FIR Differentiators for Quantized Signals

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Abstract—The impact of quantization noise on a signal whose rate is to be estimated using a FIR differentiator is analyzed, concentrating on the important constant-rate case in order that the filter be optimized for systems with low-frequency rates of change. Formulae for the mean-squared error of the filter, the corresponding spectral characteristics, and general formulae governing the filter coefficients are derived. The characteristics of four specific differentiators, including a representative wideband differentiator, are examined and compared. It is shown that a differentiator that is optimum in terms of its attenuation of white noise can also be considered optimum with respect to quantization noise attenuation in certain circumstances. An elegant relationship is derived between worst-case rms error and the fractional value of the rate at which this error occurs. Minimization of this worst-case mean-squared error is shown to be achieved with a simple differentiator. However, the corresponding average error is poor, and a simple nonlinear filter that minimizes the worst-case error, while retaining a similar average mean-squared error to that of the “optimum” differentiator, is proposed. The equivalence between FIR differentiators and the decoders used in single-loop sigma-delta modulators is also highlighted.

Index Terms—Differentiation (mathematics), FIR digital filters, quantization, sigma–delta modulation, spectral analysis, velocity measurement.

I. INTRODUCTION

Often, it can be assumed that quantization effects in digital filters are stochastic in nature, being pseudo-random over the range of the quantization [1], so that a white-noise model can conveniently be used to analyze the significance of the finite signal resolution. However, this assumption is flawed for many physical systems where the digital word represents a quantity that varies smoothly or in a deterministic fashion because there exists a strong correlation between the signal and the resultant quantization noise over a time equal to a sampling period [2]. The mathematical approach adopted in this paper to nonlinear quantized systems relies on the use of characteristic functions, a Fourier series expansion to model the quantizer, and concepts from the theory of the ergodicity of fractional operators. The technique described by Gray [2] permits an exact analysis of quantized systems for a restricted set of inputs.

The goal of this paper is to highlight differentiators that perform well when the input signals are quantized, particularly when the rates of change of the original signals are likely to vary slowly. To this end, the paper concentrates on optimization of filter characteristics for the constant-rate case. Universal properties governing such differentiators are also established.

The differentiation of quantized signals with correlated quantization noise is commonly encountered in practical applications such as image processing [4] and velocity estimation in motion control systems, using digital position information. Various differentiators have been applied in the latter field [5], [6]. The steady-state characteristics of a first-order sigma–delta converter are analogous to the first-order digital differentiator with constant-rate input [7]. Some of the extensive literature on the analysis of Σ–Δ converters (a part of which is collected in [3]) has applicability to differentiator analysis, and FIR differentiators are closely linked with encoders for such converters.

Wideband differentiators designed using standard frequency-domain design techniques will often perform poorly when estimating the rate of change of a quantized signal, as will Kumar and Roy’s maximally linear design optimized for low frequencies [8]. The design of a lowpass differentiator using a multiband design specification can help alleviate this problem. However, a more promising design technique for quantized systems is the time-domain approach in which the system is optimized for low rates of change, under the assumption that the signal at the input to the FIR differentiator is corrupted by a white-noise source [9].

The essential definition of an FIR digital differentiator adopted in this paper, encompassing both zero dc gain and zero error when processing a constant-rate signal, is introduced in Section II. The following sections deal with the rms and spectral error characteristics of such filters, two specific examples being provided in Section V. The worst-case mean-squared error for constant-rate signals is determined in Section VI, motivating the derivation in Section VII of a FIR differentiator that is optimal in terms of minimum worst-case mean-squared error. However, this filter performs poorly at most rates, as shown in Section VIII, in which the previously analyzed differentiators and a wideband device are compared. This motivates the proposal in Section IX of a nonlinear filter that has improved worst-case characteristics but very similar average mean-squared error to the optimum linear solution.

II. DEFINITION OF THE FIR DIFFERENTIATOR

The FIR differentiator consists of a transversal filter of length \( N \) (corresponding to \( N - 1 \) delay elements) with tap coefficients \( h_n, n = 0, \ldots, N - 1 \). The quantizer input sequence is assumed to be of the form \( p_k = p_0 + v_k \) (at some arbitrary time-index \( k \)), where the signal (or “position”) \( p \) and the rate (or “velocity”) \( v \) are given in units of quantization steps and quantization steps per sample time, respectively, and \( p_0 \) represents some arbitrary initial signal value. The uniformly sampled quantizer output sequence \( \{y_k\} \) is assumed to be generated by a rounding function.
\(q(\cdot)\) so that \(p_q = q(p)\) is an integer value. The filter output at sample \(i\) is the rate estimate \(\hat{v}_i\) defined by
\[
\hat{v}_i = \sum_{n=0}^{N-1} h_n p_q(i-n)
\]  
(ignoring any possible decimation at the output). The corresponding transfer function is
\[
H(z) = \sum_{n=0}^{N-1} h_n z^{-N-1-n}. 
\]  
(2)

If the quantization error \(e_q\) is associated with a quantized input \(p_q(t)\), i.e., \(p_q(t) = p_t + e_v\), the corresponding filtered output error is defined by \(e_v(t) = \hat{v}_i - v_i\).

The two fundamental characteristics that will be used to define a differentiator are i) that it should have zero gain at dc input, and ii) the response to an input ramp should be equal to the slope of that ramp. Assuming a FIR structure, these characteristics can be represented by the equations [9]
\[
\sum_{n=0}^{N-1} h_n = 0
\]  
(3)
\[
\sum_{n=0}^{N-1} nh_n = -1. 
\]  
(4)

It should be noted that most wideband differentiators designed using standard techniques will not satisfy the second criterion unless rescaling is employed or (4) is imposed as an additional constraint during the design process.

When \(h_n = -h_{N-1-n}\), the differentiator has a linear-phase characteristic with constant group delay and is of Type 3 (if \(N\) odd) or Type 4 (if \(N\) even), as commonly defined for such FIR filters [10]. This characteristic (while usual) is not a necessary condition for the FIR filter to act as a differentiator, and it is not assumed in the derivation of the general formulae derived in this paper. For brevity, it is assumed in this work that the length of the differentiator \(N\) is even. Extension to the coefficients governing differentiators of odd length is straightforward.

III. MEAN-SQUARED FILTER OUTPUT ERROR FOR A CONSTANT-RATE SIGNAL

The mean-squared error of the filter output at sample \(i\) is
\[
e_{\text{MSE}}(i) = \left(\sum_{n=0}^{N-1} h_n e_{q-i-n}\right)^2.
\]  
(5)

When the rate \(v\) is constant and irrational, the autocorrelation function of the quantization error process \(\{e\}\) is given by
\[
R_e(r) = \sum_{k \neq 0} \frac{1}{4\pi^2 k^2} e^{2\pi i k r v} = \frac{1}{12} - \frac{1}{2} \langle rv \rangle (1 - \langle rv \rangle)
\]  
(7)

where \(\langle \cdot \rangle\) represents the fractional operator, i.e., \(\langle x \rangle = x\) modulo 1, and \(e\) is the base for natural logarithms. These equations were derived by Gray [2], [11] when analyzing the quantization error associated with a single-loop sigma–delta modulator with a dc input signal (which implies a ramp input to the quantizer). It is assumed throughout this paper that \(v\) is irrational. This condition is imposed so that the quantizer error process is a quasistationary process [12], which produces an almost periodic sequence that is uniformly distributed in \([0,1)\) when an infinitely long sequence is considered and the input has constant rate. This condition is not severe because any rational number is infinitesimally close to an irrational number, and the probability that a number is irrational is identically unity, due to the density of irrational numbers in the continuum of real numbers. 1

By expanding (5) into products of individual error terms, grouping those product terms with equal delay between the error samples, and recognizing that
\[
E_e(r) = E\{e_i e_{j-r}\} = E\{e_i e_{j+r}\}, \quad \forall i, j, r \geq 0
\]  
(8)

where \(E\{\cdot\}\) represents the expectation function for quasistationary processes, the average mean-squared error associated with the filter output can be written as
\[
e_{\text{MSE}} = \sum_{n=0}^{N-1} h_n^2 R_e(0) + 2 \sum_{r=1}^{N-1-r} \sum_{n=0}^{N-1-r} h_n h_{n+r} R_e(r)
\]  
\[
= \frac{1}{12} \sum_{n=0}^{N-1} h_n^2 + \sum_{r=1}^{N-1-r} \sum_{n=0}^{N-1-r} h_n h_{n+r} \times \left(\frac{1}{6} - \langle rv \rangle (1 - \langle rv \rangle)\right)
\]  
(9)

through use of (7). This formula can be simplified by observing that
\[
\sum_{n=0}^{N-1} h_n^2 = 2 \sum_{r=1}^{N-1-r} \sum_{n=0}^{N-1-r} h_n h_{n+r} = \left(\sum_{n=0}^{N-1} h_n\right)^2 = 0
\]  
(10)

using (3) so that
\[
e_{\text{MSE}} = -\sum_{r=1}^{N-1-r} \sum_{n=0}^{N-1-r} h_n h_{n+r} \langle rv \rangle (1 - \langle rv \rangle).
\]  
(11)

The associated root-mean-squared error is \(\varepsilon_{\text{MSE}} = \sqrt{e_{\text{MSE}}}\). Decimation of the output stream will have no impact upon this error measure when the input rate is irrational. The root-mean-squared error \(\varepsilon_{\text{MSE}}\) is a function of \(\langle v \rangle\), and the slope of this function is potentially discontinuous at all rates for which \(\langle rv \rangle = 0\) for some \(r \in \{1, \ldots, N-1\}\). These rates are represented by the Farey sequence [13] of order \(N - 1\) so that a plot of \(\varepsilon_{\text{MSE}}\) will result in a curve with up to \(\phi(1) = 1\) discontinuities in slope (where \(\phi(\cdot)\) is Euler’s totient function). 2

1This reasoning is strictly valid only when no synchronization mechanism exists between the rate of change of the signal and the period of the sampler. (Techniques for the consideration of rational rates can be found in [11]). However, physical systems almost always exhibit some slight variation in rate, and the formulae derived based on the irrational assumption are found to validate practical performance.

2Therefore, a large array of \(r\) values is desirable when numerically investigating or graphically representing (11), and this array should be chosen to include all fractional values with small denominators when in reduced form.
A. Average Mean-Squared Error as Rate Varies

If it is assumed that the fractional value of the rate varies over [0, 1) with equal probability over the duration of the measurement, each of the terms (rv) will have a uniform distribution on [0, 1). Therefore, the average mean-squared error due to quantization (when β replaces (rv)) is

\[ E_{\text{ms}} = \sum_{n=0}^{N-1} h_n h_{n+r} \int_0^1 \beta(1-\beta)r\,d\beta \]

\[ = \sum_{n=0}^{N-1} h_n h_{n+r} \]

\[ = \frac{1}{12} \sum_{n=0}^{N-1} h_n^2 \] (12)

making use of (10). Minimization of (13) through suitable choice of \( \{h_n\} \) is therefore equivalent to filter optimization with respect to optimum attenuation of white-noise signal corruption. This does not imply that quantization noise can always be modeled as white noise, but rather that some average error measures due to the presence of quantization noise will be proportional to those due to the addition of white noise for some types of input.

IV. Error Spectra of Constant-Rate Signals

When \( \langle v \rangle \) is constant, the spectral content of the error is discrete and nonlinearly dependent on \( \langle v \rangle \). The standard formula providing the power spectral density of stochastic processes also holds for quasistationary processes [2], [12]. Therefore, the power spectral density at frequency \( f \) (normalized with respect to the frequency of sampling of the differentiator inputs), as a function of the autocorrelation of the filter output \( \hat{e} \), is

\[ S_b(f) = \sum_{r=-\infty}^{\infty} R_b(r)e^{-2\pi i rf}. \] (14)

It is shown in Appendix A that the magnitudes and frequencies of the power spectral components of the output of the differentiator are given by

\[ P_b(f) = \sum_{r=1}^{N-1} \left( \sum_{n=0}^{N-1} h_n h_{n+r} \right) \frac{\sin^2(\pi kr\langle v \rangle)}{\pi^2 k^2} \]

\[ f = \min(\langle kM\rangle, 1 - \langle kM\rangle), \quad k = 1, \ldots, \infty \] (15)

when the FIR filter output is downsampled by a factor \( M \) relative to the input samples (where \( M \) evenly divides \( N \)) and a two-sided spectrum is considered. Downsampling is included because it is a feature of some practical differentiator implementations that can be accommodated without added mathematical complexity, thereby increasing the generality of the derived formulae.

V. Specific FIR Differentiators

Two useful FIR differentiators, both with constant group delay, are analyzed in this section, based on the general formulae already derived.

A. DIFF1—Optimum White-Noise Attenuation

DIFF1 corresponds to an FIR differentiator with optimum attenuation with respect to added white-noise corruption and is defined by [9]

\[ h_n = \frac{6}{N(N+1)} \left( 1 - \frac{2n}{N-1} \right) \]

\[ = \frac{6(N+1-2n)}{N(N^2-1)}, \quad 0 \leq n \leq N-1 \] (16)

where \( 0 \leq n \leq N-1 \). The arithmetic progression that is evident in the filter coefficients facilitates an efficient recursive implementation of this filter [9]. Equations (9) and (15) can be simplified using

\[ \sum_{n=0}^{N-1} h_n h_{n+r} = \frac{12(N^3-N+(1-3N^2)r+2r^2)}{N^2(N^2-1)^2} \] (17)

which is obtained (after significant algebraic simplification) when the coefficients defined by (16) are substituted into the sum \( \sum_{n=0}^{N-1} h_n h_{n+r} \). This yields

\[ \bar{e}_{\text{ms}} = -12 \sum_{n=1}^{N-1} \frac{(N^3-N+(1-3N^2)r+2r^3)}{N^2(N^2-1)^2} \times \langle rv \rangle (1 - \langle rv \rangle) \] (18)

and

\[ P_b(f) = \frac{N^2(N^2-1)^2\pi^2k^2}{\frac{\sin^2(\pi kr\langle v \rangle)}{\pi^2 k^2}} \]

\[ = \min(\langle kM\rangle, 1 - \langle kM\rangle), \quad k = 1, \ldots, \infty \] (19)

The veracity of the latter formula is demonstrated in Fig. 1, which compares predicted and experimentally derived spectral estimates for an input sequence generated from a constant rate input (which is arbitrarily chosen as \( \langle v \rangle = 1.718 \)). The Bohr (almost periodic) spectrum [14], which is generated using the prior knowledge that the discrete spectral components are at frequencies \( f = \min(\langle kM\rangle, 1 - \langle kM\rangle) \), is in very close agreement with the theoretical formula. In contrast, the FFT-computed spectral peaks are in error due to the spectral leakage inherent in the FFT estimate. A cumulative spectrum would show that the total spectral energy predicted by the FFT in the neighborhood of each discrete spectral component is correct.

It is demonstrated in Appendix B that DIFF1 is equivalent to the optimum FIR interpolator that has been derived for first-order sigma–delta modulators.

B. DIFF2—Simple Accumulator-Based Differentiator

DIFF2 corresponds to a simple FIR differentiator with equal coefficients that is defined, for even \( N \), by

\[ h_n = \begin{cases} \frac{4r}{N^2}, & 0 \leq n \leq \frac{N}{2} - 1 \\ -\frac{4r}{N^2}, & N/2 \leq n \leq N - 1. \end{cases} \] (20)

This type of differentiation operation has been proposed [5] for the reduction of the effects of position sensor quantization in
motion control systems. A number of samples of the sensor output is accumulated during a sample period of the controller. The difference between the results of this accumulation over two successive sample periods provides an estimate of shaft speed. Simplification of (11) and (15) is again possible through application of the easily derived formula

\[ \frac{N^4}{16} \sum_{r=0}^{N-1} h_n h_{n+r} = \left\{ \begin{array}{l} N - 3r, \quad 1 \leq r \leq \frac{N}{2} - 1 \\ r - N, \quad \frac{N}{2} \leq r \leq N - 1 \end{array} \right. \]  

This yields

\[ \bar{\epsilon}_{\text{ms}} = \frac{16}{N^4} \left[ \sum_{r=0}^{N-1} (N - r) \langle rv \rangle (1 - \langle rv \rangle) \right. 
\left. - \sum_{r=1}^{N/2-1} (N - 3r) \langle rv \rangle (1 - \langle rv \rangle) \right] \]  

and

\[ P_b(f) = \frac{16}{\pi^2 k^2 N^4} \left( \sum_{r=0}^{N-1} (N - r) \sin^2(\pi k r v) \right. 
\left. - \sum_{r=1}^{N/2-1} (N - 3r) \sin^2(\pi k r v) \right) \] 

\[ f = \min(\langle k M v \rangle, (1 - \langle k M v \rangle)), \quad k = 1, \ldots, \infty. \]  

In a manner similar to that employed for DIFF1, it can be shown that the application of this differentiator to the quantization error of a first-order sigma–delta modulator is equivalent, in terms of the converter error, to the use of a sinc^2 filter as a decoder for such a modulator.

VI. WORST-CASE MEAN-SQUARED ERROR FOR CONSTANT-RATE SIGNAL

It will be evident from the graphs presented in Section VIII, as well as being intuitively obvious, that the worst-case error occurs when \( \langle rv \rangle \) is close to zero or one, i.e., in the first or last segments of the \( \bar{\epsilon}_{\text{ms}} \) versus \( \langle rv \rangle \) curve. Under this assumption, and noting that \( \bar{\epsilon}_{\text{ms}}(\langle rv \rangle) = \bar{\epsilon}_{\text{ms}}(1 - \langle rv \rangle) \), the first segment of the curve (when \( v < 1/(N - 1) \)) is analyzed by replacing \( \langle rv \rangle \) in (9) by \( rv \) so that

\[ \bar{\epsilon}_{\text{ms}} = - \sum_{r=1}^{N-1} \sum_{n=0}^{N-1} h_n h_{n+r}rv(1 - rv), \] 

\[ = -S_r v + S_{r2} v^2 \]  

where

\[ S_r = \sum_{r=1}^{N-1} \sum_{n=0}^{N-1} r^2 h_n h_{n+r} \]  

\[ S_{r2} = \sum_{r=1}^{N-1} \sum_{n=0}^{N-1} r^2 h_n h_{n+r} \]  

By differentiation, the maximum value of \( \bar{\epsilon}_{\text{ms}} \) is found to occur at a rate with fractional part

\[ v_{\text{wc}} = \frac{S_r}{2S_{r2}} = \frac{\sum_{r=1}^{N-1} \sum_{n=0}^{N-1} r^2 h_n h_{n+r}}{2 \sum_{r=1}^{N-1} \sum_{n=0}^{N-1} r^2 h_n h_{n+r}}. \]  

\( S_{r2} \) can be simplified by rewriting as follows:

\[ S_{r2} = \sum_{y=0}^{N-1} \sum_{x=0}^{N-1} (y - x)^2 h_x h_y \] 

\[ = \frac{1}{2} \sum_{y=0}^{N-1} \sum_{x=0}^{N-1} (y - x)^2 h_x h_y \] 

\[ = \frac{1}{2} \sum_{y=0}^{N-1} y^2 h_y \sum_{x=0}^{N-1} h_x - 2 \sum_{y=0}^{N-1} y h_y \sum_{x=0}^{N-1} x h_x \] 

\[ + \sum_{y=0}^{N-1} y h_y \sum_{x=0}^{N-1} x^2 h_x \] 

\[ = -1 \]  

for all digital differentiators obeying (3) and (4). Applying this to (24) and (27), the worst-case mean-squared error is \( \bar{\epsilon}_{\text{ms-wc}} = \langle S_r \rangle^2 / 4 \) at a fractional rate of \( v_{\text{wc}} = -S_r/2 \). Therefore, the worst-case rms errors of all FIR differentiators exhibiting zero error for constant-rate inputs are numerically equal to the fractional rates at which these worst-case conditions occur.

VII. MINIMIZATION OF THE WORST-CASE ERROR

Minimization of \( -S_r \), through appropriate choice of the FIR filter coefficients, ensures minimum worst-case mean-squared differentiator error. The Lagrangian multiplier technique can be applied via the constructed function

\[ F = -S_r + \lambda_1 \sum_{n=0}^{N-1} h_n + \lambda_2 \left( 1 + \sum_{n=0}^{N-1} n h_n \right). \]  

Expansion of the partial differential equations \( \partial F / \partial h_n = 0 \), \( m = 0, \ldots, N - 1 \) shows that they can be rewritten as

\[ \frac{\partial F}{\partial h_m} = \left( \sum_{r=0}^{N-1} -m h_{n-m} \right) + \lambda_1 + m \lambda_2 = 0 \] 

\( m = 0, \ldots, N - 1. \)
Solution of the $N+2$ simultaneous equations given by (3), (4), and (30) yields the solution

$$h_n = \begin{cases} \frac{1}{N+1}, & n = 0 \\ 0, & 1 \leq n \leq N - 2 \\ -\frac{1}{N+1}, & n = N - 1. \end{cases} \quad (31)$$

This is a simple, first-order differentiator (henceforth designated DIFF3) with the longest possible time separation between samples. The mean-squared error function for this differentiator is

$$\bar{e}_{\text{rms}} = \frac{1}{(N-1)^2} \left( (N-1)v(1 - ((N-1)v)) \right), \quad (32)$$

It is found to perform better than DIFF1 when the rate changes slowly and $\langle v \rangle$ is close to 0 or 1 but to be significantly poorer otherwise.

VIII. COMPARISON OF FIR DIFFERENTIATORS

For comparison purposes, a lowpass differentiator (DIFF4) is designed using the classical Parks–McClellan algorithm, which utilizes the Remez exchange algorithm, to produce an optimized equiripple filter [15] [but rescaled in this case to satisfy (4)]. Assuming a Nyquist frequency of 0.5, desired specifications for the passband and stopband are

$$|H(f)| = \begin{cases} 2\pi f, & f < 0.05 \\ 0, & f > 0.2 \end{cases} \quad (33)$$

when $N \leq 16$. [When $N > 16$, the transition band is narrowed by reducing the frequency of the lower edge of the stopband to $0.05 + 0.15(16/N).$]

The impulse responses of DIFF4 and the three FIR digital differentiators analyzed in Sections V and VII are displayed in Fig. 2, (assuming $N = 16$). The relatively large magnitudes of the coefficients of the wideband differentiator imply poor attenuation of white noise. The corresponding frequency responses are shown in Fig. 3. The root-mean-squared error functions that arise for constant-rate inputs are shown in Fig. 4, where it is again assumed that $N = 16$. Formule for the average mean-squared errors for specific differentiators when $\langle v \rangle$ is uniformly distributed over $[0, 1]$ are easily found using (13). Those formulae pertaining to DIFF1 to DIFF3 are displayed in Table I whereas the average rms error for various filter lengths is shown in Fig. 5. The accumulating differentiator DIFF2 is at worst 5.75 dB poorer than the “optimum” differentiator DIFF1 when the filter length is large and less so for lower order differentiators.

The critical distinction between differentiators optimized for low frequency, using time-domain techniques, and a wideband differentiator is emphasized in the time-domain plots shown in Fig. 6. The improved quantization noise attenuation achieved using DIFF1 is clearly seen, as is its inability to act as a reliable differentiator for high-frequency inputs.

3The entries in Table I are consistent with the tabulation of the properties of parabolic, triangular, and uniform interpolation for $\Sigma–\Delta$ decoding presented in [16].
Fig. 5. Plot of average rms error (over all input rates) $\tau_{rms-ave}$ of DIFF1 to DIFF4 as a function of the FIR filter length $N$.

![Fig. 5](image)

**TABLE I**

<table>
<thead>
<tr>
<th>Formulae for Average Mean-Squared Error of Specific Differentiators</th>
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<tbody>
<tr>
<td>DIFF1</td>
</tr>
<tr>
<td>$\tau_{rms-ave}$</td>
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</tbody>
</table>

Fig. 6. Temporal plots comparing the input rate $v$ of a test signal prior to quantization with the rate estimates obtained using the "optimum" differentiator DIFF1 and the wideband differentiator DIFF4.

![Fig. 6](image)

a reduced worst-case error while maintaining much of the improved performance of the other differentiators. A simple nonlinear strategy is adopted (termed DIFF5):

1) Apply DIFF3 that corresponds to

$$\hat{v} = \frac{P_y(i) - P_y(i - N + 1)}{N - 1}.$$  \hspace{1cm} (34)

2) If $\langle \langle \hat{v} \rangle \rangle > 1/(N - 1)$ and $\langle \langle \hat{v} \rangle \rangle < 1 - (1/(N - 1))$, recalculating $\hat{v}$ using the "optimum" filter DIFF1, the coefficients of which are given by (16).

Because DIFF1 and DIFF3 have the same constant group delay, DIFF5 also exhibits this property. It is nonlinear because of the inclusion of a conditional term, thereby precluding frequency-domain analysis but has good time-domain characteristics. Because the conditional decision is based on a rate estimate $\hat{v}$, rather than on the actual rate $v$, which is unknown, the nonlinear filter will inevitably reduce performance at some rates. The corresponding rms error function is shown in Fig. 7. It was numerically verified that DIFF1 and DIFF5 exhibit similar average mean-squared error; when $N \geq 12$, DIFF5 is marginally superior to DIFF1 in this regard; otherwise, it is slightly inferior.

![Fig. 7](image)

**X. CONCLUSIONS**

Characteristics common to all FIR digital differentiators that exhibit zero error for constant-rate signals have been described in this paper through the derivation of generally applicable formulae. The impact of the correlation of quantization error and input signal on the differentiator output error was clearly demonstrated for constant-rate inputs. It has been shown that differentiators derived for optimum attenuation of white noise can be effective in the differentiation of quantized signals, unless the rate changes very rapidly. If signals are expected to have a high-frequency content prior to quantization, some tradeoff is required between designs intended to minimize the deleterious effects of quantization noise and wideband differentiators that are accurate at higher frequencies. The simple accumulator-based differentiator described in Section V-B was shown to possess good error characteristics, making it attractive for applications using low processing power. The result that the worst-case rms error always occurs at a rate equal to the fractional value of that rate, besides being mathematically intriguing, also provides a simple expression for the magnitude of that error. The filter derived to generate minimum worst-case error was shown not to be useful unless combined with the "optimum" linear differentiator in a nonlinear system so that the worst-case error could be reduced without affecting the average mean-squared error. The demonstrated equivalence between some sigma–delta converters and digital differentiators with quantized inputs is also potentially useful because it implies that results obtained for both types of systems can have a wider applicability.

**APPENDIX A**

**DERIVATION OF (15)**

Expansion of the product

$$c_b(i)c_b(i - y) = \left( \sum_{n=0}^{N-1} h_n c_{i-n} \right) \left( \sum_{n=0}^{N-1} h_n c_{i-n-My} \right)$$

when the filter output is downsampled by a factor $M$, where $M$ divides evenly into $N$, easily leads to an autocorrelation expression for the differentiator output

$$R_v(y) = \left( \sum_{n=0}^{N-1} h_n^2 \right) R_v(My)$$

$$+ \sum_{r=1}^{N-1} \left( \sum_{n=0}^{N-1-w} h_n h_{n+r} \right) \left( R_v(My + r) + R_v(My - r) \right).$$
The power density of the quasistationary signal is defined by (14)

\[ S_0(f) = \sum_{y=-\infty}^{\infty} R_0(y)e^{-2\pi juf} \]

\[ = \left( \sum_{n=0}^{N-1} h_n^2 \right) \sum_{y=-\infty}^{\infty} R_e(My)e^{-2\pi juf} \]

\[ + \sum_{r=1}^{N-1} \left( \sum_{n=0}^{N-1-r} h_n h_{n+r} \right) \cdot (R_e(My + r) + R_e(My - r)) \]

\[ \times e^{-2\pi juf}. \]

Substituting for the autocorrelation terms using (6) and rearranging the order of summation

\[ S_0(f) = \left( \sum_{n=0}^{N-1} h_n^2 \right) \sum_{y=-\infty}^{\infty} \frac{1}{4\pi^2 k^2} \sum_{y=-\infty}^{\infty} e^{2\pi jy(kMy-f)} \]

\[ + \sum_{r=1}^{N-1} \left( \sum_{n=0}^{N-1-r} h_n h_{n+r} \right) \sum_{y=-\infty}^{\infty} \frac{e^{2\pi jy(kMy-f)}}{4\pi^2 k^2} \]

Therefore

\[ S_0(f) = \left[ \sum_{n=0}^{N-1} h_n^2 \right] \sum_{y=-\infty}^{\infty} \frac{1}{4\pi^2 k^2} \sum_{y=-\infty}^{\infty} e^{2\pi jy(kMy-f)} \]

\[ + \sum_{r=1}^{N-1} \left( \sum_{n=0}^{N-1-r} h_n h_{n+r} \right) \frac{\cos(2\pi y k v)}{2\pi^2 k^2} \]

\[ \times \delta(kMy - f), \quad k \neq 0 \]

due to the orthogonality of the complex exponential functions ($\delta(\cdot)$ represents the Dirac delta function). The power spectrum can be written as

\[ P_0(f) = \frac{1}{4\pi^2 k^2} \left[ \sum_{n=0}^{N-1} h_n^2 \right] + 2 \sum_{r=1}^{N-1} \left( \sum_{n=0}^{N-1-r} h_n h_{n+r} \right) \cos(2\pi y k v) \]

\[ \times \delta(kMy - f), \quad k \neq 0 \]

\[ f = \langle kMy \rangle, \quad k \neq 0 \]

\[ = \frac{1}{4\pi^2 k^2} \sum_{r=1}^{N-1} \left( \sum_{n=0}^{N-1-r} h_n h_{n+r} \right) \sin^2(\pi y k v) \]

\[ f = \langle kMy \rangle, \quad k \neq 0 \]


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