Application of Model Search to Lattice Theory

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1 Introduction

We have used the first-order model-searching programs MACE and SEM to study various problems in lattice theory. First, we present a case study in which the two programs are used to examine the differences between the stages along the way from lattice theory to Boolean algebra. Second, we answer several questions posed by Norman Megill and Mladen Pavičić on ortholattices and orthomodular lattices. The questions from Megill and Pavičić arose in their study of quantum logics, which are being investigated in connection with proposed computing devices based on quantum mechanics. Previous questions of a similar nature were answered by McCune and MACE in [2].

MACE (Models And Counter Examples) [3] and SEM (System for Enumerating Models) [6] are programs that search for finite models of first-order and equational logic statements. If the input statement is the denial of a conjecture, then any models found are counterexamples. MACE searches for models by transforming its input into an equiconsistent propositional problem, then calling a Davis-Putnam-Loveland-Logeman procedure. SEM uses a more direct method of filling in tables according to various heuristics and evaluating the input against the tables. SEM is usually more effective than MACE for problems with large formulas. Both programs are designed to be complete; that is, if the search for a model of size n terminates without finding a model, then there should be none of that size. We believe the lattices we present in this note are the smallest ones satisfying the given properties, because the programs reported that smaller examples do not exist.

This note has a companion page on the World Wide Web. The page `http://www.mcs.anl.gov/AR/aar_lattice` contains links to MACE, SEM, and EQP input files and other data files related to this work. In this note, we refer to those files with bold-faced underlined pseudolinks <u>**like this**</u>.

2 From Lattice Theory to Boolean Algebra

The standard definition of Boolean algebra is that it is a uniquely complemented distributive lattice, but other steps along the way are of interest in the study of quantum logics. As suggested by Martin Ziegler in [7], a starting point for studying the differences between the relatively well understood classical logic and the less-refined (less-understood) quantum logic, would be examining the basic underlying structures of each object. Several such structures are represented as nodes in the hierarchy shown in Figure 1. Each node represents a variety of lattices defined by the axioms listed. Each line between two nodes represents an inclusion of the top class of lattices as a subset of the class beneath it. The inclusions (B), (D), (E), and (G) are clear from the axioms alone: in each case, the axioms of the lower class are a subset of the class above it and hence immediately form a more general theory. The remaining inclusions (A), (C), and (F) were proved with the program EQP [1], with inputs available online in files **eqp-a[12].in**, **eqp-c.in**, and **eqp-f.in**,

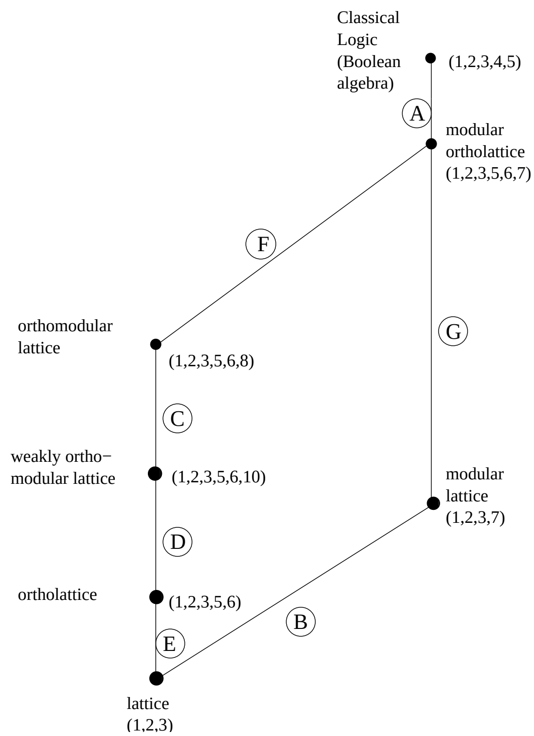

Figure 1: Structure Heirarchy

The focus of this section is on proving these inclusions to be strict inclusions by finding explicit models. An explicit model can add to an intuitive description of the theories involved. For example, each model given below resulted only after several earlier completed exhaustive searches, and so each model given is the smallest model that exists for the particular problem. Each node in Figure 1 represents a class of lattices that satisfy the axioms listed here.

Commutativity (1)

$$x \wedge y = y \wedge x$$
$$x \vee y = y \vee x$$

Associativity (2)

$$(x \wedge y) \wedge z = x \wedge (y \wedge z)$$
$$(x \vee y) \vee z = x \vee (y \vee z)$$

Absorption (3)

$$(z \vee y) \wedge x = x$$
$$(x \wedge y) \vee x = x$$

Distributivity (4)

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Invertibility (5)

$$x \vee c(x) = 1$$
$$x \wedge c(x) = 0$$
$$c(c(x)) = x$$

Compatibility (6)

$$c(x \vee y) = c(x) \wedge c(y)$$
$$c(x \wedge y) = c(x) \vee c(y)$$

Modularity (7)

$$x \vee (y \wedge (x \vee z)) = (x \vee y) \wedge (x \vee z)$$

Orthomodularity (8)

$$x \vee (c(x) \wedge (x \vee y)) = x \vee y$$

Weak Invertibility (9)

$$x \vee c(x) = 1$$
$$c(c(x)) = x$$

Weak Orthomodularity (10)

$$(c(x) \wedge (x \vee y)) \vee (c(y) \vee (x \wedge y)) = 1$$

Figure 2: Axioms

Using these equations, MACE and SEM found models that prove that none of the axiom sets for the separate types of lattice are equivalent. This was accomplished by supplying the axioms for a given type of lattice along with the negation of another axiom or set of axioms which are unique to the second type of lattice. The diagrams in Figure 3 are labeled to correspond to their use in Figure 1.

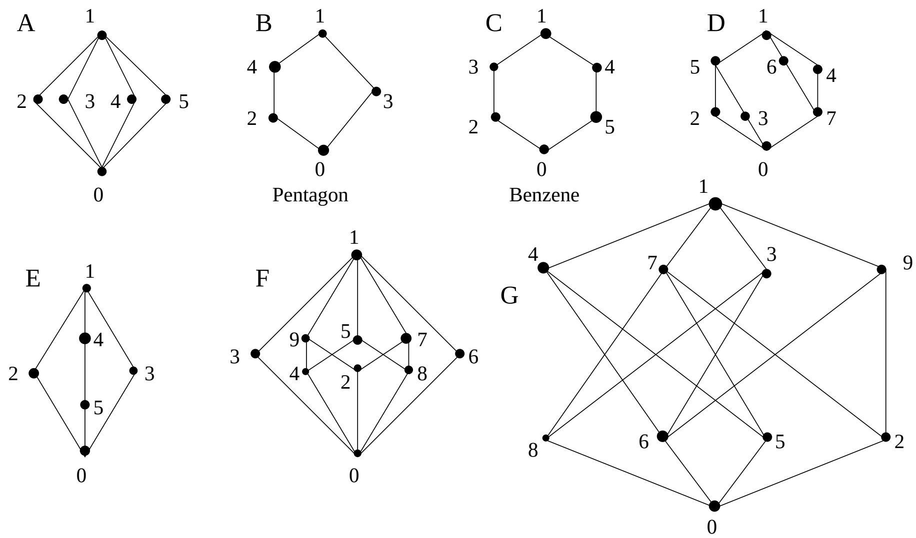

Figure 3: Lattices

For example, in order to obtain the model (A) that satisfies all the axioms of a modular ortholattice but is not necessarily a Boolean algebra, the axioms (1), (2), (3), (5), (6), and (7) are included as input while axiom (4) is denied. MACE then searched to find a model that satisfies all the input clauses including the denial. A matrix of values explicitly listing the operation values for meet, join and complement was returned that translated to model (A). The MACE inputs for models (A), (B), (C), (D), and (F), are available online in files **mace-[abcdf].in**.

The procedure used to find models (E) and (G) was different because we must find a lattice (or modular lattice) without an appropriate complement operation. We did this in two stages: first finding a lattice (or modular lattice) satisfying invertibility but not compatibility, then showing that that particular lattice does not have a complement operation. In the case of (E), for example, axioms (5) and (6), which distinguish ortholattice theory from lattice theory, introduce complementation. With axioms (1), (2), (3), and (5) with the denial of (6), MACE's output gave a model that included a list of values defining the complement operation c(x). From the output we know that the operation explicitly found by MACE does not satisfy the ortholattice axioms; however, we need to prove that no possible complement exists for the lattice that could satisfy the additional axioms of an ortholattice. The function values found in the candidate lattice were inserted into the input, forcing MACE to consider the same lattice but now with the axioms of an ortholattice included (5) and (6) in their original form (not negated). A second search with this input allowed all possible functions c(x) under axioms (5) and (6) to be considered. The search was complete with no models found, proving that the candidate lattice indeed cannot be an ortholattice. The MACE inputs for models (E) and (G) can be found online in files **mace-e[12].in** and **mace-g[12].in**.

3 Answers to Two Questions

In 1999, Megill asked [4] whether the equation

$$x \wedge (y \vee (x \wedge (c(x) \vee (x \wedge y)))) = x \wedge (c(x) \vee (x \wedge y)) \qquad \text{(*3-M68)}$$

holds in all weakly orthomodular lattices. SEM found a countermodel of size 20, which is depicted in Figure 4.

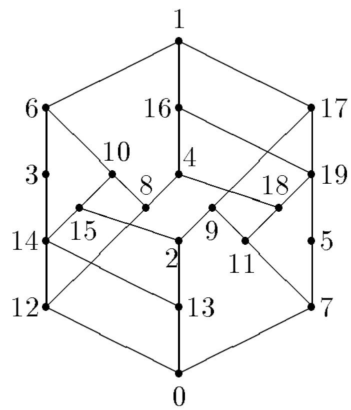

Figure 4: RW-1

This lattice (named RW-1) answers an open question published by Megill and Pavičić in [5, eq. 2.12] and several of Megill's unpublished questions [4]. The SEM input that produced this lattice is available online in file **sem-rw-1.in**.

In July 2001, Megill asked us [4] whether the equation

$$(x \vee (c(y) \wedge (c(x) \vee (c(y) \wedge (x \vee (c(y) \wedge c(x))))))) = \\ (x \vee (c(y) \wedge (c(x) \vee (c(y) \wedge (x \vee (c(y) \wedge (c(x) \vee (c(y) \wedge x)))))))) \qquad (4)$$

holds in every ortholattice. SEM found a countermodel of size 16, shown in Figure 5. The SEM input can be found online in file **sem-rw-2.in**.

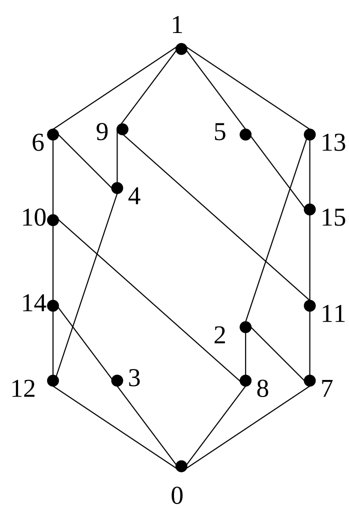

Figure 5: RW-2

Acknowledgments

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