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Secure Weakly Convex Domination in Graphs

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Abstract

In this paper, we investigate the concept of secure weakly convex domination set of some graphs. We characterized those graphs for which the secure weakly convex domination numbers are 1 and 2. Relations of this parameter with some domination parameters are also observed and a graph is constructed with a preassigned order, weakly convex domination number, secure weakly convex domination number, and secure convex domination number.

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1 Introduction

Let $G = (V(G), E(G))$ be a connected undirected graph. For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $X \subseteq V(G)$, the *open neighborhood* of X is $N(X) = \bigcup_{v \in X} N(v)$ and the *closed neighborhood* of X is $N[X] = X \cup N(X)$. For any two vertices u and v of G, the *distance* $d_G(u, v)$ is the length of the shortest u-v path in G. A u-v

path of length $d_G(u,v)$ is called *u-v geodesic*. A set $C \subseteq V(G)$ is a *weakly convex set* of G if for every two vertices $u, v \in C$ there exists a u-v geodesic whose vertices belongs to C, or equivalently, if for every two vertices $u, v \in C$, $d_{\langle C \rangle}(u,v) = d_G(u,v)$. A set C is a *convex set* of G if for every two vertices $u, v \in C$, the vertex-set of every u-v geodesic is contained in C.

A set S is a *dominating set* of G if for every $v \in V(G) \backslash S$, there exists $u \in S$ such that $uv \in E(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. A dominating set of G which is weakly convex (respectively, convex) is called a *weakly convex* (respectively, *convex*) *dominating set*. The *weakly convex* (respectively, *convex*) *domination number* of G, denoted by $\gamma_{wcon}(G)$ (respectively, $\gamma_{con}(G)$), is the smallest cardinality of a weakly convex (respectively, convex) dominating set of G.

A set S is a *secure weakly convex* (respectively, *secure convex*) *dominating set* of G if S is a weakly convex (respectively, convex) set of G and for every $u \in V(G) \backslash S$, there exists $v \in S$ such that $uv \in E(G)$ and $(S \backslash \{v\}) \cup \{u\}$ is a weakly convex dominating set of G. The *secure weakly convex* (respectively, *secure convex*) *domination number* of G, denoted by $\gamma_{swc}(G)$ (resp.,$\gamma_{scon}(G)$), is the smallest cardinality of a secure weakly convex (respectively, convex) dominating set of G.

The concept of weakly convex domination was introduced by Jerzy Topp and is discussed in [3] and [4]. Another domination parameter is the secure domination which was discussed in [1], [2], and [5]. A combination of these two concepts give rise to a new variant of domination called secure weakly convex domination.

Remark 1.1 *Let G be a connected graph of order n. Then $1 \leq \gamma_{swc}(G) \leq n$.*

2 Results

Note that if S is a secure weakly convex dominating set of a connected graph G, then $\langle S \rangle$ is connnected.

Proposition 2.1 *Let G be a connected graph of order $n \geq 3$ and let S be a secure weakly convex dominating set of G.*
(*i*) *Every cut-vertex of G is in S.*
(*ii*) *Every leaf of G is in S*

Proof: (i) Let v be a cut-vertex of G. Then $\langle V(G) \backslash \{v\} \rangle$ consists of at least two conponents. Let S be a secure weakly convex dominating set of G. Suppose $v \notin S$. Since $\langle S \rangle$ is connnected, S is contained in some component of $\langle V(G) \backslash \{v\} \rangle$. This implies that $V(G) \backslash S$ contains some vertices in the other components of $\langle V(G) \backslash \{v\} \rangle$. This contradicts the assumption that S is a dominating set of G. Therefore, $v \in S$.

(ii) Let v be a leaf of G. Then $\deg_G(v) = 1$. Let S be a secure weakly convex dominating set of G. Suppose that $v \notin S$. Since S is a dominating set of G and $\deg_G(v) = 1$, there exists a unique $u \in S$ such that $uv \in E(G)$. This implies that $\langle (S \backslash \{u\}) \cup \{v\} \rangle$ is not connected. This is a contradiction. Therefore, $v \in S$. □

Note that a star consists of a cut-vertex and leaves; and a path consists of cut-vertices and two leaves. The following results follows from Proposition 2.1

Corollary 2.2 *Let $n \geq 3$ be an integer. Then*
(i) $\gamma_{swc}(K_{1,n-1}) = n$.
(ii) $\gamma_{swc}(P_n) = n$.

The next result characterizes a graph G with $\gamma_{swc}(G) = 1$.

Theorem 2.3 *Let G be a connected graph of order $n \geq 2$. Then $G = K_n$ if and only if $\gamma_{swc}(G) = 1$.*

Proof: Clearly, if $G = K_n$, then $\gamma_{swc}(G) = 1$.

Conversely suppose that $\gamma_{swc}(G) = 1$. Let $S = \{v\}$ be a secure weakly convex dominating set of G. Suppose that $G \neq K_n$. Then there exists $u, w \in V(G)$ such that $uw \notin E(G)$. Thus, $(S \backslash \{v\}) \cup \{u\} = \{u\}$, which is not a dominating set of G. This is a contradiction. Therefore, $G = K_n$. □

The next result characterizes a graph G with $\gamma_{swc}(G) = 2$.

Theorem 2.4 *Let G be a non-complete connected graph. Then $\gamma_{swc}(G) = 2$ if and only if there exists a non-complete graph H such that $G = K_2 + H$.*

Proof: Suppose that $\gamma_{swc}(G) = 2$. Let $S = \{u, v\}$ be a secure weakly convex dominating set of G. Then $uv \in E(G)$. Define K_2 and H by the following: Take $V(K_2) = S$ and $V(H) = V(G) \backslash S$. Then $G = K_2 + H$. Since G is non-complete, it follows that H is non-complete.

Conversely, suppose there exists non-complete graph H such that $G = K_2 + H$. Then G is non-complete. By Theorem 2.3, $\gamma_{swc}(G) \neq 1$ that is, $\gamma_{swc}(G) \geq 2$. Let $S = \{u, v\}$, where $u, v \in V(K_2)$. Then S is a weakly convex set of G. By the definition of $K_2 + H$, S is a dominating set of G. Let $x \in V(G) \backslash S$. Then $x \in V(H)$ and $ux, vx \in E(G)$. Now, $(S \backslash \{u\}) \cup \{x\} = \{v, x\}$. Since $vx \in E(G)$ and x is arbitrary, $(S \backslash \{u\}) \cup \{x\}$ is a weakly convex dominating set of G. This shows that S is a secure weakly convex dominating set of G and $\gamma_{swc}(G) \leq |S| = 2$. Therefore, $\gamma_{swc}(G) = 2$. □

Corollary 2.5 *Let G be a non-complete graph and $n \geq 2$. Then $\gamma_{swc}(G + K_n) = 2$.*

Proof: Let $H = K_{n-2} + G$. Then H is a non-complete graph. Thus, $G + K_n \cong H + K_2$. By Theorem 2.4, $\gamma_{swc}(G + K_n) = \gamma_{swc}(H + K_2) = 2$. □

Since a secure weakly convex dominating set is a weakly convex dominating set and every secure convex dominating set is a secure weakly convex dominating set, we have

Remark 2.6 *Let G be a connected graph. Then $\gamma_{wcon}(G) \leq \gamma_{swc}(G) \leq \gamma_{scon}(G)$.*

Theorem 2.7 *Given integers a, b, c, and n with $3 \leq a < b < c < n$, there exists a connected graph G such that $|V(G)| = n$, $\gamma_{wcon}(G) = a$, $\gamma_{swc}(G) = b$, $\gamma_{scon}(G) = c$.*

Proof: Consider the path $P_{a+1} = [u_1, u_2, ..., u_a, u_{a+1}]$. Let G be a graph obtained fron P_{a+1} by adding the edges $u_1 v_i$ for $i = 1, 2, ..., b - a - 1$, adding the paths $[u_1, w_j, u_3]$ and $[u_2, w_j]$ for $j = 1, 2, ..., c - b$, adding the vertices $z_1, z_2, ..., z_{n-c}$ and forming the complete graph K_{n-c+2}, where $V(K_{n-c+2}) = \{z_1, ..., z_{n-c}, u_a, u_{a+1}\}$ (see Figure 1).

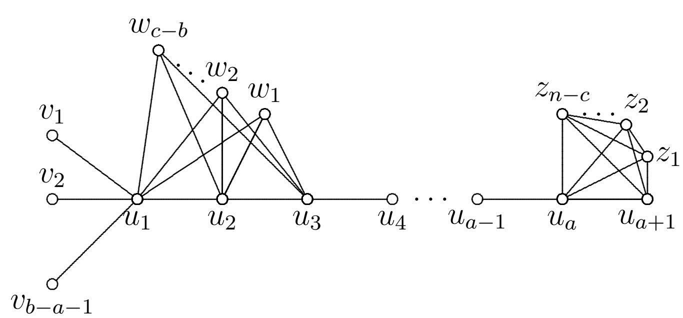

Figure 1: A graph G with $\gamma_{wcon}(G) < \gamma_{swc}(G) < \gamma_{scon}(G)$

Then $\{u_1, u_2, ..., u_a\}$ is a weakly convex dominating set of G, $\{u_1, u_2, ..., u_a, u_{a+1}\} \cup \{v_1, v_2, ..., u_{b-a-1}\}$ is a secure weakly convex dominating set of G, and $\{u_1, u_2, ..., u_a, u_{a+1}\} \cup \{v_1, v_2, ..., u_{b-a-1}\} \cup \{w_1, w_2, ..., w_{c-b}\}$ is a secure convex dominating set of G. Hence, $\gamma_{wcon}(G) = a$, $\gamma_{swc}(G) = b$, $\gamma_{scon}(G) = c$. Moreover, $|V(G)| = (a + 1) + (b - a - 1) + (c - b) + (n - c) = n$. □

The next result immediately follows from Theorem 2.7.

Corollary 2.8 *For each positive integer k, there exists a connected graph G for which $\gamma_{swc}(G) - \gamma_{wcon}(G) = \gamma_{swc}(G) - \gamma_{scon}(G) = k$.*

Corollary 2.9 *The domination parameters $\gamma_{swc}(G)$ and $\gamma_{con}(G)$ are not comparable.*

Proof: Consider a graph G in Theorem 2.7. Then $\{u_1, u_2, ..., u_a\} \cup \{w_1, w_2, ..., w_{c-b}\}$ is a convex dominating set of G. Thus, $\gamma_{con}(G) = a + c - b$. If $2b \leq a + c$, then $\gamma_{swc}(G) \leq \gamma_{con}(G)$. Otherwise, $\gamma_{swc}(G) > \gamma_{scon}(G)$. This shows that the two domination parameters are not comparable. □

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