MAXIMAL L^2 AND POINTWISE HÖLDER ESTIMATES FOR $\Box_b$ ON CR MANIFOLDS OF CLASS C^2

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Introduction

A CR manifold, as first formulated in Kohn-Rossi [KR], is a smooth $2n+1$-dimensional real manifold equipped with an integrable CR structure. It was due to Kohn [K2] who first showed that the so-called Kohn's Laplacian $\Box_b$ on (p,q)-forms satisfies subelliptic estimates on any CR manifold whose Levi form satisfies condition $Y(q)$. The results of Hölder and L^p estimates for $\Box_b$ and the Szegö projection for any smooth CR structures satisfying condition $Y(q)$ have long been established (see [K2, FK, FS, RS, BS, BG, KS]). In this paper we derive maximal L^2 estimates and pointwise Hölder and L^p estimates in the sense of Campanato for harmonic integrals for strongly pseudoconvex CR manifolds or CR manifolds satisfying condition $Y(q)$ (see Folland-Kohn [FK] or Definition 1.2) of class C^2. Our approach is a much more direct method and it also yields new estimates for $\Box_b$ and the Szegö projection.

There are many applications of the estimates for $\Box_b$ with minimal smoothness assumption. For CR manifolds with smooth structures, Boutet de Monvel [Bou] proved that any compact C^∞ smooth strongly pseudoconvex CR manifold of real dimension five is embeddable in $\mathbb{C}^N$ for some large N. Our results will immediately relax the conditions on the smoothness of CR manifolds which will be useful in the construction of bounded holomorphic functions on Kähler manifolds with negative curvature (see [Bla]). Maximal estimates under minimal smoothness assumption are also useful in the local embedding problem of CR structures (see Chapter 12 in [CS]) and the references therein). Estimates for $\Box_b$ with minimal smoothness are also important for studying nonlinear subelliptic operators. But these topics will be discussed elsewhere. In the case of the de Rham complex, the maximal estimates of harmonic integrals for Riemannian manifolds with minimal smoothness assumption have long been obtained from the classical elliptic theory for partial differential equations (see e.g. [Mor]). Previously even the L^2 maximal estimates for $\Box_b$ can only be obtained by the lifting methods used in [RS]. The maximal L^2 estimates can now be obtained directly due to new observation which greatly simplifies earlier approach and yields pointwise Hölder estimates.

In Section 1, we give precise definition of CR manifolds of class C^k, $k \geq 2$. For CR manifolds of class C^2, a complete L^2 theory of $\Box_b$ is proved in Section 2. We first derive maximal estimates in L^2 theory for $\Box_b$ on CR manifolds satisfying condition $Y(q)$. Using only commutators of vector fields, we derive the maximal estimates from the basic subelliptic estimates without pseudodifferential operators. This is achieved based on the simple observation that the vector fields between a "good" and the "bad" directions almost commute. In fact, they commute completely on the model quadric domains. Thus it is actually easier to estimate the *bad* direction first, contrary to the conventional wisdom. We use interpolation to prove estimates in *good* directions (see Proposition 2.2). This makes it possible to obtain maximal L^2 estimates using commutators of vector fields only and under minimal smoothness assumption. Even for smooth CR structures, our method is new. In earlier approaches (see [KN] or [RS]), one always estimates the good directions first and then uses pseudodifferential operators to yield estimates for the bad directions. Due to the transparent nature of our proof, we can keep track of the precise smoothness required in each step.

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in Section 3, we derive optimal pointwise Hölder estimates for $\Box_b$ and the Szegö projection on CR manifolds of class C^2. This is proved using the nonisotropic Campanato spaces and the maximal L^2 theory derived in Section 2. (For elliptic or parabolic equations using this approach, see [Gia, Shl, Wan]). When the CR manifold is smooth with arbitrary codimension, nonisotropic Hölder and L^p estimates for $\Box_b$ have been proved using Campanato spaces by the authors in [SW]. In [SW] we still rely on the known L^2 theory proved before in [K2, KN, RS] using pseudodifferential operators (thus C^∞ smoothness is required) as a starting point. In this paper, we give a much more streamlined proof which allows us to obtain both the regularity for $\Box_b$ and the Szegö projection with minimal regularity assumption.

After our paper was completed, Professor J. Bland kindly sent us a preprint on the related subject (see [BD]).

1. CR structures of class C^2 and some function spaces

A manifold is of class C^k, $k \in \mathbb{N}$, if any two coordinate systems are related by a transformation of class C^k. If M is a manifold of class C^{k+1}, then the tangent bundle of M, $T(M)$, is a C^k manifold. Recall that a vector bundle is of class C^k if the transition functions are of class C^k. Locally it is spanned by C^k sections. A vector field X is of class C^k if its coefficients in any coordinates are in C^k. This is well-defined since changing coordinates will result in multiplication of coefficients with C^k functions. Similarly, we can define differential forms or other tensor fields of class C^k.

Definition 1.1. *Let M be a real manifold of class C^{k+1} of dimension $2n+1$, where $k \in \mathbb{N}$ and $n \geq 1$. The pair $(M, T^{1,0})$ is called a CR manifold of class C^k if M is equipped with a subbundle $T^{1,0}$ of class C^k of the complexified tangent bundle $\mathbb{C}T(M)$ such that the following holds:*

(1) $dim_{\mathbb{C}} T^{1,0} = n$,

(2) $T^{1,0} \cap T^{0,1} = \{0\}$, where $T^{0,1} = \overline{T}^{1,0}$,

(3) $T^{1,0}$ is integrable in the sense that if $L_1,\ L_2$ are C^k sections of $T^{1,0}$, their Lie bracket $[L_1, L_2]$ is a C^{k-1} section of $T^{1,0}$.

We can also use the operator J to define CR structure. Let H be a subbundle of real dimension $2n$ of the real tangent bundle TM and $J : H \to H$, $J^2 = -I$. The tensor J is called integrable if for any X and Y in H, so is $[JX, Y] + [X, JY]$ and $J\{[JX, Y] + [X, JY]\} = [JX, JY] - [X, Y]$. The triple (M, H, J) is called a CR manifold of class C^k if J is integrable and of class C^k. One can check easily that these two definitions agree (see e.g. Jacobowitz [Jac]). For simplicity, we only use M to denote the CR manifold $(M, T^{1,0})$ or (M, H, J). A CR manifold M is called an embedded CR manifold in $\mathbb{C}^N$ if there exists an embedding f from M into $\mathbb{C}^N$ such that the push forward of the CR structure of M is the induced CR structure of $f(M) \subset \mathbb{C}^N$.

On any given CR manifold M of class C^k, $k \geq 1$, we set $\Lambda^{0,q}(M) = (\wedge^q T^{0,1^*})$. Sections in $\Lambda^{0,q}$ are called $(0,q)$-forms on M. We use $C^m_{(0,q)}(M)$ to denote the space of $(0,q)$-forms with C^m coefficients, where $0 \leq m \leq k$. We define $\bar\partial_b : C^k_{(0,q)}(M) \to C^{k-1}_{(0,q+1)}(M)$ by the standard derivation formula similar to the smooth case. Let $k \geq 1$ and ϕ ϕ be a C^1 function. We set $\overline{\partial}_b \phi$ to be defined by

$$\langle \overline{\partial}_b \phi, \overline{L} \rangle = \overline{L}(\phi)$$

for all C^k sections $\overline{L}$ of $T^{0,1}(M)$. We extend $\overline{\partial}_b$ to $C^k_{(0,q)}(M)$ for $q > 0$ as a derivation exactly the same as for the smooth CR manifolds. Let $\theta_{0,q}$ be the projection from $\Lambda^q \mathbb{C}T^*(M)$ onto $\Lambda^{0,q}(M)$. Then $\overline{\partial}_b = \theta_{0,q+1} \circ d$, where d is the exterior derivative on M. It is standard to see that $\bar\partial_b$ so defined satisfies $\overline{\partial}_b^2 = 0$.

On a CR manifold of class C^k, $k \geq 1$, a Riemannian metric of class C^k can be expressed locally as $g = \sum_{i,j} g_{ij} dx_i \otimes dx_j$ with coefficients $g_{i,j}$ in C^k in local coordinates $x_1, \cdots, x_n$. We equip a CR manifold M of class C^k with a Riemannian metric and then extend it as a hermitian metric on

$\mathbb{C}T(M)$ of class C^k such that $T^{1,0}$ and $T^{0,1}$ are orthogonal and isometric under conjugation. We can define the formal adjoint ϑ_b of $\bar{\partial}_b$ such that $\vartheta_b : C^k_{(0,q+1)}(M) \to C^{k-1}_{(0,q)}(M)$ by requiring

$$(\bar{\partial}_b\phi, \psi) = (\phi, \vartheta_b\psi), \quad \phi \in C^k_{(0,q)}(M),\ \psi \in C^k_{(0,q+1)}(M).$$

Note that ϑ_b is a first-order differential operator with C^{k-1} coefficients written in local coordinates.

We define the $\bar{\partial}_b$-Laplacian $\Box_b$ (or Kohn's Laplacian) by

$$\Box_b = \bar{\partial}_b\vartheta_b + \vartheta_b\bar{\partial}_b : C^k_{(0,q)}(M) \to C^{k-2}_{(0,q)}(M).$$

When $k \geq 2$, the operator $\Box_b$ is a system of second-order differential operators with C^{k-2} coefficients in local coordinates. When $k = 1$, we interpret $\Box_b$ as a second-order operator of divergence form with coefficients in C^0.

Let $N(M)$ denote the real 1-dimensional bundle such that

$$\mathbb{C}T(M) = T^{1,0}(M) \oplus T^{0,1}(M) \oplus N(M).$$

We denote the dual bundle of $N(M)$ by $N^*(M)$. Let $\tau \in N^*(M)$, then τ annihilates $T^{1,0} \oplus T^{0,1}$. Thus $N^*(M)$ is called the characteristic bundle.

For a fixed $\tau \in N^*(M)$, the *Levi form* Θ_p is defined as the quadratic form

$$\langle \Theta(L, L'), \tau\rangle_p = \sqrt{-1}\langle [L, \overline{L}'], \tau\rangle_p, \quad L, L' \in T^{1,0}(M).$$

Definition 1.2. *A CR manifold M of class C^1 is said to satisfy condition $Y(q)$ for some $0 \leq q \leq n$ if the Levi form has at least* $\max(n+1-q, q+1)$ *eigenvalues of the same sign or at least* $\min(n+1-q, q+1)$ *eigenvalues of the opposite signs.*

In particular, if the CR structure is strongly pseudoconvex (i.e, the Levi form is positive definite or negative definite), condition $Y(q)$ holds for all $1 \leq q < n$.

Suppose that M is a CR manifold of class C^k, $k \geq 1$, and M satisfies condition $Y(q)$. Let U be a sufficiently small open set in M. We choose a C^k orthonormal basis $L_1, \cdots, L_n$ for $T^{1,0}(U)$. Let $X_i = \mathrm{Re}L_i$ and $X_{n+i} = \mathrm{Im}L_i$, $i = 1, \cdots, n$. Let T be a C^k real vector field such that $X_1, \cdots, X_{2n}, T$ span the whole tangent space in U. Associated with these vector fields, there is a metric μ (see [FS]). This metric μ defines a family of natural nonisotropic balls $B_r(x_0)$ for each point $x_0 \in U$. For r sufficiently small, the ball $B_r(x_0)$ has length comparable to r in the direction of $X_1, \cdots, X_{2n}$ and length comparable to r^2 in the direction T. The size of the ball $\mu(B_r(x_0))$ is comparable to r^{2n+2}.

We also denote by $L^2_{(0,q)}(U)$ the space of $(0, q)$-forms with $L^2(U)$ coefficients. The set $W^s(U)$ denote the usual Sobolev spaces, $0 \leq s \leq k$, i.e., $W^s(U)$ consists of L^2 functions whose sth derivatives in some coordinate system are in L^2 for $1 \leq s \leq k$. This is well defined since different coordinate charts will give equivalent norms. Similarly, $W^s_{(0,q)}(U)$ denote the space of $(0, q)$-forms with $W^s(U)$ coefficients and we use W^s to denote its norm and omit the subscript $(0, q)$. The norms for noninteger s with $0 < s < k$ can be defined by interpolation norms and denoted by $\| \|_s$. We also define the nonisotripic Sobolev spaces $W^k_*(U)$. Define for any $u \in C^k(\overline{U})$,

$$\|u\|^2_{\mathcal{L}(U)} = \sum_{i=1}^n \|L_iu\|^2 + \|u\|^2, \quad \|u\|^2_{\overline{\mathcal{L}}(U)} = \sum_{i=1}^n \|\bar{L}_iu\|^2 + \|u\|^2$$

and

$$\|Xu\|^2 = \sum_{i=1}^n \|\bar{L}_iu\|^2 + \sum_{i=1}^n \|L_iu\|^2 = \sum_{i=1}^{2n} \|X_iu\|^2,$$

where $L_j = X_j + iX_{n+j}$. These norms are well defined since different choices of basis will result in equivalent norms. Thus Xu can be viewed as the gradient of u with respect to the "good" directions. We also define inductively for $k \geq 2$,

$$\|X^k u\|^2 = \sum_{i_1,\cdots,i_k=1}^{2n} \|X_{i_1}\cdots X_{i_k}u\|^2,$$

$$\|u\|^2_{W^k_*(U)} = \sum_{m=1}^{k} \|X^m u\|^2 + \|u\|^2.$$

The space $W^k_*(U)$ is the completion of $u \in C^k(\overline{U})$ under the $W^k_*(U)$ norm. We also define $W^k_{0*}(U)$ as the completion of $C^k_0(U)$ under the $W^k_*(U)$ norm. Let $W^{-1}_*(U)$ be defined as the dual of $W^1_{0*}(U)$.

Let x_0 be a point in a neighborhood U in M. Shrinking U if necessary, we choose normal coordinates $(z,t) = (z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n, t)$ such that $\partial/\partial z_j = X_j + iX_{n+j}$, $1 \leq j \leq n$ near the point $x_0 = 0$. A polynomial in (z,t) is said to be of nonisotropic order m if

$$P(z,t) = \sum_{|I|+|J|+2k\leq m} a_{IJk} z^I \bar{z}^J t^k,$$

where $I = (i_1, \cdots, i_n)$, $J = (j_1, \cdots, j_n)$ are multiindices and $k \geq 0$. A polynomial of degree 1 is a polynomial of order 1 in the z variables. A polynomial of degree two is a polynomial of degree two in z and of degree 1 in t variables.

The following is generalization of Campanato's Hölder spaces in the L^p sense at a point.

Definition 1.3. *A function $u \in \mathcal{L}^{p,m+\alpha}_*(x_0)$ for $m \in \mathbb{N} \cup \{0\}$, $0 \leq \alpha < 1$ and $1 \leq p \leq \infty$, if and only if $u \in L^p(U)$ for some open neighborhood U of x_0 and there exists a nonisotropic m-th order polynomial P_{x_0} such that*

$$\sup_{B_\rho(x_0)\subset U, 0<\rho<\rho_0} \rho^{-\alpha-m}\left\{\frac{1}{\mu(B_\rho(x_0))}\int_{B_\rho(x_0)} |u - P_{x_0}|^p d\mu\right\}^{\frac{1}{p}} < C, \tag{1.1}$$

for some constant C and ρ_0.

When $p = \infty$, (1.1) should be defined as

$$\sup_{B_\rho(x_0)} |u - P_{x_0}| \leq C\rho^{m+\alpha} \quad \text{for every } B_\rho(x_0) \subset U.$$

A function u is said to be in $\mathcal{L}^{p,m+\alpha}_*(U)$ if $u \in \mathcal{L}^{p,m+\alpha}_*(x_0)$ for every $x_0 \in U$ and the constant C in (1.1) can be chosen independent of x_0.

A function $u \in \mathcal{L}^{p,-1+\alpha}_*(x_0)$ if it can be written as a finite sum $\sum X_i f_i$ for some $f_i \in \mathcal{L}^{p,\alpha}_*(x_0)$.

2. Maximal L^2 Theory for $\Box_b$ on CR manifolds of class C^2

In this section we shall prove the subelliptic estimates for $\Box_b$ in the Hilbert spaces for a CR manifold M of class C^m, $m \geq 2$. Let U be a small neighborhood near a point $x_0 \in M$. Let $\overset{o}{C}{}^k_{(0,q)}(U)$ denote the set of $(0,q)$-forms with coefficients in $C^k_0(U)$, i.e., C^k forms with coefficients compactly supported in U. For any $\phi \in \overset{o}{C}{}^1_{(0,q)}(U)$, we set

$$Q_b(\phi,\phi) = \|\bar{\partial}_b\phi\|^2 + \|\vartheta_b\phi\|^2 + \|\phi\|^2. \tag{2.1}$$

Lemma 2.1. *Let M be a CR manifold of real dimension $2n+1$ and of class C^2. Suppose that the Levi form of M satisfies condition $Y(q)$ at a point $x_0 \in M$ for some $0 \leq q \leq n$. Then there exist a sufficiently small neighborhood U of x_0 and $C > 0$ such that*

$$\| \phi \|_{\frac{1}{2}}^2 \leq CQ_b(\phi, \phi), \tag{2.2}$$

$$\|\phi\|_{W_*^1}^2 \leq CQ_b(\phi, \phi), \tag{2.3}$$

uniformly for every $\phi \in \overset{o}{C}{}^1_{(0,q)}(U)$.

Proof. That condition $Y(q)$ holds implies that the real vector fields $X_1, \cdots, X_{2n}$ of the CR structures are of finite type two. Using arguments in Kohn [K2] one can show that (2.3) holds since only C^2 smoothness is required in the arguments. It follows from Hörmander's theorem that (2.2) holds for C^2 manifold. Though the result in [H3] was stated for a weaker norm than the Sobolev $\frac{1}{2}$-norm, it is easy to show that actually (2.2) holds by using the Hardy inequality in the arguments.

Next we prove the *nonisotropic* optimal L^2 estimates for $\Box_b$. We first prove that the solution ϕ is in W_*^3 if $\Box_b\phi$ is in W_1^*.

Proposition 2.2. *Let M be a CR manifold of real dimension $2n+1$ and of class C^3. Suppose that the Levi form of M satisfies condition $Y(q)$ at a point $x_0 \in M$ for some $0 \leq q \leq n$. Then there exist a sufficiently small neighborhood U of x_0 and $C > 0$ such that*

$$\|\phi\|_{W_*^3}^2 \leq C(\|\Box_b\phi\|_{W_*^1}^2 + \|\phi\|^2). \tag{2.4}$$

uniformly for every $\phi \in \overset{o}{C}{}^3_{(0,q)}(U)$.

Proof. From (2.3), it follows that

$$\|\phi\|_{W_*^1}^2 \leq CQ_b(\phi, \phi) \leq C(\|\Box_b\phi\|_{W_*^{-1}}\|\phi\|_{W_*^1} + \|\phi\|^2) \leq C\|\Box_b\phi\|_{W_*^{-1}}^2. \tag{2.5}$$

Next we choose a coordinate system $\{z_1, \cdots, z_n, t\}$ in a neighborhood U near $x_0 = 0$ and a basis $\{L_1, \cdots, L_n, \bar{L}_1, \cdots, \bar{L}_n, T\}$ for $\mathbb{C}TU$ near $x_0 = 0$ such that

$$\begin{cases} T = \frac{\partial}{\partial t}, \\ [T, L_i]|_0 = [T, \bar{L}_i]|_0 = 0. \end{cases} \tag{2.6}$$

If the CR manifold is a hyperquadric hypersurface M_0 in $\mathbb{C}^{n+1}$, one can choose a basis such that the vector fields T and $X_i's$ completely commute. For a general CR manifold M, there exists a nonisotropic scaling such that the manifold M can be scaled into a hyperquadric at a point x_0. Let $x_0 \in M$ and U be a small neighborhood of x_0. Using normal coordinates $(z, t) = (z_1, \cdots, z_n, t)$ and $z_j = x_j + iy_j$, $j = 1, \cdots, n$ on U such that $x_0 = 0$, we can construct a quadric manifold M_0 which osculates M at x_0.

Let $\{\overline{L}_1^0, \cdots, \overline{L}_n^0\}$ be the tangential Cauchy-Riemann equations of the osculating CR manifold M_0 where the quadric hypersurface M_0 has a natural nonisotropic dilation $x = (z, t) \to rx = (rz, r^2t)$. Let B_ϵ be the nonisotropic ball of radius ϵ centered at 0 with respect to the normal coordinates $(z, t) = x$ with nonisotropic dilation $\epsilon x = (\epsilon z, \epsilon^2 t)$. Let $\overline{L}_j^\epsilon$ be the operator corresponding to $\overline{L}_j$ such that for any $f \in C^1(B_\epsilon)$,

$$\overline{L}_j^\epsilon f(\epsilon x) = \epsilon(\overline{L}_j f)(\epsilon x) \quad \text{in } B_1.$$

It follows that

$$\overline{L}^{\epsilon}_j \to \overline{L}^{0}_j, \quad \epsilon \to 0 \quad \text{in } B_1.$$

We have for any good direction X^ϵ_i, X^ϵ_j,

$$\begin{cases} [T, X^\epsilon_i] = O(\epsilon)D, \\ [[T, X^\epsilon_i], X^\epsilon_j] = O(\epsilon)D, \end{cases}$$

for some (isotropic) first-order operator D on B_1. It is easy to see that on $U = B_\epsilon$, one has

$$\begin{cases} [T, X_i] = O(\epsilon)D, \\ [[T, X_i], X_j] = O(\epsilon)D. \end{cases} \tag{2.7}$$

Thus we have that the bad direction T almost commutes with the good directions L_i, $\overline{L}_i$'s in a small neighborhood with these basis.

Starting by substituting ϕ by Tf in (2.3) for $f \in \overset{o}{C}{}^3_{(0,q)}(U)$, one gets

$$\| XTf \|^2 + \| Tf \|^2 = \| Tf \|^2_{W^1_*} \leq CQ_b(Tf, Tf). \tag{2.8}$$

Since

$$\begin{aligned}
&(\bar\partial_b Tf,\ \bar\partial_b Tf) + (\vartheta_b Tf,\ \vartheta_b Tf) \\
&= (\bar\partial_b Tf,\ T\bar\partial_b f) - (\bar\partial_b Tf,\ \left[T,\ \bar\partial_b\right] f) \\
&\quad + (\vartheta_b Tf,\ T\vartheta_b f) - (\vartheta_b Tf,\ [T,\ \vartheta_b]\, f) \\
&= (Tf,\ \vartheta_b T\bar\partial_b f) - (\bar\partial_b Tf,\ \left[T,\ \bar\partial_b\right] f) \\
&\quad + (Tf,\ \bar\partial_b T\vartheta_b f) - (\vartheta_b Tf,\ [T,\ \vartheta_b]\, f) \\
&= (Tf,\ T\vartheta_b\bar\partial_b f) - (\bar\partial_b Tf,\ \left[T,\ \bar\partial_b\right] f) \\
&\quad + (Tf,\ T\bar\partial_b\vartheta_b f) - (\vartheta_b Tf,\ [T,\ \vartheta_b]\, f) \\
&\quad + (Tf,\ [\vartheta_b,\ T]\, \bar\partial_b f) + (Tf,\ \left[\bar\partial_b,\ T\right] \vartheta_b f) \\
&= (T'Tf,\ \Box_b f) + O(\epsilon)(\| f \|_1 (\| \bar\partial_b Tf \| + \| \vartheta_b Tf \|)) + \mathcal{R},
\end{aligned}$$

where T' is the adjoint of T and

$$\mathcal{R} = (Tf,\ [\vartheta_b,\ T]\, \bar\partial_b f) + (Tf,\ \left[\bar\partial_b,\ T\right] \vartheta_b f).$$

The term $(Tf,\ [\vartheta_b,\ T]\, \bar\partial_b f)$ in $\mathcal{R}$ can be estimated by

$$\begin{aligned}
&|(Tf,\ [\vartheta_b,\ T]\, \bar\partial_b f)| \\
&= |(Tf,\ \bar\partial_b\, [\vartheta_b,\ T]\, f) + (Tf,\ \left[[\vartheta_b,\ T],\ \bar\partial_b\right] f)| \\
&= |(\vartheta_b Tf,\ [\vartheta_b,\ T]\, f) + (Tf,\ \left[[\vartheta_b,\ T],\ \bar\partial_b\right] f)|.
\end{aligned}$$

From (2.7), the term $|(\vartheta_b Tf,\ [\vartheta_b,\ T]\, f)|$ is estimated by $O(\epsilon)(\| f \|_1 \| \vartheta_b Tf \|)$.

To estimate the term $(Tf,\ \left[[\vartheta_b,\ T],\ \bar\partial_b\right] f)$, we write $\left[[\vartheta_b,\ T],\ \bar\partial_b\right]$ as a combination of terms of the type $[[X_i,\ T],\ X_j]$. Thus we have from (2.7),

$$|(Tf,\ \left[[\vartheta_b,\ T],\ \bar\partial_b\right] f)| \leq C\epsilon \|f\|_1.$$

Similarly, one can estimate the term $(Tf,\ \left[\bar\partial_b,\ T\right] \vartheta_b f)$ in $\mathcal{R}$. We have

$$|\mathcal{R}| \leq C\epsilon(\| f \|_1 \| \bar\partial_b Tf \| \ + \| f \|_1 \| \vartheta_b Tf \| \ + \| f \|^2_1)$$

Using condition $Y(q)$, we can write $T' = -T$ as a linear combination of vectors of the type $X_iX_j - X_jX_i$ modulo the good directions. Using integration by parts, one gets

$$(T'Tf,\ \Box_b f) \le C\|XTf\|(\|X\Box_b f\| + \|\Box_b f\|) \le \epsilon\|XTf\|^2 + C_\epsilon(\|X\Box_b f\|^2 + +\|\Box_b f\|^2).$$

Thus we have

$$Q_b(Tf, Tf) \le C_\epsilon(\|X\Box_b f\|^2 + +\|\Box_b f\|^2) + \epsilon(\|Tf\|^2_{W^1_*} + \|\ f\ \|^2_1).$$

Choosing ϵ sufficiently small, we have from (2.3) and (2.8) that

$$\begin{aligned} \|Tf\|^2_{W^1_*} + \|f\|^2_1 &\le C_\epsilon(\|X\Box_b f\|^2 + +\|\Box_b f\|^2) + \epsilon(\|Tf\|^2_{W^1_*} + \|\ f\ \|^2_1) \\ &\le C\left(\|X\Box_b f\|^2 + \|\Box_b f\|^2 + \|\ f\ \|^2\right). \end{aligned} \tag{2.9}$$

since the term $\epsilon(\|Tf\|^2_{W^1_*} + \|\ f\ \|^2_1)$ can be absorbed by the left-hand side.

Substituting T by X in (2.8), we obtain easily using the above argument

$$\|X^2 f\|^2 \le C\left(\|X\Box_b f\|^2 + \|\Box_b f\|^2 + \|\ f\ \|^2\right). \tag{2.10}$$

Next we substitute Tu by X^2u in (2.8), i.e., forms of the type X_iX_ju. We have

$$\|\ X^3 f\ \|^2 + \|\ X^2 f\ \|^2 = \|\ X^2 f\ \|^2_{W^1_*} \le CQ_b(X^2 f, X^2 f). \tag{2.11}$$

Repeating the same argument as above and using (2.11), we obtain

$$\begin{aligned} &(\bar\partial_b X^2 f,\ \bar\partial_b X^2 f) + (\vartheta_b X^2 f,\ \vartheta_b X^2 f) \\ &= (X^2 f,\ X^2\vartheta_b\bar\partial_b f) - (\bar\partial_b X^2 f,\ \left[X^2,\ \bar\partial_b\right] f) \\ &\quad + (X^2 f,\ X^2\bar\partial_b\vartheta_b f) - (\vartheta_b X^2 f,\ \left[X^2,\ \vartheta_b\right] f) \\ &\quad + (X^2 f,\ \left[\vartheta_b,\ X^2\right]\bar\partial_b f) + (X^2 f,\ \left[\bar\partial_b,\ X^2\right]\vartheta_b f) \\ &\le \|X^3 f\|\|X\Box_b f\| + C(\|\ X^2 f\ \| + \|XTf\|)(\|\ \bar\partial_b X^2 f\ \| + \|\ \vartheta_b X^2 f\ \|) + \mathcal{R}, \end{aligned}$$

where we only need to estimate the terms in $\mathcal{R}$ of the form

$$(X^2 f,\ \left[\vartheta_b,\ X^2\right]\bar\partial_b f) + (X^2 f,\ \left[\bar\partial_b,\ X^2\right]\vartheta_b f).$$

Writing

$$\begin{aligned} &|(X^2 f,\ \left[\vartheta_b,\ X^2\right]\bar\partial_b f)| \\ &= |(X^2 f,\ \bar\partial_b\left[\vartheta_b,\ X^2\right] f) + (X^2 f,\ \left[\left[\vartheta_b,\ X^2\right],\ \bar\partial_b\right] f)| \\ &= |(\vartheta_b X^2 f,\ \left[\vartheta_b,\ X^2\right] f) + (X^2 f,\ \left[\left[\vartheta_b,\ X^2\right],\ \bar\partial_b\right] f)|, \end{aligned} \tag{2.12}$$

the first term in (2.12) is estimated by

$$|(\vartheta_b X^2 f,\ \left[\vartheta_b,\ X^2\right] f)| \le C\|\vartheta_b X^2 f\|(\|X^2 f\| + \|XTf\|).$$

We write $\left[\left[\vartheta_b,\ X^2\right],\ \bar\partial_b\right] u = T^2 u + \cdots$ where the dots represent terms of the type X^2u and XTu, which have already been estimated by (2.9) and (2.10). It remains to estimate terms of the form $(X^2 f, T^2 f)$. Expressing $T = X_iX_j - X_jX_i$ modulo good directions, integration by parts gives

$$|(X^2 f, T^2 f)| \le \|X^3 f\|(\|XTf\| + \|X^2 f\|).$$

Thus

$$|\mathcal{R}| \le C(\|\ Xf\ \|_1\ (\|\ \bar\partial_b X^2 f\ \|\ \ + \|\vartheta_b X^2 f\ \|\ \ + \|\ X^3 f\ \|).$$

We have

$$\begin{aligned} \|X^3 f\|^2 &\le C_\epsilon(\|X\Box_b f\|^2 + \|Xf\|^2_1 + \|f\|^2_1)\ \ + \epsilon\|X^3 f\|^2 \\ &\le C\left(\|X\Box_b f\|^2 + \|\Box_b f\|^2 + \|\ f\ \|^2\right). \end{aligned} \tag{2.13}$$

This finishes the proof for (2.4).

Theorem 2.3. *Let M be a CR manifold of real dimension $2n+1$ and of class C^m, where $m \geq 2$. Suppose that the Levi form of M satisfies condition $Y(q)$ at a point $x_0 \in M$ for some $0 \leq q \leq n$. Then there exist a sufficiently small neighborhood U of x_0 and $C > 0$ such that*

$$\|\phi\|^2_{W^k_*} \leq C(\|\Box_b \phi\|^2_{W^{k-2}_*} + \|\phi\|^2), \quad k = 1, 2 \cdots m. \tag{2.14}$$

uniformly for every $\phi \in \overset{o}{C}{}^k_{(0,q)}(U)$.

Proof. We will prove (2.14) by induction on m. To prove the theorem for $m = 2$, we first note that from (2.5), (2.14) holds for $k = 1$. Notice that for a C^2 CR structure, C^3 functions are well defined. Also for a fixed coordinate system, the norm $X^3 u$ is well defined for C^2 vector fields since only two derivatives are differentiated. Thus we can carry out the arguments used in the proof of Proposition 2.2 to show that (2.14) holds for $k = 3$ under some fixed coordinates. Notice that in the arguments, at most two derivatives of the coefficients of the vector fields are used. But only the W^2_*-norms are well-defined (i.e., invariant of the coordinates). Since W^k_*'s are interpolation spaces (see e.g. Lions-Magenes [LM]), we have $(\Box_b + I)^{-1}$ is a bounded operator from W^{k-2}_* to W^k_* when $k = 1$ and $k = 3$. Using interpolation between W^1_* and W^3_* as well as W^{-1}_* and W^1_*, we have that $(\Box_b + I)^{-1}$ is bounded from W^{s-2}_* to W^s_* for all $1 \leq s \leq 3$. Thus (2.14) also holds for $k = 2$ also. This proves the theorem for $m = 2$.

Next we assume that $m \geq 3$ and assume that (2.14) holds for $k = 1, 2, \cdots, m-1$. Let $\Box_b f = g \in W^{m-2}_*$, $m \geq 3$. From the induction hypothesis, we already have that $f \in W^{m-1}_*$. We claim that

$$X^{m-2} T f \in L^2. \tag{2.15}$$

To prove (2.15), we observe that

$$T \Box_b f = \Box_b T f + [T, \Box_b] f = T g \in W^{m-4}_*. \tag{2.16}$$

Using

$$[T, \Box_b] = [T, \bar{\partial}_b]\vartheta_b + \bar{\partial}_b [T, \vartheta_b] + [T, \vartheta_b]\bar{\partial}_b + \vartheta_b [T, \bar{\partial}_b],$$

the commutator $[T, \Box_b]$ involves terms of the form XT or X^2. Thus

$$\Box_b T f = -[T, \Box_b] f + T g \in W^{m-4}_*. \tag{2.17}$$

From induction, we have $Tf \in W^{m-2}_*$. This proves the claim (2.15). Next we use (2.15) to show $\Box_b X f = -[X, \Box_b] f + X g \in W^{m-3}_*$. By induction we conclude that $Xf \in W^{m-1}_*$ and that $f \in W^m_*$. This proves (2.14) for m. By induction, Theorem 2.3 is proved.

Theorem 2.4. *Let M be a CR manifold of class C^m, where $m \geq 2$ with real dimension $2n+1$. Suppose that the Levi form of M satisfies condition $Y(q)$ at a point $x_0 \in M$ for some $0 \leq q \leq n$. Then there exist a sufficiently small neighborhood U of x_0 and $C > 0$ such that*

$$\|f\|^2_{W^k} \leq C(\|\Box_b f\|^2_{W^{k-1}} + \|f\|^2), \quad k = 1, 2 \cdots m. \tag{2.18}$$

uniformly for every $f \in \overset{o}{C}{}^k_{(0,q)}(U)$.

Proof. We prove (2.18) by induction on m. Using (2.14) for $m = 2$, (2.18) holds for $k = 1$. For $k = 2$, we use the previous argument to conclude that $Tf \in W^2_*$ and $f \in W^2$. This proves (2.18) for $m = 2$.

For $m \geq 3$, we assume that (2.18) holds for $m - 1$. Let $\Box_b f = g$ and let $g \in W^{m-1}$. We shall prove

$$X^{m-l} T^l f \in L^2, \quad \text{for } l = 1, \cdots, m. \tag{2.19}$$

From the induction hypothesis and Theorem 2.3, we have $f \in W^{m-1} \cap W_*^{m+1}$. Thus (2.19) holds for $l = 1$. To prove (2.19) for $m \geq l \geq 2$, we write

$$\Box_b T^l f = -[T^l, \Box_b]f + T^l g. \tag{2.20}$$

The term $T^l g$ is in $W^{m-1-l} \subset W_*^{m-1-l}$. The commutator $[T^l, \Box_b]$ can be written as

$$[T^l, \Box_b] = [T^l, \bar{\partial}_b]\vartheta_b + \bar{\partial}_b[T^l, \vartheta_b] + [T^l, \vartheta_b]\bar{\partial}_b + \vartheta_b[T^l, \bar{\partial}_b]. \tag{2.21}$$

Thus $[T^l, \Box_b]$ consists terms of the form $T^l X$ and terms of the form $T^{l-1}X^2$ type. By an induction procedure on l, we may assume that (2.19) holds for $l-1$. This implies that

$$T^l X f = T(T^{l-1}Xf) \in T(W_*^{m-l}) \subset W_*^{m-l-2}. \tag{2.22}$$

From (2.20)-(2.22), we have $\Box_b T^l f \in W_*^{m-l-2}$. Using Theorem 2.3 to the equation (2.20), it follows that $T^l f \in W_*^{m-l}$. This proves (2.19) for all $l \leq m$. We have proved that $f \in W^m$ and (2.18) holds. By induction, Theorem 2.4 is proved.

We denote by $\mathcal{H}_{(0,q)}(M) = \mathrm{Ker}(\Box_b)$, the harmonic space of $(0,q)$-forms on M. Let $\mathcal{S}_{(0,q)}$ be the projection operator from $L^2_{(0,q)}(M)$ to $\mathcal{H}^b_{(0,q)}(M)$. The projection $\mathcal{S}_{(0,0)}$ is called the Szegö projection, denoted simply by $\mathcal{S}$. From Theorem 2.3 and 2.4, we have the following strong Hodge decomposition theorem for $\Box_b$.

Theorem 2.5. *Let M be a compact CR manifold of class C^m, $m \geq 2$. Suppose that M satisfies condition $Y(q)$ for some $1 \leq q \leq n$. There exists a compact operator $G_b : L^2_{(0,q)}(M) \to L^2_{(0,q)}(M)$ such that*

1. *For any $f \in L^2_{(0,q)}(M)$, $f = \overline{\partial}_b \overline{\partial}_b^* G_b f + \overline{\partial}_b^* \overline{\partial}_b G_b f + \mathcal{S}_{(0,q)} f$.*
2. *$G_b \Box_b = \Box_b G_b = I - \mathcal{S}_{(0,q)}$ on Dom$(\Box_b)$. $G_b \mathcal{S}_{(0,q)} = \mathcal{S}_{(0,q)} G_b = 0$.*
3. *$G_b(W^s(M)) \subset W^{s+1}(M)$, $s = 0, \cdots, m-1$.*
4. *$G_b(W_*^s(M)) \subset W_*^{s+2}(M)$, $s = -1, 0, \cdots, m-2$.*

Corollary 2.6. *Under the same assumption as in Theorem 2.5, we have $\mathcal{H}^b_{(0,q)}(M) \subset W^m_{(0,q)}$ is finite dimensional. For any $(0,q)$-form $f \in W_*^s(M)$, $s \leq m-1$, with $\overline{\partial}_b f = 0$ and $\mathcal{S}_{(0,q)} f = 0$, there is an unique $(0,q-1)$-form $\phi \in W_*^{s+1}(M)$ of $\overline{\partial}_b \phi = f$ with $\phi \perp$ Ker$(\overline{\partial}_b)$.*

Corollary 2.7. *Let M be a compact strongly pseudoconvex CR manifold of class C^m, $m \geq 2$, with real dimension $2n+1$ where $n \geq 2$. Then the Szegö projection $\mathcal{S}$ on M is given by*

$$\mathcal{S} = I - \overline{\partial}_b^* G_b \overline{\partial}_b \tag{2.23}$$

and $\mathcal{S}(W_*^m(M)) \subset W_*^m(M)$. Furthermore, the Szegö projection $\mathcal{S}$ is bounded from $W^m(M)$ to $W^{m-2}(M)$.

Remarks: When M is C^∞-smooth, (1)-(3) in Theorem 2.5 are proved in [K2] and [KN] while (4) and the estimates in W_*^m in Corollaries 2.6, 2.7 in Theorem 2.5 is proved in [FS] and [RS] using pseudodifferential operators.

3. Pointwise Hölder estimates for $\Box_b$ and the Szegö projection

Let M be a CR manifold satisfying condition $Y(q)$ for some q, $0 \leq q \leq n$. Let $x_0 \in M$ and U be a small neighborhood of x_0. We now analyze $\Box_b$ in the normal coordinates in a small ball B_ϵ in $\mathbb{R}^{2n+1}$ where $0 < \epsilon < \epsilon_0$ for a fixed ϵ_0. Setting $(z,t) = x$ and $\epsilon x = (\epsilon z, \epsilon^2 t)$ as before, let $\overline{L}_j^\epsilon$ be the operator on B_1 corresponding to $\overline{L}_j$ on B_ϵ by scaling arguments. We define a CR structure $T_\epsilon^{1,0}$ which is spanned by sections $\overline{L}_1^\epsilon, \cdots, \overline{L}_n^\epsilon$.

Let $\bar\partial_b^\epsilon$ be the complex associated with the CR structure $T_\epsilon^{1,0}$ and $\Box_b^\epsilon$ be the corresponding $\bar\partial_b^\epsilon$-Laplacian. If $\Box_b u = f \quad \text{in } B_\epsilon$, then by scaling, we have

$$\Box_b^\epsilon u(\epsilon x) = \epsilon^2 f(\epsilon x) \quad \text{in } B_1,$$

and

$$\Box_b^\epsilon \to \Box_b^0, \quad \epsilon \to 0.$$

Thus the operator $\Box_b$ can be scaled to the scaling invariant $\bar\partial_b$-Laplacian on M_0. Let Q_b^ϵ denote the energy form corresponding to $\bar\partial_b^\epsilon$

$$Q_b^\epsilon(\phi,\phi) = \|\bar\partial_b^\epsilon\phi\|^2 + \|\vartheta_b^\epsilon\phi\|^2 + \|\phi\|^2.$$

The norm $\| \ \|_{W^1_{\epsilon*}} = \| \ \|_{\mathcal{L}_\epsilon} + \| \ \|_{\overline{\mathcal{L}}_\epsilon} + \| \ \|$ corresponds to the W^1_* norm with respect to $T_\epsilon^{1,0}$. Since $\Box_b$ satisfies condition $Y(q)$ at $x_0 \in M$, using Lemma 2.1, there is a sufficiently small $\epsilon_0 > 0$ such that for every $\epsilon < \epsilon_0$,

$$\| \phi \|^2 \le C \| \Box_b^\epsilon \phi \|^2, \tag{3.1}$$

$$\| \phi \|^2_{\frac{1}{2}} \le C Q_b^\epsilon(\phi,\phi), \tag{3.2}$$

$$\| \phi \|^2_{\mathcal{L}_\epsilon} + \| \phi \|^2_{\overline{\mathcal{L}}_\epsilon} \le C Q_b^\epsilon(\phi,\phi), \tag{3.3}$$

$$\| \phi \|_1^2 \le C(\|\Box_b^\epsilon\phi\|^2 + \|\phi\|^2), \tag{3.4}$$

for every $\phi \in \overset{o}{C}{}^2_{(0,q)}(B_1(0))$. The constant C can be chosen independent of ϵ if ϵ_0 is chosen sufficiently small.

By identifying a $(0,q)$-form as a vector with respect to the fixed basis, we can compare the norms of $(0,q)$-forms on M with respect to different CR structure $T_\epsilon^{1,0}$ for different ϵ. In the following, the notation $⨍_B$ is used to denote $\frac{1}{\mu(B)}\int_B$.

Lemma 3.1. *Let M be a CR manifold of class C^2 with real dimension $2n+1$. Suppose that the Levi form of M satisfies condition $Y(q)$ at x_0, where $0 \le q \le n$. For any $\eta > 0$, there exists $\delta > 0$ such that if $u \in W^1_{\epsilon*}(B_1(0)))$ is a solution of the inhomogeneous equation $\Box_b^\epsilon u = f$ in $B_1(0)$, where $0 \le \epsilon < \epsilon_0$, $⨍_{B_1} |u|^2 \, d\mu \le 1$, $⨍_{B_1} |f|^2 \le \delta$, then there exists a $(0,q)$-form h in $B_{\frac{1}{2}}$ satisfying $\Box_b^0 h = 0$ and*

$$⨍_{B_{\frac{1}{2}}} |u-h|^2 \le \eta. \tag{3.5}$$

Proof. Suppose that the lemma is not true. Then there exist an $\eta_0 > 0$ and a sequence of $(0,q)$-forms u_n and f_n with $\Box_b^{\epsilon_n} u_n = f_n$ on $B_1(0)$, where $0 \le \epsilon_n \le \frac{1}{n}$, $⨍_{B_1} |u_n|^2 \, d\mu \le 1$ and $⨍_{B_1} |f_n|^2 \, d\mu \le \frac{1}{n}$ such that

$$⨍_{B_{\frac{1}{2}}} |u_n - h|^2 > \eta_0 \tag{3.6}$$

for every h satisfying $\Box_b^0 h = 0$ in $B_{\frac{1}{2}}(0)$. Since

$$\fint_{B_1} |u_n|^2 \, d\mu \le 1,$$

there exists a subsequence u_n converges weakly to some element u_∞ in $L^2(B_1)$. Let ζ be a cut-off function such that $\zeta = 1$ on $B_{\frac{1}{2}}$ and $\zeta \in C_0^\infty(B_1)$. Using (3.4) applied to ζu_n, we have

$$\| u_n \|^2_{W^1(B_{\frac{1}{2}})} \le C \quad \text{for all } n. \tag{3.7}$$

Thus there exists a subsequence, still denoted by u_n, which converges strongly in $L^2(B_{\frac{1}{2}})$ to u_∞.

We claim that $\Box_b^0 u_\infty = 0$ in $B_{\frac{1}{2}}$ in the distribution sense. It follows from (3.7) that (a subsequence of) $\bar\partial_b^{\epsilon_n} u_n$ converges weakly to some form $g_1 \in L^2(B_{\frac{1}{2}})$ and $\vartheta_b^{\epsilon_n} u_n$ converges to some form $g_2 \in L^2(B_{\frac{1}{2}})$ weakly. It is easy to check that $g_1 = \bar\partial_b^0 u_\infty$ and $g_2 = \vartheta_b^0 u_\infty$. Let $\phi \in \overset{o}{C}{}^1_{(0,q)}(B_{\frac{1}{2}})$ be a test form. We have

$$(\bar\partial_b^{\epsilon_n}\phi, \bar\partial_b^{\epsilon_n} u_n) + (\vartheta_b^{\epsilon_n}\phi, \vartheta_b^{\epsilon_n} u_n) = (\phi, f_n).$$

Letting $n \to \infty$, using the fact that $\bar\partial_b^{\epsilon_n}\phi \to \bar\partial_b^0\phi$ and $\vartheta_b^{\epsilon_n}\phi \to \vartheta_b^0\phi$ strongly, it follows that

$$(\bar\partial_b^0\phi, \bar\partial_b^0 u_\infty) + (\vartheta_b^0\phi, \vartheta_b^0 u_\infty) = 0.$$

This proves that $\Box_b^0 u_\infty = 0$ in $B_{\frac{1}{2}}$ in the distribution sense. But this implies that

$$\fint_{B_{\frac{1}{2}}} |u_n - u_\infty|^2 \to 0,$$

which is a contradiction to (3.6).

Lemma 3.2. *Let M be a CR manifold of class C^2 with real dimension $2n+1$. Suppose that the Levi form of M satisfies condition $Y(q)$ at x_0, where $0 \le q \le n$. There exist $0 < \rho < 1$ and $\delta = \delta(\rho) > 0$ such that if $\fint_{B_1} |u|^2 \, d\mu \le 1$, $\fint_{B_1} |f|^2 \le \delta$ and if $u \in W^1_{\epsilon *}(B_1(0)))$ is a solution of the inhomogeneous equation $\Box_b^\epsilon u = f$ in $B_1(0)$, $0 \le \epsilon < \epsilon_0$, then there exists a nonisotropic second-order polynomial P in normal coordinates (z,t) satisfying $\Box_b^0 P = 0$ and*

$$\fint_{B_\rho} |u - P|^2 \le \rho^{2(2+\alpha)}.$$

Proof. Let h be the $(0,q)$-form obtained in Lemma 3.1 for some $\eta > 0$, where η is to be decided later. Then $\Box_b^0 h = 0$ in $B_{\frac{1}{2}}$ and

$$\fint_{B_{\frac{1}{2}}} |u - h|^2 \le \eta.$$

It follows from the interior regularity of $\Box_b^0$ that $h \in C^\infty(B_{\frac{1}{2}}(0))$ and

$$|h(0)| + |\nabla h(0)| + \left|\nabla^2 h(0)\right| \le C \tag{3.8}$$

where C is a constant depending only on n, k and the Levi form at 0. Let P be the nonisotropic second-order Taylor expansion of h componentwise at 0, i.e.,

$$\begin{aligned} P(z,t) = P(z,\bar z,t) &= h(0) + \sum_{m=1}^n \left(\frac{\partial h}{\partial \bar z_m}(0)\bar z_m + \frac{\partial h}{\partial z_m}(0) z_m \right) + \frac{\partial h}{\partial t}(0) t \\ &\quad + \frac{1}{2}\left(\sum_{i,j=1}^n \frac{\partial^2 h}{\partial z_i \partial z_j}(0) z_i z_j + \sum_{i,j=1}^n \frac{\partial^2 h}{\partial z_i \partial \bar z_j}(0) z_i \bar z_j + \sum_{i,j=1}^n \frac{\partial^2 h}{\partial \bar z_i \partial \bar z_j}(0) \bar z_i \bar z_j \right) \\ &= A + Bz + Ct + \frac{1}{2} z^t D z, \end{aligned}$$

where A, B, C, D are matrices. We have that from (3.8), for ρ sufficiently small,

$$|h - P| \le C\rho^3 \quad \text{on } B_\rho.$$

Thus, choosing ρ small, we have that

$$\begin{aligned}
\fint_{B_\rho} |u-P|^2 &\le 2\fint_{B_\rho} |u-h|^2 + 2\fint_{B_\rho} |h-P|^2 \\
&\le C\rho^{-2n-2} \int_{B_{\frac{1}{2}}} |u-h|^2 + 2C\rho^6 \\
&\le 2C'\rho^{-2n-2}\eta + 2C\rho^6.
\end{aligned}$$

We can choose ρ and then η, δ sufficiently small, where both η and δ depends on ρ. It follows that

$$\fint_{B_\rho} |u-P|^2 \le \rho^{2(2+\alpha)}.$$

The lemma is proved.

Theorem 3.3. *Let M be a CR manifold of class C^2 with real dimension $2n+1$. Suppose that the Levi form of M satisfies condition $Y(q)$ at x_0, where $0 \le q \le n$. Let $u \in W^1_{*(0,q)}(B_R(x_0))$ be a solution of the inhomogeneous equation $\Box_b u = f$ in $B_R(x_0)$, $R > 0$. Assume that for some $0 < r_0 \le R$, f satisfies*

$$\fint_{B_r(x_0)} |f - f_{B_r(x_0)}|^2 \le Cr^{2\alpha}, \quad \textit{for every } r < r_0.$$

Then u is in $\mathcal{L}^{2,2+\alpha}_*(x_0)$; i.e., there exist $0 < r_1 \le r_0$ and a nonisotropic second-order polynomial P in normal coordinates (z,t) originating from x_0 such that

$$\fint_{B_r(x_0)} |u-P|^2 \le Cr^{2(2+\alpha)}, \quad \textit{for every } r < r_1.$$

Proof. Let (z,t) be the normal coordinates centered at x_0. Let ρ and δ be as in Lemma 3.2. Without loss of generality, we may assume that $f(0) = 0$ and there exists $\delta > 0$ such that for $\rho < r_0$,

$$\fint_{B_{\rho^m}} |f|^2 \le \frac{\delta}{2}\rho^{2\alpha m} \quad \text{for every } m \in \mathbb{N}. \tag{3.9}$$

Since $\Box_b$ is linear, it suffices to prove the theorem for δ small. We claim that for each $m = 1, \cdots$, there exists a nonisotropic second-order polynomial P_m in (z,t) such that each P_m has the form

$$P_m = A_m + B_m z + C_m t + \frac{1}{2} z^t D_m z$$

with

$$\begin{aligned}
&\Box_b^0 P_m = 0, \\
&\fint_{B_{\rho^m}} |u - P_m|^2 \le \rho^{2(\alpha+2)m} \quad \text{for every } m \in \mathbb{N}, \\
&|A_m - A_{m-1}| \le C\rho^{(m-1)(\alpha+2)}, \\
&|B_m - B_{m-1}| \le C\rho^{(m-1)(\alpha+1)}, \\
&|C_m - C_{m-1}| + |D_m - D_{m-1}| \le C\rho^{(m-1)\alpha}
\end{aligned} \tag{3.10}$$

where $P_0 = 0$ and C depends only on n and the Levi form.

To prove the claim, we use induction on m. Let $x = (z, t)$ and $\rho^m x = (\rho^m z, \rho^{2m} t)$. We may assume that $R = 1$ and $\|u\|_{B_1} = 1$ by considering $u(Rx)/\|u\|_{B_1}$ instead of u. The claim is already proved for $m = 1$ from Lemma 3.2. Assume that the claim is true for $m \geq 1$.

Consider

$$\tilde{u}(z, t) = \tilde{u}(x) = \frac{u(\rho^m x) - P_m(\rho^m x)}{\rho^{(2+\alpha)m}}.$$

Then we have

$$\Box_b^{\rho^m} \tilde{u}(x) = \frac{\rho^{2m} f(\rho^m x) - \Box_b^{\rho^m} P_m(\rho^m x)}{\rho^{(2+\alpha)m}} \equiv \tilde{f}(x).$$

Using $\Box_b^0 P_m = 0$, we have

$$\left| \Box_b^{\rho^m} P_m(\rho^m x) \right| \leq C \rho^{3m}$$

where C depends only on n. If we choose $C\rho^{1-\alpha} \leq \frac{\delta}{2}$, from (3.9), it follows that

$$\fint_{B_1} \left| \tilde{f} \right|^2 d\mu \leq \delta.$$

By the induction hypothesis that (3.7) holds for m, we have

$$\fint_{B_1} |\tilde{u}|^2 d\mu = \fint_{B_{\rho^m}} \frac{|u(x) - P_m(x)|^2 d\mu}{\rho^{(2+\alpha)m}} \leq 1.$$

Using Lemma 3.2, there exists a $(0, q)$-form $\tilde{h}$ with $\Box_b^0 \tilde{h} = 0$ in $B_{\frac{1}{2}}$ such that the second-order (nonisotropic) Taylor polynomial expansion of $\tilde{h}$ at 0 $\tilde{P}$ satisfies $\Box_b^0 \tilde{P} = 0$ and

$$\fint_{B_\rho} \left| \tilde{u} - \tilde{P} \right|^2 \leq \rho^{2(2+\alpha)}.$$

This implies that after scaling

$$\fint_{B_{\rho^{m+1}}} \left| \frac{u(x) - P_m(x)}{\rho^{m(2+\alpha)}} - \tilde{P}(\frac{1}{\rho^m} x) \right|^2 \leq \rho^{2(2+\alpha)}.$$

Setting

$$P_{m+1}(x) = P_m(x) + \rho^{m(2+\alpha)} \tilde{P}(\frac{1}{\rho^m} x),$$

we have

$$\fint_{B_{\rho^{m+1}}} |u - P_{m+1}|^2 \leq \rho^{2m(2+\alpha)} \rho^{2(2+\alpha)} = \rho^{2(m+1)(2+\alpha)}.$$

Notice that $\tilde{h}$ is infinitely differentiable at 0 and all the derivatives at 0 are bounded by the $L^2(B_{\frac{1}{2}})$-norm of $\tilde{h}$. Using

$$\|\tilde{h}\|^2_{B_{\frac{1}{2}}} \leq \|\tilde{u}\|^2_{B_{\frac{1}{2}}} + \|\tilde{h} - \tilde{u}\|^2_{B_{\frac{1}{2}}} \leq C,$$

we have that the coefficients of P_m satisfies

$$\begin{aligned}
&|A_{m+1} - A_m| + \rho^m |B_{m+1} - B_m| + \rho^{2m}(|C_{m+1} - C_m| + |D_{m+1} - D_m|) \\
&\leq \rho^{m(2+\alpha)}(|\tilde{h}(0)| + |\nabla \tilde{h}(0)| + |\nabla^2 \tilde{h}(0)|) \leq C\rho^{m(2+\alpha)}.
\end{aligned}$$

Thus from induction, (3.10) is proved for all m. It is easy to see that P_m converges to a second-order polynomial $P(x)$ since the coefficients A_m, B_m, C_m and D_m converge. Also we have using (3.10) that

$$\begin{aligned} \fint_{B_{\rho^m}} |P_m - P|^2 &\le \sum_{i=m}^{\infty} \fint_{B_{\rho^m}} |P_i - P_{i+1}|^2 \\ &\le C\rho^{2m(2+\alpha)}. \end{aligned}$$

This implies that for each $m \in \mathbb{N}$,

$$\fint_{B_{\rho^m}} |u - P|^2 \le 2\fint_{B_{\rho^m}} |u - P_m|^2 + 2\fint_{B_{\rho^m}} |P_m - P|^2 \le C\rho^{2m(2+\alpha)}.$$

For $0 < r < \rho$ with ρ fixed, one can choose $m \in \mathbb{N}$ such that $\rho^{m+1} \le r < \rho^m$. Since $\mu(B_r) = Cr^{2n+2}$, the doubling property for the balls holds. There exists C depending on ρ but independent of m such that $\frac{\mu(B_{\rho^m})}{\mu(B_r)} \le C$. Thus we have that for $0 < r < \rho$ with fixed ρ,

$$\fint_{B_r} |u - P|^2 \le \frac{\mu(B_{\rho^m})}{\mu(B_r)} \fint_{B_{\rho^m}} |u - P|^2 \le C\rho^{2m(2+\alpha)} \le Cr^{2(2+\alpha)}.$$

This implies that $u \in \mathcal{L}_*^{2,2+\alpha}(x_0)$. Theorem 3.3 is proved.

Next we will prove the pointwise Hölder estimates for $\Box_b$ in the divergence form. We study the regularity of the equation

$$\Box_b u = Xf,$$

where u and f are $(0,q)$-form and X is some derivative in the space $T^{1,0}(U) + T^{0,1}(U)$. This divergence form of $\Box_b$ will give the regularity for the Szegö projection. First we need the following compactness lemma. We use X^ϵ to denote the corresponding derivative with respect to the nonisotropic scaling ϵx.

Lemma 3.4. *Let M be a CR manifold of class C^2 with real dimension $2n+1$. Suppose that the Levi form of M satisfies condition $Y(q)$ at $x_0 \in M$, where $0 \le q \le n$. For any $\eta > 0$, there exists $\delta > 0$ such that if $u \in W^1_{\epsilon *}(B_1(0)))$ is a solution of the inhomogeneous equation $\Box_b^\epsilon u = X^\epsilon f$ in $B_1(0)$, where $0 \le \epsilon < \epsilon_0$, $\fint_{B_1} |u|^2 \, d\mu \le 1$, $\fint_{B_1} |f|^2 \le \delta$, then there exists a $(0,q)$-form h in $B_{\frac{1}{2}}$ satisfying $\Box_b^0 h = 0$ and*

$$\fint_{B_{\frac{1}{2}}} |u - h|^2 \le \eta. \tag{3.11}$$

Proof. The proof is similar to the proof of Lemma 3.1. Suppose that the lemma is not true. Then there exist an $\eta_0 > 0$ and a sequence of $(0,q)$-forms u_n, $(0,q)$-form f_n with $\Box_b^{\epsilon_n} u_n = X^{\epsilon_n} f_n$ on $B_1(0)$, where $0 < \epsilon_n \le \frac{1}{n}$, $\fint_{B_1} |u_n|^2 \, d\mu \le 1$, $\fint_{B_1} |f_n|^2 \, d\mu \le \frac{1}{n}$, such that

$$\fint_{B_{\frac{1}{2}}} |u_n - h|^2 > \eta_0 \tag{3.12}$$

for every h satisfying $\Box_b^0 h = 0$ in $B_{\frac{1}{2}}(0)$.

There exists a subsequence u_n converges weakly to some element u_∞ in $L^2(B_1)$. Let $\zeta \in C_0^\infty(B_1)$ be a cut-off function such that $\zeta \ge 0$ in B_1 and $\zeta = 1$ on $B_{\frac{1}{2}}$ and $\zeta \in C_0^\infty(B_1)$. We use the inequality $ab \leq (\mathrm{sc})a^2 + (\mathrm{lc})b^2$ for small constant sc and large constant lc. Since $\Box_b^{\epsilon_n} u_n = X^{\epsilon_n} f_n$ in the distribution sense, we get from (2.3),

$$
\begin{aligned}
&(\bar\partial_b^{\epsilon_n}\zeta u_n, \bar\partial_b^{\epsilon_n}\zeta u_n) + (\vartheta_b^{\epsilon_n}\zeta u_n, \vartheta_b^{\epsilon_n}\zeta u_n) \\
&\leq C(\left|(\zeta^2 u_n, X^{\epsilon_n} f_n)\right| + \|u_n\|^2) \\
&\leq \mathrm{sc}\|X^{\epsilon_n}\zeta u_n\|^2 + \mathrm{lc}(\|f_n\|^2 + \|u_n\|^2) \\
&\leq \mathrm{sc}(\|\bar\partial_b^{\epsilon_n}\zeta u_n\|^2 + \|\vartheta_b^{\epsilon_n}\zeta u_n\|^2) + \mathrm{lc}(\|f_n\|^2 + \|u_n\|^2) \\
&\leq C'(\|f_n\|^2 + \|u_n\|^2) \\
&\leq C''.
\end{aligned}
\tag{3.13}
$$

Using (2.2) applied to ζu_n, we have that

$$\| u_n \|^2_{W^{\frac{1}{2}}(B_{\frac{1}{2}})} \leq \| \zeta u_n \|^2_{W^1_*(B_1)} \leq C \quad \text{for all } n.$$

Thus there exists a subsequence, still denoted by u_n, which converges strongly in $L^2(B_{\frac{1}{2}})$ to u_∞.

We claim that $\Box_b^0 u_\infty = 0$ in $B_{\frac{1}{2}}$ in the distribution sense. Using (3.13), we have that (a subsequence of) $\bar\partial_b^{\epsilon_n} u_n$ converges to the form $\bar\partial_b^0 u_\infty \in L^2(B_{\frac{1}{2}})$ and $\vartheta_b^{\epsilon_n} u_n$ converges to the form $\vartheta_b^0 u_\infty \in L^2(B_{\frac{1}{2}})$ weakly. Let $\phi \in C^2_{(0,q)}(B_{\frac{1}{2}})$ be any $(0,q)$-form with compact support. We get

$$(\bar\partial_b^{\epsilon_n}\phi, \bar\partial_b^{\epsilon_n} u_n) + (\vartheta_b^{\epsilon_n}\phi, \vartheta_b^{\epsilon_n} u_n) = (\tilde X^{\epsilon_n}\phi, f_n),$$

where $\tilde X^{\epsilon_n}$ is the formal adjoint of X^{ϵ_n}. Using $\fint_{B_1} |f_n|^2\, d\mu \leq \frac{1}{n}$, it follows easily that

$$(\bar\partial_b^0\phi, \bar\partial_b^0 u_\infty) + (\vartheta_b^0\phi, \vartheta_b^0 u_\infty) = 0.$$

Thus $\Box_b^0 u_\infty = 0$ in $B_{\frac{1}{2}}$ in the distribution sense. This gives a contradiction to (3.12). Lemma 3.4 is proved.

Theorem 3.5. *Let M be a CR manifold of class C^2 with real dimension $2n+1$. Suppose that the Levi form of M satisfies condition $Y(q)$ at $x_0 \in M$, where $0 \leq q \leq n$. Let $u \in W^1_{*(0,q)}(B_R(x_0))$ be a solution of the equation $\Box_b u = Xf$, where $f \in L^2_{(0,q)}(B_R(x_0))$ and $R > 0$. Assume that for some $0 < \alpha < 1$ and $0 < r_0 \leq R$, f satisfies*

$$\fint_{B_r(x_0)} |f - f_{B_r(x_0)}|^2 \leq C r^{2\alpha}, \quad \text{for every } r < r_0. \tag{3.14}$$

Then u is in $\mathcal{L}_*^{2,1+\alpha}(x_0)$; i.e., there exist $0 < r_1 \leq r_0$ and a first-order polynomial $L(z)$ in normal coordinates (z,t) originating from x_0 such that

$$\fint_{B_r(x_0)} |u - L|^2 \leq C r^{2(1+\alpha)}, \quad \text{for every } 0 < r < r_1.$$

Proof. Without loss of generality, we may assume that $R = 1$ and $f(0) = 0$ and the constant in (3.14) is small. Using Lemma 3.4, for any $\eta > 0$, one can choose $\delta > 0$ such that there exists a $(0,q)$-form h satisfying $\Box_b^0 h = 0$ in $B_{\frac{1}{2}}$ and

$$\fint_{B_{\frac{1}{2}}} |u - h|^2 \leq \eta.$$

Let L_1 be the nonisotropic first-order polynomial expansion of h at 0. Then there exists $0 < \rho < 1$ such that

$$\fint_{B_\rho} |u-L|^2 \le 2\fint_{B_\rho} |u-h|^2 + 2\fint_{B_\rho} |h-L|^2 \le 2\rho^{-Q}\eta + C\rho^4 \le \rho^{2(1+\alpha)} \tag{3.15}$$

if we choose first ρ, then η small. From the scaling equations, we have

$$\Box_b^{\rho^m} u(\rho^m x) = \rho^{2m}(\Box_b u)(\rho^m x) = \rho^m(\bar\partial_b f)(\rho^m x) = \rho^m \bar\partial_b^{\rho^m} f(\rho^m x). \tag{3.16}$$

Using (3.15) and (3.16), one can prove by induction that there exists a sequence of first-order polynomials $L_m = A_m + B_m z$ satisfying

$$\begin{aligned} &\Box_b^0 L_m = 0 \\ &\fint_{B_{\rho^m}} |u-L_m|^2 \le \rho^{2m(1+\alpha)} \\ &|A_m - A_{m-1}| + \rho^m |B_m - B_{m-1}| \le C\rho^{(m-1)(\alpha+1)} \end{aligned} \tag{3.17}$$

and L_m converges to some first-order polynomial L. The rest of the proof is similar to the proof of Theorem 3.3.

Theorem 3.6. *Let M be a CR manifold of class C^2 and with real dimension $2n+1$. Suppose that the Levi form of M satisfies condition $Y(q)$ at $x_0 \in M$, where $0 \le q \le n$. Let $u \in W^1_{*(0,q)}(B_R(x_0))$ be a solution of the equation $\Box_b u = f$ for some $R > 0$. If $f \in L^p_{(0,q)}(B_R(x_0))$ with $n+1 < p < \infty$ and $p \ne 2n+2$, then $u \in \mathcal{L}^{2,2-\alpha}_*(x_0)$, where $\alpha = \frac{2n+2}{p}$; that is, there exist a $[2-\alpha]$-th order polynomial $L(z)$ and $0 < R_1 \le R$ such that*

$$\fint_{B_\rho} |u-L|^2 \le C\rho^{2(2-\alpha)} \quad \textit{for every } \rho \le R_1.$$

Proof. For any $0 < \rho < r < r_0$, we have from Hölder's inequality,

$$\int_{B_\rho} |f|^2 \le \|f\|^2_{L^p(B_\rho)} \rho^{Q(1-\frac{2}{p})},\ Q = 2n+2.$$

Thus it follows that

$$\fint_{B_{\rho^m}} |f|^2 \le \delta\rho^{\frac{-2Qm}{p}} = \delta\rho^{-2m\alpha}, \quad m \in \mathbb{N}$$

where $\delta = \|f\|^2_{L^p(B_\rho)}$ can be taken small. We claim that for each $m = 1, 2, \cdots$ there exists a first-order polynomial $L_m(z)$ such that $\Box_b^0 L_m = 0$ and

$$\fint_{B_{\rho^m}} |u-L_m|^2 \le \rho^{2m(2-\alpha),} \quad m \in \mathbb{N}.$$

This is proved by induction on m similar to the proof of Theorem 3.3. The theorem is proved.

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