leq i \leq 6$, $(G - F_i)$ is an $(r-1)$-graph.

Proof of Claim. First assume that $r \geq 5$. By Theorem 7 $(G-F)$ contains a 1-factor, say $F_k$, such that $(G-F-F_k)$ is an $(r-2)$-graph. Note that $(G-F_k)$ is an $(r-1)$-graph.

Now assume that $r = 4$, so that $(G-F)$ is isomorphic to $P$ and $(G-F-F_i)$ consists of two disjoint 5-cycles for each $i = 1, 2, \ldots, 6$. Suppose for the sake of contradiction that for each $i = 1, 2, \ldots, 6$, $\phi(G - F_i) > (r-1)$. Then, for each $i = 1, 2, \ldots, 6$, $F$ must contain two edges induced by each 5-cycle of $(G - F - F_i)$. Since there are 12 such 5-cycles, $F$ is forced to have at least 24 edges, including multiple counts. But it is straightforward to check that any edge of $F$ can be induced in no more than four 5-cycles of $P$, so $F$ must have at least $\frac{24}{4} = 6$ distinct edges, a contradiction. \Box

Now, without loss of generality suppose that $F_1$ is a 1-factor of $P$ such that $(G - F_1)$ is an $(r-1)$-graph. As before by Corollary 2.1 and Theorem 7 we may assume that $(G - F_1)$ is a non-shrinkable $(r-1)$-graph and that $(G-F_1)^S$ is isomorphic to $P$. We will show that by ‘combining’ the 1-factors $F$ and $F_1$ we can find a 1-factor $F'$ of $G$ such that $(G - F')$ is an $(r-1)$-graph and $(G-F')^S$ is not isomorphic to $P$. Then, iterating this procedure will imply that $\chi'(G) = \phi(G) = r$.

Let $(G-F)^S$ be labelled as in Fig. 1, and assume without loss of generality that $F_1$ consists of the five spokes $u_1v_1, u_2v_2, u_3v_3, u_4v_4, u_5v_5$. Since $(G-F)^S$ is isomorphic to $P$ but $G^S$ is not, $F$ must contain an edge $e$ that has multiplicity one in $G$. Since $F_1$ is a matching of $G-F$, edge $e$ cannot be in $F_1$. Without loss of generality, let $e$ be the edge $u_5v_1$, so that the graph in Fig. 2 is a subgraph of $(G-F_1)^S$. But $(G-F_1)^S$ is isomorphic to $P$, and the graph in Fig. 2 can be embedded within the Petersen graph in only two ways. Thus, $(G-F_1)^S$ must be one of the two graphs in Fig. 3. If $(G-F_1)^S$ is as in Fig. 3(a), let $F' = \{u_1v_2, u_2u_3, v_1v_3, u_4v_4, u_5v_5\}$; if $(G-F_1)^S$ is as in Fig. 3(b), let
In either case, we see that \((G - F')\) has the following three properties.

(i) \((G - F')\) is an \((r - 1)\)-regular multigraph, since \(F'\) is a matching of \(G\),

(ii) \((G - F')\) has no overfull subgraph, because \(|F' - F_1| \leq 3\) and \((G - F_1)\) has no full subgraph. Note that to make both \((S)\) and \((S')\) of \((G - F_1)\) overfull, we would need to replace at least two edges in each.

(iii) \((G - F')^S\) is not isomorphic to the Petersen graph. In Case 3a, \((G - F')^S\) contains the triangle \(u_1v_1u_5\). In Case 3b, \((G - F')^S\) contains the 4-cycle \(u_1u_2v_4v_1\).

The result now follows from these properties of \(F'\).

Case 2: \(G\) is not an \(r\)-graph. Let \(\phi(G) = \min\{r(G)\} = r' > r\). Let \(I(G) = \max\{t(S) : S \subseteq V(G), |S| \geq 3, |S| \text{ odd}\} = t(\langle S^* \rangle)\). Since \(\langle S^* \rangle\) is overfull, by Proposition 2.1, \(\langle S^* \rangle\) is
also overfull and so, without loss of generality, we may assume that \(|S^*| = 3\) or 5. Note that if \(|S^*| = 3\), then \(\tau(S^*)\) is an integer. If \(|S^*| = 5\) and if \(\tau(S^*)\) is not an integer then we can add an edge to \(S^*\) to obtain a multigraph \(G'\) such that \(A(G') \leq (r + 1)\) and \(\Gamma(G') = \tau(S^*; G') = \phi(G)\). Since \(\phi(G') > r\), we have that \(\phi(G') = \phi(G'') = \phi(G)\).

Now, by Theorem 8, there exists an \(r\)-graph \(G^*\) of order 10 such that \(G \subseteq G' \subseteq G^*\) and \(\langle S^* \rangle\) is full in \(G^*\). Hence, by Corollary 2.1 we have that \(\chi'(G^*) = \phi(G^*) = r' = \phi(G)\) and hence, \(\chi'(G) = \phi(G)\).

With the aid of Theorem 8 we now expand Theorem 1 to include the cases of non-regular multigraphs of order 10 and all multigraphs of order 9, thus obtaining a complete classification of multigraphs of order at most 10 according to whether \(\chi'(G) = \phi(G)\) or \(\chi'(G) = \phi(G) + 1\).

**Theorem 2.** Let \(G\) be a multigraph of order at most 10. Then,

\[
\chi'(G) = \begin{cases} 
\phi(G) + 1 & \text{if there exists } G' \in P^* \text{ and } v \in V(G') \text{ such that } G' - v \subseteq G \subseteq G', \\
\phi(G) & \text{otherwise.}
\end{cases}
\]

**Proof.** Suppose that there exists \(G' \in P^*\) and \(v \in V(G')\) such that \(G' - v \subseteq G \subseteq G'\). Let \(\phi(G - v) = r\). By Theorem 8, there exists an \(r\)-graph \(G^*\) of order 10 such that \((G - v) \subseteq G^*\). Clearly, \(G^* = G'\). Hence, \(\phi(G' - v) = \phi(G) = \phi(G')\).

Now, suppose for contradiction that \(\chi'(G) = \phi(G)\). Then, \(\chi'(G' - v) \leq \phi(G) = r\). Consider any coloring \(\mathcal{C}\) of the edges of \((G' - v)\) in \(r\) colors. Note that \(|E((G' - v))| = 4r\). Each color in \(\mathcal{C}\) is therefore absent at exactly one vertex of \((G - v)\). Thus, the coloring \(\mathcal{C}\) of the edges of \((G' - v)\) in \(r\) colors can be extended to a coloring of the edges of \(G'\) in \(r\) colors, giving that \(\chi'(G') = r = \phi(G')\), and contradicting Theorem 1. Thus, \(\chi'(G) = \phi(G) + 1\).

Now, assume that there does not exist \(G' \in P^*\) and \(v \in V(G')\) such that \(G' - v \subseteq G \subseteq G'\). We need to show that \(\chi'(G) = \phi(G)\). If \(G\) has order no more than 8 then \(\chi'(G) = \phi(G)\) by Theorem 5. Thus, by adding an isolated vertex if necessary, we may assume that \(G\) has order 10.

Let \(r = \phi(G)\). By Theorem 8 there exists an \(r\)-graph, say, \(G^*\) of order 10 that contains \(G\). If \(G^* \notin P^*\), \(\chi'(G) = \chi'(G^*) = r\) by Theorem 1. Hence, we may now assume that \(G^* \in P^*\). Note that \(G^*\) cannot contain a shrinkable subgraph because by Theorem 1, \(\chi'(G^*) = (r + 1)\). Since there does not exist \(G' \in P^*\) and \(v \in V(G')\) such that \(G' - v \subseteq G \subseteq G'\), \(G^*\) must contain at least two independent edges, \(e_1 = (x, y)\) and \(e_2 = (w, z)\) that are not in \(G\). Let \(e_3\) and \(e_4\) be the edges \((x, w)\) and \((y, z)\) and let \(G^{**} = G^* - e_1 - e_2 + e_3 + e_4\). Since \(G^*\) contains no shrinkable subgraph, no subgraph of \(G^{**}\) can be overfull (the number of edges of any induced subgraph increases by at most one when going from \(G^*\) to \(G^{**}\)), and hence \(G^{**}\) is also an \(r\)-graph. But since \(e_1, e_2, e_3,\) and \(e_4\) form a 4-cycle and the simple graph underlying \(G^*\) is isomorphic to \(P\), \(G^*\) cannot contain both \(e_3\) and \(e_4\). Therefore, by Proposition 2.2, the simple graph underlying \(G^{**}\) is not isomorphic to \(P\) because at least three edges need to be
removed from $G^* + e_3 + e_4$ to eliminate 3-cycles and 4-cycles. Thus, by Theorem 1, $\chi'(G) = \chi'(G^{**}) = r = \phi(G)$. \qed

References