2-PLACEMENT OF \((p, q)\)-TREES

Beata Orchel

University of Mining and Metallurgy
Al. Mickiewicza 30
30–059 Kraków, Poland

Abstract

Let \(G = (L, R; E)\) be a bipartite graph such that \(V(G) = L \cup R\), \(|L| = p\) and \(|R| = q\). \(G\) is called \((p, q)\)-tree if \(G\) is connected and \(|E(G)| = p + q - 1\).

Let \(G = (L, R; E)\) and \(H = (L', R'; E')\) be two \((p, q)\)-trees. A bijection \(f : L \cup R \rightarrow L' \cup R'\) is said to be a biplacement of \(G\) and \(H\) if \(f(L) = L'\) and \(f(x)f(y) \notin E'\) for every edge \(xy\) of \(G\). A biplacement of \(G\) and its copy is called 2-placement of \(G\). A bipartite graph \(G\) is 2-placeable if \(G\) has a 2-placement. In this paper we give all \((p, q)\)-trees which are not 2-placeable.

Keywords: tree, bipartite graph, packing graph.

2000 Mathematics Subject Classification: 05C35.

1. Definitions

We shall use standard graph theory notation. All graphs will be assumed to have neither loops nor multiple edges. Let \(G = (L, R; E)\) be a bipartite graph with a vertex set \(V(G) = L \cup R\), where \(L \cap R = \emptyset\) \(L(G) = L\), \(R(G) = R\) are left and right set of bipartition of the vertex set, an edge set \(E(G) = E\) and size \(e(G)\). For a vertex \(x \in V(G)\) by \(N(x, G)\) and \(d(x, G)\) we denote the set of its neighbors in \(G\) and the degree of the vertex \(x\) in \(G\), respectively. \(\Delta_L(G)\) and \(\Delta_R(G)\) are the maximum vertex degree in the set \(L(G)\) and \(R(G)\), respectively. By \(P_n\) we denote the path of length \(n - 1\). Bipartite graph \(G = (L, R; E)\) is said \((p, q)\)-bipartite if \(|L| = p\) and \(|R| = q\). \(K_{p,q}\) is the complete \((p, q)\)-bipartite graph. \(\bar{G}\) is the complement of
A bipartite graph \( G = (L, R; E) \) is a subgraph of bipartite graph \( H = (L', R'; E') \) if \( L \subseteq L' \), \( R \subseteq R' \) and \( E \subseteq E' \).

Let \( G = (L, R; E) \) and \( H = (L', R'; E') \) be two \((p, q)\)-bipartite graphs. We say that \( G \) and \( H \) are mutually placeable (for short \( \text{mp} \)) if there is a bijection \( f : L \cup R \rightarrow L' \cup R' \) such that \( f(L) = L' \) and \( f(x)f(y) \) is not edge in \( H \) whenever \( xy \) is an edge of \( G \). The function \( f \) is called the biplacement of \( G \) and \( H \). Thus \( G \) and \( H \) are \( \text{mp} \) if and only if \( G \) is contained in the graph \( \overline{H} \), i.e., \( G \) is subgraph of \( \overline{H} \). 2-placement of \( G \) is a biplacement of \( G \) and its copy. If such a 2-placement of \( G \) exists then we say that \( G \) is 2-placeable.

In the proof of the main theorem of this paper we use the adjacency matrices defined as follows.

Let \( G = (L, R; E) \) be a \((p, q)\)-bipartite graph, \( L = \{x_1, \ldots, x_p\} \) and \( R = \{y_1, \ldots, y_q\} \). The matrix \( M_G = (a_{ij})_{i=1,\ldots,p; \ j=1,\ldots,q} \) where:

\[
a_{ij} = \begin{cases} 
1, & \text{if } x_ix_j \in E(G), \\
0, & \text{if } x_ix_j \not\in E(G) 
\end{cases}
\]

is called adjacency matrix of the graph \( G \). Let \( G \) and \( H \) be mutually placeable \((p, q)\)-bipartite graphs and let \( f \) be a biplacement of \( G \) and \( H \). We may define the new \( p \times q \) matrix \( M_{G,H} = (b_{ij}) \) by the formula

\[
b_{ij} = \begin{cases} 
1, & \text{if } x_ix_j \in E(H), \\
2, & \text{if } x_ix_j \in E(f(G)), \\
0, & \text{if } x_ix_j \not\in E(H) \text{ and } x_ix_j \not\in E(f(G)). 
\end{cases}
\]

The matrix \( M_{G,H} \) is said to be the matrix of biplacement of \( G \) and \( H \). Next, instead of looking for biplacement of \( G \) and \( H \) we shall look for a matrix \( M_{G,H} \).

A \((p, q)\)-bipartite graph \( G \) is called \((p, q)\)-tree if \( G \) is connected and \( |E(G)| = p + q - 1 \). Thus each \((p, q)\)-tree is a tree and for each tree \( T \) there exist integers \( p \) and \( q \) such that \( T \) is \((p, q)\)-tree.

Let \( T \) be a \((p, q)\)-tree and \( y \in V(T) \). Let us denote by \( U_y \) the set of all \( z \in N(y, T) \) such that \( d(z, T) = 1 \). We shall call \( U_y \) the bough with the center \( y \). We say that \( \{x, y\} \subseteq L \) (or \( \{x, y\} \subseteq R \)) is a good pair of vertices (for short good pair) if there exist vertices \( w \) and \( z \) such that \( x \in U_w \), \( y \in U_z \) and \( w \neq z \).
2. Results

Let \( G \) be a general graph of order \( n \). The following theorem has been proved in [2].

**Theorem 1.** If \( e(G) \leq n - 1 \) and \( n \geq 8 \) then either \( G \) is contained in \( \overline{G} \) or \( G \) is isomorphic to one of the following graphs: \( K_{1,n-1}, K_{1,n-4} \cup K_{3} \).

Wang and Saver proved the following result in [6].

**Theorem 2.** A tree of order \( n \geq 7 \) is not \( 3 \)-placeable if and only if it is isomorphic to the star \( S_n \) or the graph obtained from \( S_{n-1} \) by inserting a new vertex into an edge of \( S_{n-1} \).

Makheo, Saclé and Woźniak in [4] characterized all triples of trees \( \{T_n, T'_n, T''_n\} \) which are not mutually placeable in \( K_n \).

For bipartite graphs, J.L. Fouquet and A.P. Wojda in [3] characterized those \((p, q)\)-bipartite graphs of size \( p + q - 2 \) which are not \( 2 \)-placeable in \( K_{p,q} \).

All pairs of \((p, q)\)-bipartite graphs \( G, H \) which are not placeable, \( e(G) \leq p + q - 1, e(H) \leq p \) and \( p \leq q \) are given in [5].

The main result to be presented in this paper is that any \((p, q)\)-tree \( T \) such that \( \Delta_L(T) < q, \Delta_R(T) < p, p \geq 3, q \geq 3 \) and \( p + q \geq 7 \) is either \( 2 \)-placeable or \( T \) is in the family \( T(p, q) \) of graphs which are defined below:

\( T'(p, q, k) \) is the \((p, q)\)-tree \( T \) such that, there are three vertices \( v, w, w' \) such that \( v \in L \) and \( d(v, T) = q - 1, w' \in R \setminus N(v, T), d(w', T) = k, w \in N(v, T) \) and \( d(w, T) = p - k + 1 \) (see Figure 1). We shall called the vertex \( v \) the left center of \( T \).

It is not difficult to see that \( T'(p, q, k) \) is \( 2 \)-placeable if and only if \( 1 < k \leq \frac{p}{2} \). Let \( TL(p, q) = \bigcup \{T'(p, q, k); k > \frac{p}{2} \} \). Analogically we define the tree \( T''R(p, q, k) \) and the set \( TR(p, q) = \{T''R(p, q, k); k > \frac{q}{2} \} \). The tree \( T''R(p, q, k) \) is shown in Figure 2.

By \( T(p, q) \) we denote the set \( TR(p, q) \cup TL(p, q) \).

Now, we can formulate our main result.

**Theorem A.** Let \( T = (L, R; E) \) be a \((p, q)\)-tree such that \( \Delta_L(T) < q, \Delta_R(T) < p, p \geq 3, q \geq 3 \) and \( p + q \geq 7 \). Then either \( T \) is \( 2 \)-placeable or \( T \in T(p, q) \).
\[ T'(p, q, k) \]

Figure 1

\[ T'(p, q, k) \]

Figure 2
3. Proof of Theorem A

To prove Theorem A we shall need two lemmas and some observations.

**Lemma 3.1.** Let $T = (L, R; E)$ be a $(p, q)$-tree such that there are two different vertices $y$ and $y'$ such that either $y, y' \in L$ or $y, y' \in R$, $U_y \neq \emptyset$ and $U_y' \neq \emptyset$. Let $|U_y| = k$, $U_y = \{x_1, \ldots, x_k\}$, $|U_y'| = k', U_y' = \{x_1', \ldots, x_{k'}\}$, and $k \leq k'$. Denote by $U_y^*$ the set $\{x_1', \ldots, x_k\}$. If $T \setminus (U_y \cup U_y^*)$ is 2-placeable, then $T$ is 2-placeable, too.

**Proof.** Let $T' = T \setminus (U_y \cup U_y^*)$ and let $f$ be a 2-placement of $T'$. We may define a 2-placement $f^*$ of $T$ in the following way:

- $f^*(v) = f(v)$, for each vertex $v$ of $T'$,
- if $f(x') = y'$ or $f(y) = y$ then $f^*(U_y) = U_y^*$, $f^*(U_y^*) = U_y$,
- if $f(y') \neq y'$ and $f(y') \neq y$ then $f^*(U_y) = U_y$, $f^*(U_y^*) = U_y^*$.

**Lemma 3.2.** Let $T = (L, R; E)$ be a $(3, q)$-tree, $\Delta_L(T) < q$, $\Delta_R(T) < 3$ and $q \geq 4$. Then $T$ is 2-placeable unless $T \in T(3, q)$.

**Proof.** Let $T = (L, R; E)$ be a $(3, q)$-tree, $\Delta_L(T) < q$ and $\Delta_R(T) < 3$. Let $L = \{a, b, c\}$, $d(a, T) = k_1$, $d(b, T) = k_2$ and $d(c, T) = k_3$. Note that two of sets $N(a, T) \cap N(b, T)$, $N(b, T) \cap N(c, T)$, $N(c, T) \cap N(a, T)$ are 1-sets, while the third is empty. We assume that $N(a, T) \cap N(b, T) \neq N(b, T) \cap N(c, T)$, otherwise $\Delta_R(T) = 3$. Let $z$ be a common neighbor of vertices $a$ and $b$, and let $y$ be a common neighbor of vertices $b$ and $c$. Let $N(a, T) = \{x_1, \ldots, x_{k_1}\}$, $x_1 = z$, $N(b, T) = \{x_{k_1}, \ldots, x_{k_1+k_2-1}\}$, $x_{k_1+k_2-1} = y$ and $N(c, T) = \{x_{k_1+k_2-1}, \ldots, x_q\}$. The tree $T$ and the matrix $M_T$ is shown in Figure 3.

Observe that $k_1 \geq 1$, $k_3 \geq 1$, $k_2 \geq 2$ and $k_1 + k_2 + k_3 - 2 = q$. If $k_1 = 1$ and $k_3 > \frac{q}{2}$ or $k_3 = 1$ and $k_1 \geq \frac{q}{2}$ then $T \in T(3, q)$. If $k_1 = 1$ and $k_3 \leq \frac{q}{2}$ then any function $f : L \cup R \to L \cup R$ such that $f(N(b, T)) = \{x_q-k_2+1, \ldots, x_q\}$ and $f(N(c, T)) = \{x_1, \ldots, x_q-k_2+1\}$, $f(b) = a$, $f(a) = b$, $f(c) = c$ is 2-placement of $T$. For $k_3 = 1$ and $k_1 \leq \frac{q}{2}$ we define a 2-placement of $T$ analogically.

So, we assume that for each $i \in \{1, 2, 3\}$ $k_i \geq 2$. Let $k = \max\{k_1, k_2, k_3\}$. We consider two cases.

**Case 1.** $k \neq k_2$

We may assume that $k = k_3$. The function $f$ such that $f(c) = a$, $f(b) = b$, $f(c) = a$, $f(N(a, T)) = \{x_1, \ldots, x_k\}$, $f(N(b, T)) = \{x_1, x_{k_1+k_3}, \ldots, x_q\}$ and
$f(N(c,T)) = \{x_{k_1+1}, \ldots, x_{k_1+k_3-1}, x_q\}$ is a 2-placement of $T$. For $k_1 = 4$, $k_2 = 4$ and $k_3 = 6$ the matrix $M_{T,T}$ is shown in the Figure 4.

![Figure 3](image1)

![Figure 4](image2)

**Case 2. $k = k_2$**

Without loss of the generality, we may suppose that $k_1 \leq k_3 < k_2$. The 2-placement of $T$ we may define as follows: $f(a) = b$, $f(b) = a$, $f(c) = c$, $f(N(b,T)) = \{x_{q-k_2+1}, \ldots, x_q\}$, $f(N(a,T)) = \{x_1, \ldots, x_{k_1-1}, x_q\}$, $f(N(c,T)) = \{x_{k_1}, \ldots, x_{q-k_2+1}\}$. The matrix of $M_{T,T}$ when $k_1 = 4$, $k_2 = 6$ and $k_3 = 5$ is shown in Figure 5.
Let $T$ be $(p, q)$-tree, such that $\Delta_R(T) < p < \Delta_L(T) < q$, $5 \leq p \leq q$ and $6 \leq q$.

Let $\{x, y\}$ be a good pair of vertices. We say that $\{x, y\}$ is a very good pair if either $\Delta_L(T \setminus \{x, y\}) < q - 2$ and $T \setminus \{x, y\} \notin T(p, q - 2)$ when $\{x, y\} \subset R$ or $\Delta_R(T \setminus \{x, y\}) < p - 2$ and $T \setminus \{x, y\} \notin T(p - 2, q)$ when $\{x, y\} \subset L$.

**Observations.**

1. If $T \in T(p, q)$ then if $v$ is the left (or right) center of $T$, then there is exactly one vertex which is not pendent in $N(v, T)$.
2. If $T \in T(p, q)$ and $z$ is the common neighbor of the vertices $w$ and $w'$ then $d(z, T) = 2$.

**Proof of Theorem A.** We shall give the main idea of the proof, leaving to reader long but easy verification of some details. The proof is by induction on $p + q$.

Without the loss of the generality we may assume that $p \leq q$. By Lemma 3.2 the theorem holds if $p = 3$ and $q \geq 5$. So, we assume that $p \geq 4$, $q \geq p$ and the theorem is true for every $(p', q')$-tree if $p' + q' < p + q$.

Let $T$ be a $(p, q)$-tree verifying assumptions of the theorem. Then there is a pendent vertex in $R$.

To prove that $T$ is 2-placeable unless $T \in T(p, q)$ we shall distinguish two cases.

**Case 1.** There are two pendent vertices in $R$, say $x$ and $y$, having different neighbors — $\{x, y\}$ is a good pair in $R$. When $q = 4$ then the theorem is easy to check. So, we may assume that $q \geq 5$.

Let $T' = T \setminus \{x, y\}$. If $\{x, y\}$ is a very good pair, then by the induction hypothesis $T'$ is 2-placeable. The 2-placement of $T$ we have by the Lemma 3.1. Now, we suppose that $\{x, y\}$ is not a very good pair. We consider three subcases.
Subcase 1.1. $\Delta_L(T') = q - 2$

Let $v$ be a vertex in $L$ such that $d(v, T') = q - 2$. First, we assume that $d(v, T) = q - 2$. Let $N(x) = \{z\}$ and $N(y) = \{z'\}$ (see Figure 6). Observe that if $p \leq q - 2$ then there is a pendant vertex, say $x'$, in the set $N(v, T)$ and $\{x, x'\}$ is a very good pair in $R$. In fact, if $T'' = T \setminus \{x, x'\}$ then $\Delta_L(T'') = q - 3 < q - 2$ and $\Delta_R(T'') = \Delta_R(T) < p$. Suppose that $T'' \in TL(p, q - 2)$. Then the only possible center is the vertex $v$. But then $R(T'') \setminus N(v, T'') = \{y\}$ and $d(y, T'') = 1$, a contradiction.

Now, we suppose that $p = q \geq 6$ or $p = q - 1 \geq 5$ and each neighbor of the vertex $v$ has the degree at least two. In this case either $T = T_1$ or $T = T_2$ else $T = T_3$ where $T_1$, $T_2$ and $T_3$ are the graphs defined in the Figure 7.

Note that there is a very good pair of vertices in $L$. Let $\{x', y'\}$ be a very good pair in $L$. By induction hypothesis $T \setminus \{x', y'\}$ has 2-placement. $T$ is 2-placeable by the Lemma 3.1.

When $p = q = 5$ and there are no very good pairs in $L$ and each neighbor of the vertex $v$ has the degree at least two or if $p = 4$ the proof may be completed by checking all possible cases.
Let us suppose now, that \(d(v, T) = q - 1\) and \(y \notin N(v, T)\) (see Figure 8).

If there is a 2-placement \(f\) of \(T \setminus \{x\}\) then \(f(v) \neq \{v\}\) and the map defined by \(f^*(z') = f(z')\), for \(z' \neq x\), \(f^*(x) = x\) is 2-placement of \(T\).

Observe that \(T \setminus \{x\}\) is \((p, q-1)\)-tree, \(\Delta_L(T \setminus \{x\}) = q - 2 < q - 1\) and \(\Delta_R(T \setminus \{x\}) = \Delta_R(T) < p\). There are at least two vertices of the degree at least two in the set \(N(v, T)\). In the other case \(\Delta_R(T) = p\). Therefore, by Observation 1, \(T \setminus \{x\} \notin TL(p, q - 1)\). If there is a vertex of degree \(p - 1\) in \(N(v, T) \setminus \{y_1\}\), where \(\{y_1\} = N(v, T) \cap N(z, T)\), then \(T \setminus \{x\} \in TR(p, q - 1)\).
But the degree of the vertex \( z \), which is not adjacent to the right center of \( T \), is two. Hence we conclude that \( T \setminus \{ x \} \not\in TR(p, q - 1) \) and, by the induction hypothesis, there is a 2-placement \( f \) of \( T \setminus \{ x \} \).

\[
\begin{array}{c}
\text{x} \\
\text{v} \\
\text{y} \\
\text{z} \\
\text{y}_1 \\
\text{y} \\
\text{\ldots} \\
\text{\ldots} \\
\text{v} \\
\text{x} \\
\end{array}
\]

Figure 8

**Subcase 1.2.** \( T' \in TR(p, q - 2) \)

First we assume that \( d(w, T') \geq 3 \). Then either \( T = T_1 \), or \( T = T_2 \), or \( T = T_3 \), else \( T = T_4 \) (see Figure 9).

Let \( T = T_1 \) and let \( x' \) be a pendent neighbor of the vertex \( w' \). The tree \( T \setminus \{ x', y \} \) has two neighbors of vertex \( v \) of degree at least two. Hence, by Observation 1, \( T \setminus \{ x', y \} \not\in T(p, q - 2) \) and \( \{ x', y \} \) is very good pair.

Analogically, we may show that \( \{ x', y \} \) is a very good pair if \( T = T_2 \) and \( x' \) is pendent in \( N(w') \) or if \( T = T_3 \), \( x' \in N(w) \) and \( d(x', T) = 1 \). When \( T = T_4 \) then \( T \in TR(p, q) \).

If \( d(w, T') = 2 \) and \( T = T_3 \) then there is no very good pair in \( V(T) \). Let then the tree \( T = T_3' \). The matrix \( M_{T_4', T_4} \) is shown in Figure 10.

**Subcase 1.3.** \( T' \in TL(p, q - 2) \)

At the beginning we assume that \( d(w', T') = p - 1 \). In this case either there are very good pair in \( R \) or \( T \in TR(p, q) \) else \( T = T_3' \) (See Figure 10).

For \( d(w', T') = p - 2 \), unless \( T = T_5 \) or \( T = T_6 \) (See Figure 11), there is a very good pair of vertices in \( T' \). The matrices \( M_{T_5, T_5} \) and \( M_{T_6, T_6} \) are not difficult to find.

If \( d(w', T) \leq p - 3 \) then there is very good pair of vertices \( V(T) \).

**Case 2.** There is a vertex in \( L \), say \( z_0 \), such that each pendent vertex in \( R \) is its neighbor.
Figure 9
\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure10}
\caption{T_3'}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure11}
\caption{T_5, T_6}
\end{figure}

\[\begin{bmatrix}
1 & 2 & \cdots & q \\
1 & 1 & 2 & \cdots & 2 \\
2 & 2 & 1 & 2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
p - 1 & 2 & 1 & 1 & 0 & \cdots & 0 \\
0 & 2 & 2 & 1 & 1 & \cdots & 1
\end{bmatrix}\]
Let us denote by $U_{z_0}$ the bough with center $z_0$ and let $|U_{z_0}| = m$. Note that $d(z_0, T) \geq m$. If $d(z_0, T) = m$ then $m = q$ and $T = K_{1,q}$. So, we suppose now, that $d(z_0, T) \geq m + 1$. Observe, that there is at least one pendent vertex in $L$. In the other case there is a good pair of the vertices in $R$.

First, we assume that there is a good pair, say $x'$ and $y'$, in $L$. When $p = 4$ then $m = q - 2$ or $m = q - 3$ and is easy to check the theorem.

For $p \geq 5$ $T'' = T \setminus \{x', y'\}$ is $(p - 2, q)$-tree, $(p - 2 \geq 3)$ and if $\{x', y'\}$ is very good pair then $T''$ is 2-placeable by the induction hypothesis. $T$ has 2-placement by Lemma 3.1.

Now, we suppose that there is no very good pair in $L$ — i.e., $\{x', y'\}$ is a good pair but either $\Delta_R(T'') = p - 2$ or $T'' \in T(p - 2, q)$. Observe that $\Delta_R(T'') < p - 2$. In the other case either $\Delta_L(T) = q$ or there is a cycle $C_4$ in $T$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure12.png}
\caption{Figure 12}
\end{figure}
If $T'' \in TR(p - 2, q)$ or $T'' \in TL(p - 2, q)$, then either $T \in TL(p, q)$ or $T = T_3'$.  

Finally, we assume that all pendent vertices in $L$ have a common neighbor. Let $x_0$ be a vertex in $R$ such that if $v' \in L$ and $d(v', T) = 1$ then $v' \in N(x_0)$ and let $|U_{x_0}| = l$. Observe, that $T''' = T \setminus U_{x_0} \setminus U_{x_0} = P_{2n}$, where $n = q - m = p - l$. When $n = 1$ then $\Delta_L(T) = q$. If $n = 2$ then $T \in TL(p, q)$. For $n \geq 3$ the tree $T = T_{10}$ and the matrix $M_{T_{10}, T_{10}}$ shown in Figure 12.

This completes the proof of the theorem.  

Acknowledgements

The author gratefully acknowledges the many helpful suggestions of Professor A.P. Wojda during the preparation of the paper.

References


Received 19 December 2000
Revised 7 March 2002